CLASSICAL INTEGRABLE SYSTEMS AND THEIR QUANTUM COUNTERPARTS

1. Content

The lectures will start from examples of quantum integrable systems. Basic examples are Heisenberg spin chain, quantum Bose gas with δ-interaction, and the Toda system. After the introducing natural algebraic structures related to these examples we will pass to classical integrable systems. Then we will focus on the notion of integrability in classical and quantum case, will see how classical counterparts of our examples are related to the geometry of corresponding Poisson Lie groups. Then the list of subject depends on how fast the course will go. This list will evolve during the semester.

The next section contains the outline of the material and a sequence in which it will appear. All notions will be introduced and explained. It is desirable that students know basic algebra, have the knowledge of basic facts in geometry.

Section 3 contain the list of projects. This part will also evolve as well as the list of references.

2. Lectures

2.1. Examples of quantum integrable systems. Here we will focus on the Heisenberg spin chain model and on the 6-vertex model in statistical mechanics. After brief discussion of physical motivations and of most interesting problems related to these models (from physics perspective) we will focus on an algebraic problem of diagonalizing transfer-matrices in the 6-vertex model. The main algebraic result here is the Bethe anstaz.

(1) The Heisenberg spin chain. Isotropic, anisotropic. Isotropic: the sum of elementary permutations. Physically interesting problems, $N \to \infty$, ground state, separable subspace in $\mathbb{C}^{2^N}$ when $N \to \infty$, asymptotical spectrum, correlation functions.

The 6-vertex model. States, weights, partition function with open boundary conditions, boundary states; periodic boundary conditions and partition functions of the cylinder or torus, higher genus surfaces. Row-to-row transfer-matrix for tori and cylinders. Physically interesting problems: the asymptotical behavior of the partition function on large graphs, why the transfer-matrix representation is important. The spectrum of the transfer-matrix for large $N$. The asymptotic of the partition function on $N \times M$ torus when $N, M \to \infty$. The dependence of this asymptotic on weights; asymptotic of correlation functions.

The dependence of the asymptotic of the partition function on the boundary state. General discussion.


(3) The spectrum of the 6-vertex transfer-matrix. Algebraic Bethe ansatz for quasiperiodic boundary conditions. The ”coordinate” form of the Bethe ansatz. Bethe equations. Other ”integrable” boundary conditions (reflection b.c.).
2.2. Algebraic perspective.

(1) Two Hopf algebras $U_\pm$ (trigonometric and rational cases) related to the 6-vertex model. Yangian of $sl_2$. Commutative subalgebras. Rational quotients. Irreducible minimal representations. $U_q(sl_2)$. Tensor product representations. Tensor product of two dimensional representations and the transfer-matrix for the 6-vertex model. Real forms and positivity.

(2) Symmetric higher spin generalizations of the 6-vertex model as an irreducible finite dimensional representation of $U_\pm$. "Integrable" high spin generalization of the Heisenberg spin chain. Higher commuting Hamiltonians. Diagonalization of the higher spin transfer-matrix for quasiperiodic boundary conditions.

(3) Quantum 1-dimensional Bose particles with $\delta$-function interaction. Bethe ansatz. The existence of infinitely many commuting Hamiltonians. Quantum Lax operator and the quantum monodromy matrix. The system as an infinite-dimensional representation of the Yangian type Hopf algebra. The algebraic Bethe ansatz. Quantum Bose gas as a limit of higher spin Heisenberg spin chain.

(4) Quantum periodic Toda chain. The Lax operator and the quantum monodromy matrix. Corresponding representation of the Yangian.

2.3. From quantum to classical and back.

(1) Symplectic and Poisson manifolds. Symplectic leaves. Example: constant symplectic and Poisson structures on $\mathbb{R}^n$, $T^*N$, $g^*$. Isotropic, co-isotropic and Lagrangian submanifolds, examples. The structure of a neighborhood of Lagrangian submanifold.

(2) Integrable Hamiltonian systems. Liouville theorem: the structure of level surfaces of integrals. Examples: Toda system, classical Heisenberg spin chain. $r$-matrix Poisson brackets and Poisson Lie structures on loop groups.

(3) The general machinery of how to obtain integrable systems from Poisson Lie groups. Degenerate integrable systems. Finite dimensional simple Lie groups, their symplectic leaves and double Bruhat cells. Symplectic leaves in loop groups and polynomial loop groups. Finite dimensional and infinite dimensional systems, Nonlinear Schrodinger (NS) system.

(4) Quantization of Poisson manifolds. Correspondence between symplectic leaves of Poison manifolds and irreducible representations of their quantized algebras of functions. Examples: $g^*$ and universal enveloping algebra, Poisson Lie group $SL_n^*$ and $U_q(sl_2)$. Quantization of $T^*N$ and differential operators. Quantization of a neighborhood of a Lagrangian submanifold and differential operators.

(5) Integrable quantization of integrable Hamiltonian systems. Examples: Toda system, higher spin Heisenberg spin chain. Infinite dimensional example: the NS equation and Bose gas with $\delta$-interaction.

(6) The semiclassical quantization of integrable systems. WKB asymptotic of joint eigenfunctions.

2.4. Algebraic integrable systems.

(1) Complex analytic integrable systems, algebraic integrable systems. General structure. Examples: integrable systems related to symplectic leaves in loop groups. Spectral curve.

(2) Solutions to corresponding integrable flows in terms of $\theta$-functions.

(3) The semiclassical limit in Bethe vectors and the spectral curve.

2.5. The thermodynamical limit and asymptotical representation theory. Most likely we will have no time for this material. Some of the topics can be projects for the final workshop presentation or research projects.

(2) The discussion of the thermodynamical limit for lattice models. The dependance of the partition function on boundary conditions. Scales, limit shapes and phase transitions.

(3) Large $N$ limit in the spectrum of transfer-matrices. The conjectural structure of the spectrum. Correlation functions and the partition function in the thermodynamical limit. The ground state ($\Delta < -1$) of the 6-vertex model.

(4) Large $N$ limit in classical integrable systems. Large $N$ limit and the semiclassical limit.

(5) Antiferroelectric vacuum. Vertex operators and the Fock space. The computation of correlation functions as matrix elements of vertex operators, and quantum Knizhnik-Zamolodchikov equation.

Related subjects: Knizhnik-Zamolodchikov (KZ) connection. Hitchin systems and KZ connections as their quantization. Isomonodromic deformations and KZ connections as their quantization.

3. Possible project

Some of these projects go beyond 20-30 min presentations and can be research projects for those who want to start working in this area.

(1) Completeness of the spectrum for generic inhomogeneities, non-degeneracy of the spectrum (for generic parameters), when the transfer-matrix is positive definite, when the transfer-matrix is Hermitian.

(2) Reflection boundary conditions, corresponding algebra of transfer-matrices, Bethe ansatz for reflection boundary conditions, completeness of eigenvectors. Fock space, boundary state and correlation functions.

(3) Special degenerations of the 6-vertex model such as dimer models, 5-vertex model, limit shapes in dimer models.

(4) The universal R-matrix for the double of a Hopf algebra. The 6-vertex R-matrix and the algebra $U_q(\widehat{sl}_2)$.

(5) Highest weights and evaluation representations of $U_q(\widehat{sl}_2)$, quantum Knizhnik-Zamolodchikov equation.

(6) Double Bruhat cells for simple Lie groups and for polynomial loop groups and cluster variables.

(7) Other examples of integrable systems.

(8) The 6-vertex model and the asymmetric exclusion process (ASEP). Semiclassical limit in the ASEP.

(9) Asymptotic of the partition function on the torus in the thermodynamical limit.

(10) Conformally invariant quantum field theories in 2-dimensions, on a plane, on a domain, on a cylinder.

(11) Criticality and conformal invariance in 2-dimensions.

(12) Integrable perturbations of 2-dimensional CFT (on a cylinder).

References


