1. (a) Since \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 8 > 1 \), part (b) of the Ratio Test tells us that the series \( \sum a_n \) is divergent.

(b) Since \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0.8 < 1 \), part (a) of the Ratio Test tells us that the series \( \sum a_n \) is absolutely convergent (and therefore convergent).

(c) Since \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 \), the Ratio Test fails and the series \( \sum a_n \) might converge or it might diverge.

2. The series \( \sum_{n=1}^{\infty} \frac{2^n}{2^n} \) has positive terms and

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \left[ \frac{(n+1)^2 \cdot 2^n}{n^2 \cdot 2^n} \right] = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^2 \cdot \frac{1}{2} = \frac{1}{2} < 1 , \text{ so the series is absolutely convergent by the Ratio Test.}
\]

3. \( \sum_{n=0}^{\infty} \frac{(-10)^n}{n!} \). Using the Ratio Test,

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-10)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-10)^n} \right| = \lim_{n \to \infty} \left| \frac{-10}{n+1} \right| = 0 < 1 , \text{ so the series is absolutely convergent.}
\]

4. \( \sum_{n=1}^{\infty} (-1)^{-n-1} \frac{2^n}{n^4} \) diverges by the Test for Divergence. \( \lim_{n \to \infty} \frac{2^n}{n^4} = \infty \), so \( (-1)^{-n-1} \frac{2^n}{n^4} \) does not exist.

9.

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{1/(2n+2)!}{1/(2n)!} = \lim_{n \to \infty} \frac{(2n)!}{(2n+2)!} = \lim_{n \to \infty} \frac{(2n)!}{(2n+2)(2n+1)(2n)} = \lim_{n \to \infty} \frac{1}{(2n+2)(2n+1)} = 0 < 1 , \text{ so the series } \sum_{n=1}^{\infty} \frac{1}{(2n)!} \text{ is absolutely convergent by the Ratio Test. Of course, absolute convergence is the same as convergence for this series, since all of its terms are positive.}
\]

10.
25. Use the Ratio Test with the series

\[
1 \cdot \frac{3}{5!} + \frac{1 \cdot 3 \cdot 5}{7!} + \cdots + \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n-1)!} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n-1)!}
\]

By the Ratio Test,

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-1) [2(n+1)-1]}{(2n+1)(2n+2)(2n+3) \cdots (2n+1)!} \cdot \frac{(2n-1)!}{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-1)} \right| = \lim_{n \to \infty} \frac{1}{2n} = 0 < 1,
\]

so the given series is absolutely convergent and therefore convergent.

29. By the recursive definition,

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{5n+1}{4n+3} \right| = \frac{5}{4} > 1,
\]

so the series diverges by the Ratio Test.

31. (a) \[ \lim_{n \to \infty} \left| \frac{1/(n+1)^3}{1/n^3} \right| = \lim_{n \to \infty} \left| \frac{n^3}{(n+1)^3} \right| = \lim_{n \to \infty} \frac{1}{(1+1/n)^3} = 1. \quad \text{Inconclusive.} \]
(b) \( \lim_{n \to \infty} \left| \frac{(n+1)^{2n}}{2^{n+1}} \cdot \frac{2^n}{n} \right| = \lim_{n \to \infty} \frac{n+1}{2n} = \lim_{n \to \infty} \left( \frac{1}{2} + \frac{1}{2n} \right) = \frac{1}{2} \). Conclusive (convergent).

(c) \( \lim_{n \to \infty} \left| \frac{(-3)^n \cdot \sqrt{n}}{\sqrt{n+1} \cdot (-3)^{n-1}} \right| = 3 \lim_{n \to \infty} \sqrt{\frac{n}{n+1}} = 3 \lim_{n \to \infty} \sqrt{\frac{1}{1+1/n}} = 3 \). Conclusive (divergent).

(d) \( \lim_{n \to \infty} \left| \frac{\sqrt{n+1}}{1+(n+1)^2} \cdot \frac{1+n^2}{\sqrt{n}} \right| = \lim_{n \to \infty} \left[ \sqrt{1+\frac{1}{n}} \cdot \frac{1/n^2 + 1}{1/n^2 + (1+1/n)^2} \right] = 1 \). Inconclusive.

33. (a) \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n \to \infty} \left| \frac{x}{n+1} \right| = |x| \lim_{n \to \infty} \frac{1}{n+1} = |x| \cdot 0 = 0 < 1 \), so by the Ratio Test the series \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \) converges for all \( x \).

(b) Since the series of part (a) always converges, we must have \( \lim_{n \to \infty} \frac{x^n}{n!} = 0 \) by Theorem .2.6.

39. (a) Since \( \sum a_n \) is absolutely convergent, and since \( \left| a_n^+ \right| \leq \left| a_n \right| \) and \( \left| a_n^- \right| \leq \left| a_n \right| \) (because \( a_n^+ \) and \( a_n^- \) each equal either \( a_n \) or 0), we conclude by the Comparison Test that both \( \sum a_n^+ \) and \( \sum a_n^- \) must be absolutely convergent. (Or use Theorem .2.8.)

(b) We will show by contradiction that both \( \sum a_n^+ \) and \( \sum a_n^- \) must diverge. For suppose that \( \sum a_n^+ \) converged. Then so would \( \sum \left( a_n^+ - \frac{1}{2} a_n \right) \) by Theorem .2.8. But

\[
\sum \left( a_n^+ - \frac{1}{2} a_n \right) = \sum \left[ \frac{1}{2} a_n^+ - \frac{1}{2} a_n \right] = \frac{1}{2} \sum \left| a_n \right|,
\]

which diverges because \( \sum a_n \) is only conditionally convergent. Hence, \( \sum a_n^+ \) can’t converge. Similarly, neither can \( \sum a_n^- \).