REPRESENTATIONS OF THE ALGEBRA $U_q(sl(2))$, q-ORTHOGONAL POLYNOMIALS AND INVARIANTS OF LINKS

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INTRODUCTION

This work shows how quantized universal enveloping algebras are connected with other areas of mathematics, using algebra sl(2) as an example. It is shown that in the representation theory of the algebra $U_q(sl(2))$ the q-analogues of $6j$-symbols which were introduced by Askey and Wilson [1] in connection with q-orthogonal polynomials, appear naturally. The connection between the quantized universal algebras and the theory of invariants of links, discovered in [2], is considered in more detail. With the help of q-analogues of $6j$ -symbols we propose a new representation for the invariants of links, related to $U_q(sl(2))$, which is to a great extend similar to SOS models of statistical physics. The representation theory of algebra $U_q(sl(2))$ is closely connected with Temperly-Lieb-Jones algebra, which emerged in statistical mechanics [3] and in the theory of von-Neumann algebras [4]. It happens that the matrix elements of generators in irreducible representations of Jones algebras are special values of q-analogues of $6j$ -symbols.

Let us make some historical comments. Quantum universal enveloping algebras appeared as a result of research into on algebraic aspects of quantum integrable systems [5, 6]. The first example of such an algebra was the algebra $U_q(sl(2))$ found by Kulish and Reshetikhin [7]. The structure of Hopf algebra on $U_q(sl(2))$ was discovered independently by Sklyanin [8], Drinfeld [9], Jumbo [10] who built the q-deformation of the universal enveloping algebra for any simple Lie algebra $\mathfrak{g}$ [9, 10]. A new approach to $U_q(\mathfrak{g})$ algebras, which reflects most adequatly their connection with quantum integrable systems, was proposed by Faddeev, Reshetikhin and Takhtadjan [11], (see also [12]). Finite dimensional representationsoof $U_q(sl(2))$ are described in [7]. First substantial results in the representation theory of algebras $U_q(\mathfrak{g})$ were obtained by Lusztig [13] and Rosso [14] for simple $\mathfrak{g}$. The q-analogues of the Rakah-Fock formulae for $3j$ -symbols were obtained by Vaksman [15]. Other representation of the q-analogues of $3j$ -symbolds, as well as the properties of their symmetry were found by Kirillov [16].

The connection between universal enveloping algebras with the link theory was established in [9]. We should also mention that the invariants of links, connected with tensor representations of $U_q(\mathfrak{g})$ for classical Lie algebras, can be obtained with the help of cabling (Murakami [17]). Cabling invariants correspond in the terminology of [2] to invariants parametrized by the tensor products of the vector representations of $U_q(\mathfrak{g})$. In particular, the invariants corresponding to finite dimensional representations of $U_q(sl(2))$ are built by Akutzu and Wadati [18] with help of braid representation using the results from vertex models of statistical mechanics. The main results obtained in this direction, are represented in the review by V. Jones [29].

Then studying classical orthogonal polynomials Askey and Wilson [1], proposed q-analogues of $6j$ -symbols. As it turns out the q-analogues of $6j$ -symbols, that occur in the representation theory of $U_q(sl(2))$ are proportional to the ones defined in [1]. The orthogonality of these polynomials as well as the recurrent relations for them follows from equalities for $q-6j$-symbols in the representation theory of the algebra $U_q(sl(2))$.

Let us briefly consider the content of this paper. Section 1 contains the description of the algebra $U_q(sl(2))$, of the corresponding universal R-matrix and also presents some useful formulae. The irreducible representations of $U_q(sl(2))$ and the q-analog of the Weyl element are described in Section 2. In the same section an extension of algebra $U_q(sl(2))$ by the q-analog of the Weyl element is introduced.

The decomposition of tensor product of two irreducible finite-dimensional representation of $U_q(sl(2))$ is given in Section 3. It also contains the relations between R-matrices and Clebsh-Gordan coefficients (CGC). We prove that the extension of Uq(sl(2)) by the Weyl element is a Hopf algebra. In Section 4, following [2], a graphical representation of relations between R-matrices and GGC is proposed. In Section 5 the q-analogues of $6j$-symbols are described. It is shown that they are defined by q-hypergeometrical function ${}_4\varphi_3$ and the symmetries of them are found. Note that the q-analogues of CGC and $6j$-symbols correspond to the function ${}_3\varphi_2$ and ${}_4\varphi_3$ for such values of arguments when these functions are polynomials. Graphical representations for $q-6j$-symbols are introduced in Section 6. Relations between $q-6j$-symbols, particularly their orthogonality, are easily obtained with their help.

In Section 7 a new representation for the invariants of links connected with $U_q(sl(2))$ is built with the help of $q-6j$-symbols. This representation is an exact analogue of SOS models in statistical mechanics [19]. The relation of SOS models in statistical mechanics to q-analogue of $6j$-symbols was recently found also by Pasquier [31].

1. Algebra $U_q(sl(2))$

The algebra $U_q(sl(2))$ [7-10] is generated by elements H, $X^{\pm}$ with the commutation relations

$$[X^{\pm},H]=\mp 2X^{\pm}\,,\quad [X^{+},X^{-}]=\frac{q^{H/2}-q^{-H/2}}{q^{1/2}-q^{-1/2}}\,. \tag{1.1}$$

As a linear algebra $U_q(sl(2))$ space consists of convergent power series in H and of polynomials in $X^{\pm}$. The following formulae for the comultiplication, the antipode and counit on the generators define the structure of a Hopf algebra on $U_q(sl(2))$:

$$\Delta(X^{\pm})=X^{\pm}\otimes q^{H/4}+q^{-H/4}\otimes X^{\pm}\,,\quad \Delta(H)=H\otimes 1+1\otimes H\,, \tag{1.2}$$

$$S(X^{\pm})=-q^{\pm 1/2}X^{\pm}\,,\quad S(H)=-H\,, \tag{1.3}$$

$$\varepsilon(H)=\varepsilon(X^{\pm})=0\,. \tag{1.4}$$

The maps $\Delta'=\sigma\circ\Delta$, $S'=S^{-1}$ where σ is the permutation in $U_q(sl(2))^{\otimes 2}$, $\sigma(a\otimes b)=b\otimes a$ also define the structure of Hopf algebra on $U_q(sl(2))$. Let us denote this Hopf algebra as by $U_q(sl(2))'$. It is evident from (1.2), (1.3) that

$$U_q(sl(2))'=U_{q^{-1}}(sl(2))\,. \tag{1.5}$$

Comultiplications Δ and Δ' are connected in $U_q(sl(2))^{\otimes 2}$ by the following automorphism [9]:

$$\Delta'(a)=R\,\Delta(a)\,R^{-1}\,, \tag{1.6}$$

where $R \in U_q(sl(2))^{\otimes 2}$ and

$$R = \exp\left(\frac{h}{4} H \otimes H\right) \sum_{n \geq 0} \frac{(1-q^{-1})^n}{[n]!} e^n \otimes f^n, \quad q = e^h. \quad (1.7)$$

Here $[n] = (q^{n/2} - q^{-n/2})/(q^{1/2} - q^{-1/2})$, $e = \exp(\frac{h}{4} H) X^+$, $f = \exp(-\frac{h}{4} H) X^-$.

The element R is called the universal R-matrix. It satisfies the relations:

$$(\Delta \otimes id) R = R_{13} R_{23}, \quad (1.8)$$

$$(id \otimes \Delta) R = R_{13} R_{12}, \quad (1.9)$$

$$(S \otimes id) R = R^{-1}, \quad (1.10)$$

where the indices show the embeddings of R into $U_q(sl(2))^{\otimes 3}$.

Formula (1.8) (or (1.9)) imply the Yang-Baxter equation for R :

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}. \quad (1.11)$$

The center of the algebra $U_q(sl(2))$ is generated by the q-analog of Casimir's elements [8, 10]:

$$C = \left(\frac{q^{\frac{H+1}{4}} - q^{-\frac{H+1}{4}}}{q^{1/2} - q^{-1/2}} \right)^2 + X^- X^+. \quad (1.12)$$

For real q one can introduce $*$-antiinvolution

$$(X^{\pm})^* = X^{\mp}, \quad H^* = H. \quad (1.13)$$

The real form of $U_q(sl(2))$ corresponding to this $*$-antiinvolution is denoted by $U_q(su(2))$. The element (1.12) is invariant under the action of involution (1.13) and therefore all finite-dimensional representations of $U_q(su(2))$ are completely reducible.

Let us give now some useful formulae:

$$\Delta^{(N)}((X^{\pm})^m) = \sum_{0\le a_1\le \dots \le a_{N-1}\le m} \begin{bmatrix} & m & \\ a_1, a_2-a_1, a_3-a_2, \dots, m-a_{N-1} \end{bmatrix}_q$$

$$\bigotimes_{i=1}^{N} (X^{\pm})^{a_i-a_{i-1}} \, q^{-(m-a_i-a_{i-1})\cdot\frac{H}{4}}, \quad a_N = m, \tag{1.14}$$

$$[X_+, X_-^m] = X_-^{m-1} [m]_1 \frac{q^{\frac{m-1}{2}} q^{H/2} - q^{-\frac{m-1}{2}} q^{-H/2}}{q^{1/2} - q^{-1/2}}, \tag{1.15}$$

where $\Delta^{(N)} : U_q(sl(2)) \longrightarrow U_q(sl(2))^{\otimes N}$ is the composition of comultiplications and

$$\begin{bmatrix} & m & \\ b_1 & b_2 \dots & b_N \end{bmatrix}_q = \frac{[m]!}{[b_1]! \dots [b_N]!}, \tag{1.16}$$

$b_1 + \dots + b_N = m$.

2. Irreducible Representations Of $U_q(su(2))$ And Finite Dimensional R-matrices

All irreducible representations of $U_q(su(2))$ are finite dimensional. It is useful to describe them in the weight basis. In this basis the element H is a diagonal matrix. The irreducible representations of $U_q(su(2))$ are parametrized by integer (or halfinteger) numbers j. The representation π^j have dimension $2j+1$. The generators $H, X^{\pm}$ act in weight basis e^j_m in the following way:

$$\pi^j(X^{\pm})e^j_m = \left([j\mp m][j\pm m+1]\right)^{1/2} e^j_{m\pm 1}, \quad (2.1)$$

$$\pi^j(H)e^j_m = 2m e^j_m, \quad (2.2)$$

where $-j\le m\le j$, $2m\equiv 2j \pmod 2$. We shall denote by V^j the representation space of the representation π^j. From (1.1) and (1.2) at $m=j$ we find the value of the central element (1.12) in the irreducible representation:

$$c\,V^j = \left(\frac{q^{\frac{2j+1}{4}} - q^{-\frac{2j+1}{4}}}{q^{1/2} - q^{-1/2}}\right)^2 V^j = [j+\tfrac{1}{2}]^2 V^j. \quad (2.3)$$

The following important formula holds

$$\pi^j((X^{\pm})^a)e^j_n = \left(\frac{[j\mp n]!\,[j\pm n+a]!}{[j\mp n-a]!\,[j\pm n]!}\right)^{1/2} e^j_{n\pm a}. \quad (2.4)$$

Here the r.h.s. is nonzero only for $|n\pm a|\le j$. Let us consider the matrix $\mathbb{R}^{j_1 j_2} = (\pi^{j_1}\otimes\pi^{j_2})R$ acting in $V^{j_1}\otimes V^{j_2}$. It is not difficult find the matrix elements of $\mathbb{R}^{j_1 j_2}$:

$$\left(\mathbb{R}^{j_1 j_2}\right)^{n_1\, n_2}_{n_1+n,\, n_2-n} = \frac{(1-q^{-1})^n}{[n]!}\, q^{\frac{n_1-n_2+2n}{2}n}.$$

$$\left(\frac{[j_1-n_1]!\,[j_1+n_1+n]!\,[j_2+n_2]!\,[j_2-n_2+n]!}{[j_1-n_1-n]!\,[j_1+n_1]!\,[j_2+n_2-n]!\,[j_2-n_2]!} \right)^{1/2} . \qquad (2.5)$$

Define the matrix w^j acting in V^j with the elements:

$$w^j_{mm'} = q^{-\frac{c_j}{2}} (-1)^{j-m} \delta_{m,-m'}\, q^{-\frac{m}{2}}, \qquad (2.6)$$

where $c_j = j(j+1)$ and consider the algebra $\widetilde{U_q}(su(2))$ which is the extension of $U_q(su(2))$ by the element with the value (2.6) in any representation. Let τ be the linear antiautomorphism of $U_q(su(2))$ which is the transposition in

$$\tau(X^{\pm}) = X^{\mp}, \quad \tau(H) = H. \qquad (2.7)$$

PROPOSITION 2.1. In $\widetilde{U_q}(su(2))$ we have:

$$w a w^{-1} = \tau S(a), \quad \forall a \in U_q(su(2)). \qquad (2.8)$$

To prove (2.8) it is sufficient to check it in any irreducible $U_q(su(2))$-module.

From (1.10) and (2.8) we obtain the crossing-symmetry of universal R-matrix:

$$(\tau \otimes id) R^{-1} = (w \otimes 1) R (w^{-1} \otimes 1), \qquad (2.9)$$

and finite dimensional R-matrices $\mathbb{R}^{j_1 j_2}$,

$$\left(\left(\mathbb{R}^{j_1 j_2} \right)^{-1} \right)^{t_1} = w_1 \mathbb{R}^{j_1 j_2} w_1^{-1}. \qquad (2.10)$$

Here t_1 is the transposition in the first space in $V^{j_1} \otimes V^{j_2}$ and $w_1 = w^{j_1} \otimes 1$.

3. q-Analog Of Clebsch-Gordan Coefficients

Let us consider the tensor product $\pi^{j_1} \otimes \pi^{j_2}$ of two irreducible representations of $U_q(su(2))$. In accordance with the complete reducibility of representations of $U_q(su(2))$, $\pi^{j_1} \otimes \pi^{j_2}$ is decomposed into the following sum of irreducible components [10]:

$$V^{j_1} \otimes V^{j_2} = \sum^{\oplus}_{\substack{|j_1 - j_2| \le j \le j_1 + j_2 \\ 2j \equiv 2j_1 + 2j_2 \pmod 2}} V^{j} \quad . \tag{3.1}$$

Let $e^j_m(j_1 j_2)$ be the weight basis in the irreducible component V^j. The coordinates of the vectors $e^j_m(j_1 j_2)$ in the basis $e^{j_1}_{m_1} \otimes e^{j_2}_{m_2}$, by analogy with $q=1$ case, will are called the Clebsch-Gordan coefficients (CGC):

$$e^j_m(j_1 j_2) = \sum_{m_1, m_2} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q e^{j_1}_{m_1} \otimes e^{j_2}_{m_2} \,. \tag{3.2}$$

Since π^{j_1} are $*$ -representations of $U_q(su(2))$ we have also

$$e^{j_1}_{m_1} \otimes e^{j_2}_{m_2} = \sum_{j, m} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q e^j_m(j_1 j_2)\, \delta(j_1 j_2 j). \tag{3.2'}$$

Here $\delta(j_1 j_2 j) = 1$ if $|j_1 - j_2| \le j \le j_1 + j_2$, $2j \equiv 2j_1 + 2j_2 (2)$ and $\delta(j_1 j_2 j) = 0$ in other cases.

The coefficients in (3.2), (3.2') can be considered as q -analog of CGC for $su(2)$. The coefficients $\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q$ can be calculated using the formulae of Section 2. We shall omit the calculations and present only the final formulae:

q-analog of the Majumbar formula [16]:

$$\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q = \delta_{m_1+m_2,m}\, q^{-\frac{1}{4}(j_1+j_2-j)(j_1-j_2+j+1)-j\frac{m_2}{2}-j_2\frac{m}{2}}$$

$$(-1)^{j_1-m_1}\left\{\frac{[j-m]!\,[j+m]!\,[j_1+m_1]!\,[j_2+m_2]!\,[j_1+j_2-j]!\,[2j+1]}{[j_1-m_1]!\,[j_2-m_2]!\,[j_1-j_2+j]!\,[j_2-j_1+j]!\,[j_1+j_2+j+1]!}\right\}^{1/2}$$

$$\sum_{r\geq 0}(-1)^r\, q^{\frac{r(m-j-1)}{2}}\,\frac{[2j_2-r]!\,[j_1-j_2+j+r]!}{[r]!\,[j_2+m_2-r]!\,[j-j_2+m_1+r]!\,[j_1+j_2-j-r]!}. \tag{3.3}$$

q-analog of the Rakah-Fock formula [15]:

$$\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q = \delta_{m_1+m_2,m}\,(-1)^{j_1-m_1}\, q^{\frac{1}{4}\left(j_2(j_2+1)-j_1(j_1+1)-j(j+1)\right)+\frac{m_1(m_1+1)}{2}}$$

$$\left\{\frac{[j+m]!\,[j-m]!\,[j_1-m_1]!\,[j_2-m_2]!\,[j_1+j_2-j]!\,[2j+1]}{[j_1+m_1]!\,[j_2+m_2]!\,[j_1-j_2+j]!\,[j_2-j_1+j]!\,[j_1+j_2+j+1]!}\right\}^{1/2} \tag{3.4}$$

$$\sum_{r\geq 0}(-1)^r\, q^{\frac{1}{2}r(m+j+1)}\,\frac{[j_1+m_1+r]!\,[j_2+j-m_1-r]!}{[r]!\,[j-m-r]!\,[j_1-m_1-r]!\,[j_2-j+m_1+r]!}.$$

q-analog of the Van der Waerder formulae[16]:

$$\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q = \delta_{m_1+m_2,m}\, \Delta(j_1 j_2 j)\, q^{\frac{1}{4}(j_1+j_2-j)(j_1+j_2+j+1)+\frac{j_1 m_2 - j_2 m_1}{2}}$$

$$\left\{[j_1+m_1]!\,[j_1-m_1]!\,[j_2+m_2]!\,[j_2-m_2]!\,[j+m]!\,[j-m]!\,[2j+1]\right\}^{1/2}$$

$$\sum_{r\geqslant 0} (-1)^r\, q^{-\frac{1}{2} r (j_1+j_2+j+1)}\,. \qquad (3.5)$$

$$\frac{1}{[r]!\,[j_1+j_2-j-r]!\,[j_1-m_1-r]!\,[j_2+m_2-r]!\,[j-j_2+m_1+r]!\,[j-j_1+r-m_2]!},$$

where

$$\Delta(abc) = \left\{\frac{[-a+b+c]!\,[a-b+c]!\,[a+b-c]!}{[a+b+c+1]!}\right\}^{1/2}. \qquad (3.6)$$

The proofs of these formulae are given in [16].

The following relations between CGC reflect the completness and the orthogonality of the basis $e^j_m(j_1 j_2)$ in $V^{j_1}\otimes V^{j_2}$:

$$\sum_{m_1 m_2} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q \begin{bmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m' \end{bmatrix}_q = \delta_{jj'}\,\delta_{mm'}\,\delta(j_1 j_2 j). \qquad (3.7)$$

$$\sum_{\substack{|j_1-j_2|\le j\le j_1+j_2 \\ |m|\le j}} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q \begin{bmatrix} j_1 & j_2 & j \\ m_1' & m_2' & m \end{bmatrix}_q = \delta_{m_1 m_1'}\,\delta_{m_2 m_2'} \,. \qquad (3.8)$$

Applying operators $\Delta(X^{\pm})$ to both sides of (3.2) we obtain the following recurrence relations for $q-3j$ -symbols

$$\{[j\mp m][j\pm m+1]\}^{1/2} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m\pm 1 \end{bmatrix}_q =$$

$$= q^{\frac{m_2}{2}} \{[j_1\pm m_1][j_1\mp m_1+1]\}^{1/2} \begin{bmatrix} j_1 & j_2 & j \\ m_1\mp 1 & m_2 & m \end{bmatrix}_q + \qquad (3.6')$$

$$+ q^{-\frac{m_1}{2}} \{[j_2+m_2][j_2\mp m_2+1]\}^{1/2} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2\mp 1 & m \end{bmatrix}_q \,.$$

It is interesting that there is a simple relation between $q-3j$ -symbols and Hahn q -polynomials. Let us remind that Hahn polynomials $Q_n(x)$ and dual Hahn polynomials $R_n(x)$ are defined by the following expressions $(n, N\in\mathbb{Z}_+ \;,\; n\le N)$:

$$Q_n(x) := Q_n(x;a,b,N|q) = {}_3\varphi_2\left(\begin{matrix} q^{-n}, & abq^{n+1}, x \\ aq, & q^{-N} \end{matrix} ; q, q\right),$$

$$R_n(\mu(x)):=R_n(\mu(x);a,b,N|q)={}_3\varphi_2\left(\begin{matrix}q^{-n}, abq^{x+1}, q^{-x} \\ aq, q^{-N}\end{matrix};q,q\right),$$

where $\mu(x)=q^{-x}+abq^{x+1}$. For $0\le x, n\le N$ we have:

$$R_n(\mu(x);a,b;N|q)=Q_x(q^{-n};a,b,N|q),$$

and $Q_n(x)$ (or $R_n(\mu(x))$) is the polynomial of degree n of the variable x (or $\mu(x)$).

These polynomials satisfy the following orthogonality relations

$$\sum_{x=0}^{N} Q_m(q^{-x})Q_n(q^{-x})\rho(x)=\delta_{nm}\, d_n^2,$$

$$\sum_{x=0}^{N} R_m(\mu(x))R_n(\mu(x))\rho(x)=\delta_{nm}\, d_n^2,$$

where $\rho(x)$ and d_n are weight and norm of Hahn polinomials.

$$\rho(x):=\rho(x;a,b,N|q)=\frac{(aq;q)_x\,(bq;q)_x}{(q;q)_x\,(q;q)_{N-x}}(aq)^{-x},$$

$$d_n^2:=d_n^2(a,b,N|q)=\frac{(1-abq)\,(abq^2;q)_N\,(aq)^{-N}(-aq)^n}{(1-abq^{2n+1})\,(q;q)_N}\cdot$$

$$\cdot\frac{(q, bq, abq^{N+2};q)_n}{(aq, abq, q^{-N};q)_n}\cdot q^{\binom{n}{2}-nN}$$

for Hahn polynomials $Q_n(x)$, and

$$\rho(x):=\rho(x;a,b,N|q)=\frac{(1-abq^{2x+1})(aq,abq,q^{-N};q)_x}{(1-abq)(q,bq,abq^{N+2};q)_x}\cdot$$
$$\cdot(-aq)^{-x}q^{Nx-\binom{x}{2}},$$

$$d_n^2:=d_n^2(a,b,N|q)=\frac{(abq^2;q)_N\,(q,b^{-1}q^{-N};q)_n}{(bq;q)_N\,(aq,q^{-N};q)_n}(aq)^{-N}(abq)^n$$

for dual Hahn polynomials $R_n(\mu(x))$.

From the Racah-Fock representation (3.4) we obtain the relation between $q-3j$-symbols and Hahn polynomials:

$$\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q = (-1)^{j_1-m_1}\frac{\{\rho(x)\}^{1/2}}{d_n}Q_n(q^{-x};a,b,N|q), \tag{3.8'}$$

where $n=j-m$, $a=q^{j_2-j+m_1}$, $b=q^{j_1-j_2+m}$, $x=j_1-m_1$, $N=j+j_2-m_1$ $(0\le x,n\le N)$ and the relation between $q-3j$-symbols and dual Hahn polynomials

$$\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q = (-1)^{j_1-m_1}\frac{\{\rho(x)\}^{1/2}}{d_n}R_n(\mu(x);a,b,N|q), \tag{3.8"}$$

where n, x, N, are the same as in (3.8'), $b=q^{j-j_2+m_1}$, $\mu(x)=q^{m_1+1/2}(q^{j_1+1/2}-q^{-j_1-1/2})$. Here $\rho(x)$ and d_n are weight and norm of Hahn polynomials.

The orthogonality relations for Hahn polynomials and dual Hahn polynomials follows immediately from (3.8), (3.8"). The recurrence relations (3.6') for $q-3j$-symbols imply the recurrence relations for Hahn and dual Hahn polynomials.

THEOREM 3.1. There are the following relations between CGC:

$$\sum_{m_1' m_2'} \left(\mathbb{R}^{j_1 j_2}\right)_{m_1 m_2}^{m_1' m_2'} \begin{bmatrix} j_1 & j_2 & j \\ m_1' & m_2' & m \end{bmatrix}_q = \tag{3.9}$$

$$= (-1)^{j_1+j_2-j}\, q^{\frac{1}{2}(c_j - c_{j_1} - c_{j_2})} \begin{bmatrix} j_2 & j_1 & j \\ m_2 & m_1 & m \end{bmatrix}_q ,$$

$$\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q = (-1)^{j_1+j_2-j} \begin{bmatrix} j_2 & j_1 & j \\ m_2 & m_1 & m \end{bmatrix}_{q^{-1}} , \tag{3.10}$$

$$\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q = (-1)^{j_1 - m_1} q^{-\frac{m_1}{2}} \left\{ \frac{[2j+1]}{[2j_2+1]} \right\}^{1/2} \begin{bmatrix} j_1 & j & j_2 \\ m_1, & -m, & -m_2 \end{bmatrix}_{q^{-1}} , \tag{3.11}$$

where the numbers c_j and $[n]$ defined in (2.6), (1.7).

The proof of this theorem follows from the representations (3.3) -(3.5) (see also [2]).

THEOREM 3.2.

$$\sum_{m'} \left(\mathbb{R}^{j j_3}\right)_{m' m_3}^{m m_3'} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m' \end{bmatrix}_q = \sum_{m_1' m_2' m_3''} \begin{bmatrix} j_1 & j_2 & j \\ m_1' & m_2' & m \end{bmatrix}_q \cdot \tag{3.12}$$

$$\left(\mathbb{R}^{j_1 j_3}\right)_{m_1 m_3}^{m_1' m_3''} \left(\mathbb{R}^{j_2 j_3}\right)_{m_2 m_3''}^{m_2' m_3'} .$$

To prove this formula it is sufficient to consider relation (1.8) in the representation $\pi^{j_3} \otimes \pi^{j_1} \otimes \pi^{j_2}$ (for more details see [2]).

From (1.11) it follows that the matrices $\mathbb{R}^{j_1 j_2}$ satisfy the Yang-Baxter relation:

$$\mathbb{R}_{12}^{j_1 j_2} \mathbb{R}_{13}^{j_1 j_3} \mathbb{R}_{23}^{j_2 j_3} = \mathbb{R}_{23}^{j_2 j_3} \mathbb{R}_{13}^{j_1 j_3} \mathbb{R}_{12}^{j_1 j_2} . \tag{3.13}$$

Here all matrices act in $V^{j_1} \otimes V^{j_2} \otimes V^{j_3}$ and the indices show how these matrices are acting in the product space.

To conclud this section let us prove that the extension of $U_q(su(2))$ by the Weyl element w is a Hopf algebra.

THEOREM 3.3. Formulae

$$\Delta(w) = R^{-1}(w \otimes w), \quad \varepsilon(w) = 1 \tag{3.14}$$

define the structure of a Hopf algebra on $\widetilde{U_q}(su(2))$.

PROOF. Let us check (3.14) in all irreducible representations. In accordance with the definition of CGC we have:

$$(\pi^{j_1} \otimes \pi^{j_2}) \Delta(w)_{m_1' m_2'}^{m_1 m_2} = \sum_{j,m,m'} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q w_{mm'}^{j} \begin{bmatrix} j_1 & j_2 & j \\ m_1' & m_2' & m' \end{bmatrix}_q . \tag{3.15}$$

The symmetries of CGC imply the identity

$$\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q = \sum_{m_1' m_2'} \left((\mathbb{R}^{j_1 j_2})^{-1} \right)_{m_1' m_2'}^{m_1 m_2} \begin{bmatrix} j_1 & j_2 & j \\ -m_1' & -m_2' & -m \end{bmatrix}_q (-1)^{j_1+j_2-j} q^{\frac{1}{2}(c_j - c_{j_1} - c_{j_2})} .$$

Comparing this identity with (3.15) we obtain the equality (3.14) in the representation

$$(\pi^{j_1} \otimes \pi^{j_2}) \Delta w = (\mathbb{R}^{j_1 j_2})^{-1} w^{j_1} \otimes w^{j_2} .$$

Formulae (1.8) and (1.9) imply thecoassociativity of the action (3.14).

Let $\Delta(a)=\sum_i a^i\otimes a_i$. The Hopf axiom for the antipode is $S(a^i)a_i = a^i S(a_i)=\varepsilon(a)\mathbb{1}$. To define the action of the antipode on w we need the following lemma.

LEMMA. Let A be the quasitriangle Hopf algebra (see [9][12][36] and $R=e_i\otimes e^i$ is the universal R-matrix; then

1. The element $u=\sum_i S(e^i)e_i$ is invertible and $u^{-1}=\sum_j e^j S^2(e_j)$.
2. $S^2(a)=u\,a\,u^{-1}$ for any $a\in A$.

We do not give here the proof of this lemma, because it would have demanded the description of many auxiliary constructions. The proof of this lemma for arbitrary quasitriangle Hopf algebra was given by V.G.Drinfeld [36] and for quasitriangle algebras of special structure (for dubles of Hopf algebras) by one of the authors (N.R. u published). Let us check now the property of the antipode in $\widetilde{U}_q(su(2))$. From (3.14) we obtain $(S\otimes id)\Delta w = S(w)S^2(e^i)\otimes e_i w$ and therefore we must have

$$S(w)S^2(e_i)e^i w=\varepsilon(w)\cdot\mathbb{1}\,. \tag{3.16}$$

Lemma 1 implies that $S(u^{-1})=S^2(e_i)e^i$. Comparing with (3.16) we get

$$S(w)w=\varepsilon(w)u\,.$$

Further, the equalities

$$\varepsilon(w)S(u)u=S(w)u\,w=S(w)^2w^2=$$
$$=S(w)w\cdot wS(w)\,,\quad S(u)u=u\,S(u)$$

prove the relation

$$w\,S(w)=\varepsilon(w)S(u)\,.$$

Using this relation we have:

$$e_i w S(w) e^i = \varepsilon(w) e_i S(u) e^i = \varepsilon(w) S(u) S(u^{-1}) =$$
$$= \varepsilon(w)\cdot 1 .$$

So, the conditions of the axiom for an antipode in $\widetilde{U_q}(su(2))$ are satisfied. To calculate $\varepsilon(w)$ we substitute w into (2.8). Since $\tau^2 = id$ we obtain

$$S(w) = \tau(w).$$

Therefore in every irreducible representation we have

$$(w^j)^t w^j = \varepsilon(w)\pi^j(u), \quad w^j (w^j)^t = \varepsilon(w)\pi^j(S(u)).$$

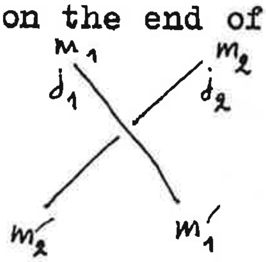
Comparing this formula with (2.6) we get

$$\varepsilon(w) = 1, \quad \pi^j(u) = q^{-c_j + H/2}, \quad \pi^j(S(u)) = q^{-c_j - H/2} .$$

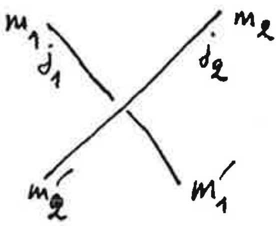
4. Graphical Representation of R-matrices And q-Glebsch-Gordan Coefficients

The relations (3.7)-(3.13) between CGC and R-matrices can be represented graphically [2].

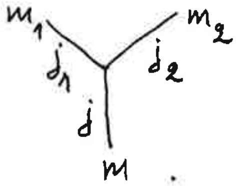
Let us represent the R-matrices and CGC by graphs with strings colored by numbers j and with states $\{m_i\}$ on the end of strings:



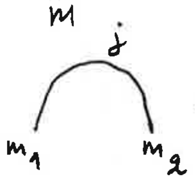
$$\longrightarrow \left(R^{j_2 j_1}\right)^{m_2' m_1'}_{m_2 m_1} . \qquad (4.1)$$



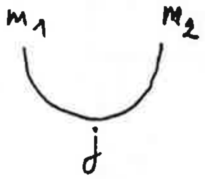
$$\longrightarrow \left(\left(R^{j_2 j_1}\right)^{-1}\right)^{m_1' m_2'}_{m_1 m_2} . \qquad (4.2)$$



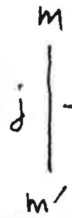
$$\longrightarrow \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q . \qquad (4.3)$$



$$\longrightarrow \delta_{m_1,-m_2} (-1)^{-j+m_2}\, q^{\frac{m_2}{2}} , \qquad (4.4)$$

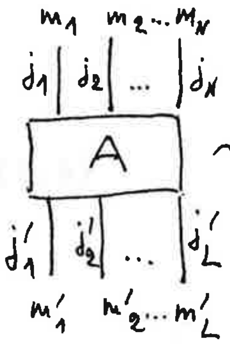


$$\longrightarrow \delta_{m_1,-m_2} (-1)^{j-m_1}\, q^{-\frac{m_1}{2}} ,$$

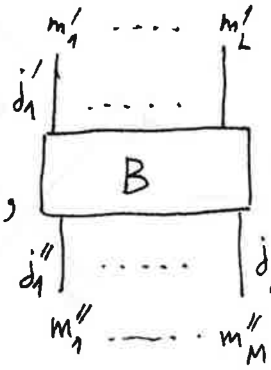


$$\rightarrow \delta_{mm'} . \qquad (4.5)$$

The multiplication of matrices correspond to the joining of graphs of these matrices together. For example if the matrix A acts from $V^{j_1'} \otimes \dots \otimes V^{j_L'}$ into $V^{j_1} \otimes \dots \otimes V^{j_N}$, the matrix B acts from $V^{j_1''} \otimes \dots \otimes V^{j_M''}$ to $V^{j_1'} \otimes \dots \otimes V^{j_L'}$ and they are represented by graphs:

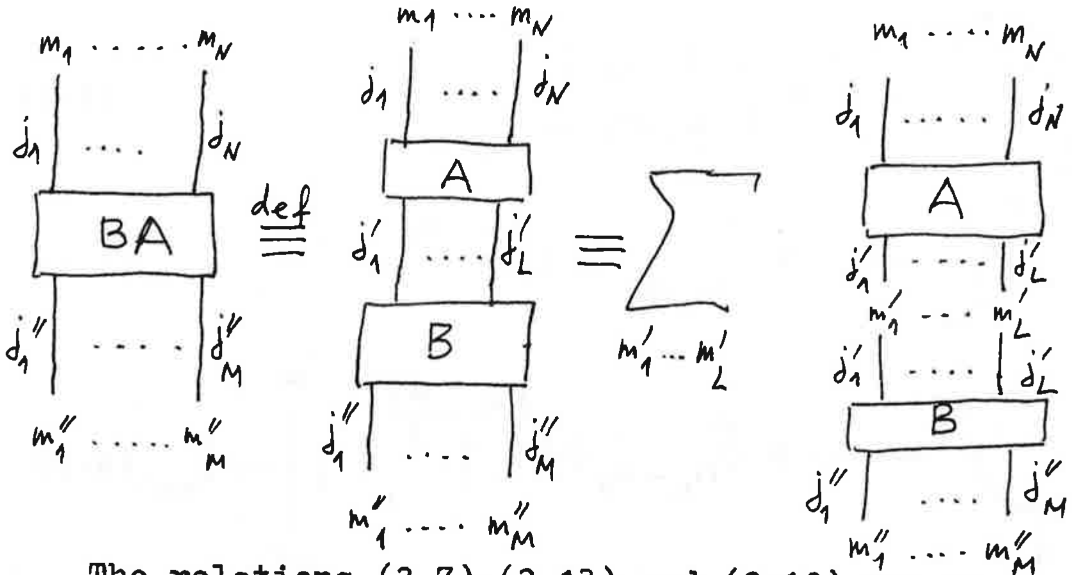


$$\rightsquigarrow A^{m_1' \dots m_L'}_{m_1 \dots m_N} ,$$

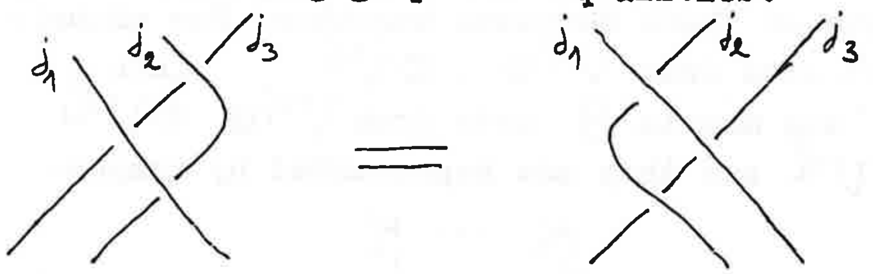


$$\rightsquigarrow B^{m_1'' \dots m_M''}_{m_1' \dots m_L'} ,$$

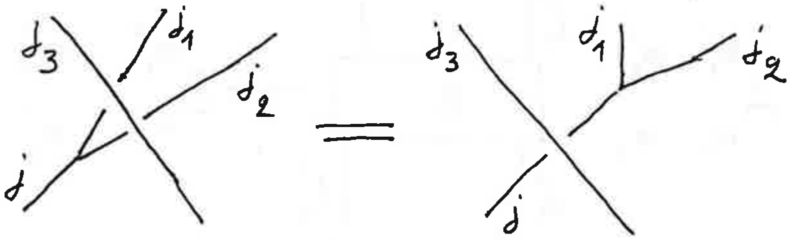
then the product BA is represented by the joining of strings $j_1' \dots j_L'$ connected with B with strings $j_1', \dots, j_L'$ connected with A. The joining means the summation over the states corresponding to the ends of the joining strings:



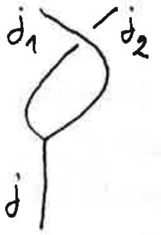
The relations (3.7)-(3.13) and (2.10) are represented by the following graphical equalities:



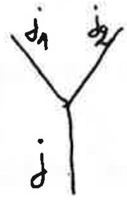
(4.6)



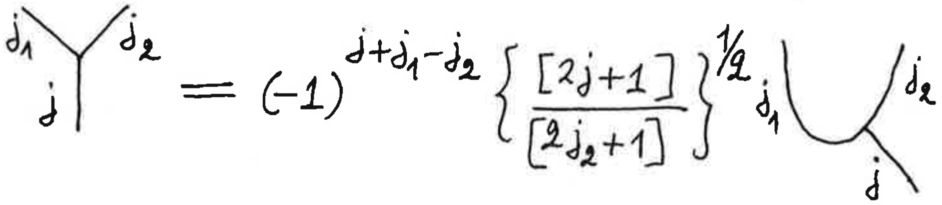
(4.7)



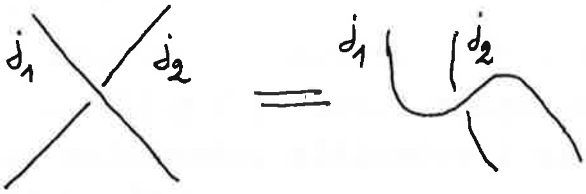
$$= (-1)^{j_1+j_2-j}\, q^{\frac{1}{2}(c_j - c_{j_1} - c_{j_2})}$$



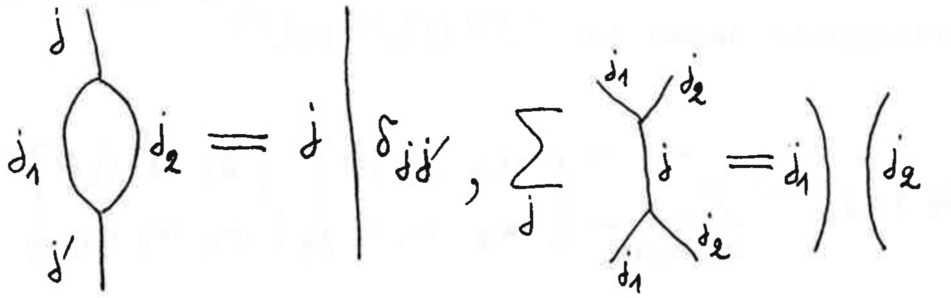
(4.8)



(4.9)



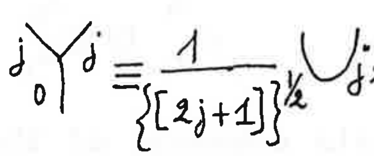
(4.10)



(4.11)

The relations (4.11), (4.8), (4.9), (4.7), (4.6) represent the formulae (3.7)-(3.13) respectively and (4.10) represents the crossing-symmetry (2.10) of R-matrices. In the relation (4.8) the value $j=0$ is very important for the further application to links theory. In this case:

$$\begin{bmatrix} j & j & 0 \\ m_1 & m_2 & 0 \end{bmatrix}_q = \frac{w^{j}_{m_1 m_2}}{\{[2j+1]\}^{1/2}}\, q^{\frac{1}{2}c_j} ,$$



(4.12)

and therefore from (4.8) with $j=0$ it follows that

(4.13)

5. The q-Analogs Of The Wigner-Rakah $6j$ -Symbols

Let us consider now the q-analog of $6j$-symbols and the properties of these $q-6j$ -symbols. For this purpose let us consider tensor product $V^{j_1}\otimes V^{j_2}\otimes V^{j_3}$ of three irreducible representations of $U_q(su(2))$. There are two symplest ways to obtain irreducible components in this representation. One is to decompose first $V^{j_1}\otimes V^{j_2} = \sum\limits^{\oplus} V^{j_{12}}$ and then to take irreducible submodules in $V^{j_{12}}_{j_{12}}\otimes V^{j_3}$. The other is to decompose first $V^{j_2}\otimes V^{j_3} = \sum\limits^{\oplus}_{j_{23}} V^{j_{23}}$ and then $V^{j_1}\otimes V^{j_{23}}$. These two ways give two complete orthogonal bases in $V^{j_1}\otimes V^{j_2}\otimes V^{j_3}$:

$$e^{j_{12}j}_{m}(j_1 j_2 | j_3) = \sum_{m_1, m_2 m_3} \begin{bmatrix} j_{12} & j_3 & j \\ m_{12} & m_3 & m \end{bmatrix}_q \begin{bmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{bmatrix}_q \cdot$$
$$e^{j_1}_{m_1}\otimes e^{j_2}_{m_2}\otimes e^{j_3}_{m_3} , \tag{5.1}$$

$$e^{j_{23}j}_{m}(j_1 | j_2 j_3) = \sum_{m_1 m_2 m_3} \begin{bmatrix} j_1 & j_{23} & j \\ m_1 & m_{23} & m \end{bmatrix}_q \begin{bmatrix} j_2 & j_3 & j_{23} \\ m_2 & m_3 & m_{23} \end{bmatrix} \cdot$$
$$e^{j_1}_{m_1}\otimes e^{j_2}_{m_2}\otimes e^{j_3}_{m_3} . \tag{5.2}$$

The matrix elements of the matrix, connecting these bases will be called $q-6j$ -symbols :

$$e^{j_{12}j}_{m}(j_1, j_2 | j_3) = \sum_{j_{23}} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}_q e^{j_{23}j}_{m}(j_1 | j_2 j_3) . \tag{5.3}$$

In the case $q=1$ we have

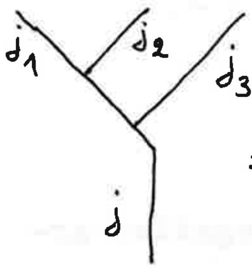
$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}_{q=1} = \{[2j_{12}+1][2j_{23}+1]\}^{1/2} (-1)^{j_1+j_2-j-j_3-2j_{12}} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix},$$

where $\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}$ is Racah-Wigner $6j$ -symbol.

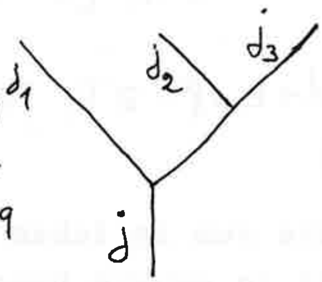
For $q \neq 1$ we shall also use Racah-Wigner normalization:

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}_q^{RW} = \{[2j_{12}+1][2j_{23}+1]\}^{-1/2} (-1)^{j_1+j_2-j-j_3-2j_{12}} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}_q .$$

If we use the graphical technique of the previous section the definition (5.3) of $q-6j$-symbols will have the form



$$= \sum_{j_{23}} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}_q \qquad (5.4)$$

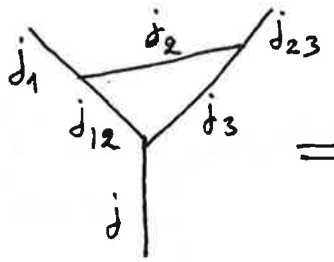
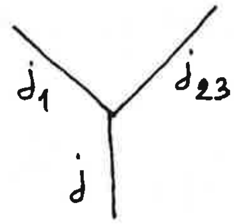


Using the orthogonality (4.11) of CGC we obtain an expression of $q-6j$-symbols in terms of CGC:

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}_q = \begin{bmatrix} j_1 & j_{23} & j \\ m_1 & m_{23} & m \end{bmatrix}_q^{-1} \sum_{\substack{m_2, m_3 \\ m_2+m_3=m-m_1}} \begin{bmatrix} j_2 & j_3 & j_{23} \\ m_2 & m_3 & m_{23} \end{bmatrix}_q \cdot \qquad (5.5)$$

$$\begin{bmatrix} j_{12} & j_3 & j \\ m_{12} & m_3 & m \end{bmatrix}_q \begin{bmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{bmatrix}_q ,$$

or

 $= \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}_q$  . (5.6)

THEOREM 5.1.

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q^{RW} = \Delta(abe)\Delta(acf)\Delta(ced)\Delta(dbf) \cdot$$

$$\sum_z (-1)^z [z+1]! \Big\{ [z-a-b-e]! [z-a-c-f]! \tag{5.7}$$

$$[z-b-d-f]! [z-d-c-e]! [a+b+c+d-z]!$$

$$[a+d+e+f-z]! [b+c+e+f-z]! \Big\}^{-1} .$$

Here the sum is taken only over z with nonnegative arguments in square brackets, $[0]! \equiv 1$.

The proof of this stheorem is given in .

REMARK 1. The sum (5.7) can be expressed through the generalized hypergeometric function ${}_4\varphi_3$ (see [1]):

$$\frac{(-1)^{a+b+c+d} [a+c+b+d+1]!}{[a+b-e]! [a+c-f]! [c+d-e]! [b+d-f]! [e+f-c-b]! [e+f-a-d]!} \cdot$$

$${}_4\varphi_3 \left(\begin{matrix} q^{-a-b+e}, & q^{-a-c+f}, & q^{-c-d+e}, & q^{-b-d+f} \\ q^{-a-b-c-d-1}, & q^{e+f-c-b+1}, & q^{e+f-a-d+1} & \end{matrix} ; q, q \right)$$

Here we use the following notation

$$ {}_{p+1}\varphi_p\left(\begin{matrix} a_1,\dots,a_{p+1} \\ b_1,\dots,b_p \end{matrix}; q, z\right) = \sum_{k=0}^{\infty} \frac{(a_1,\dots,a_{p+1};q)_k}{(b_1,\dots,b_p,q;q)_k} z^k, $$

where

$$ (a;q)_k = \prod_{i=0}^{k-1} (1-aq^i), $$

$$ (a_1,\dots,a_p;q)_k = \prod_{j=1}^{p} (a_j;q)_k . $$

Let us suppose that one of the numbers qa , qbd or qc is equal to q^{-M}. Then

$$ R_n(\lambda(x); a,b,c,d\,|q) = {}_4\varphi_3\left(\begin{matrix} q^{-n}, q^{n+1}ab, q^{-x}, q^{x+1}cd \\ aq, bdq, cq \end{matrix}; q, q\right) $$

is the polynomial of degree n on $\lambda(x) = q^{-x} + q^{x+1}cd$. These polynomials are called the Racah q-polynomials. The Racah-Wilson polynomials correspond to $c = q^{-M-1}$ and are denoted by $W_n(x)$:

$$ W_n(x) := W_n(x; a,b,c,M\,|q) = {}_4\varphi_3\left(\begin{matrix} q^{-n}, abq^{n+1}, q^{-x}, cq^{x-M} \\ aq, q^{-M}, bcq \end{matrix}; q, q\right), $$

$n = 0, 1, \dots, M$. This is a polynomial of degree n in $\mu(x) = q^{-x} + cq^{x-M}$, and $W_n(x; a, b, 0, M\,|q) = Q_n(q^{-x}; a, b, M\,|q)$.

From the theorem 5.1 we have

$$ \left\{\begin{matrix} a & b & e \\ d & c & f \end{matrix}\right\}_q = \frac{\{\rho(x)\}^{1/2}}{d_n} W_n(x; \alpha, \beta, \gamma, M|q), $$

where $n = a+b-e$, $x = c+d-e$, $M = a+b+c+d+1$, $\alpha = q^{-a-d+e+f}$, $\beta = q^{-a-b-c+d-1}$, $\gamma = q^{a+e+f-d+1}$, $\rho(x)$ and d_n are the weight and norm of the Racah-Wilson polynomials:

$$\rho(x) := \rho(x;\alpha,\beta,\gamma,M|q) =$$

$$= \frac{(1-\gamma q^{2x-M})(\alpha q, \beta\gamma q, \gamma q^{-M}, q^{-M}; q)_x}{(1-\gamma q^{-M})(q, \beta^{-1}q^{-M}, \gamma q, \alpha^{-1}\gamma q^{-M}; q)_x} (\alpha\beta q)^{-x},$$

$$d_n^2 := d_n^2(\alpha,\beta,\gamma,M|q) =$$

$$= \frac{(1-\alpha\beta q)(q, \beta q, \alpha\gamma^{-1}q, \alpha\beta q^{M+2}; q)_n}{(1-\alpha\beta q^{2n+1})(\alpha\beta q, \alpha q, \beta\gamma q, q^{-M}; q)_n} \frac{(\alpha^{-1}\gamma, \beta^{-1}; q)_M}{(\gamma q, \alpha^{-1}\beta^{-1}q^{-1}; q)_M} \cdot$$

$$\cdot (\gamma q^{-M})^n .$$

The symmetries of q- $6j$-symbols follow from (A.18)

1. $$\begin{Bmatrix} b & a & e \\ c & d & f \end{Bmatrix}_q^{RW} = \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q^{RW}$$

2. $$\begin{Bmatrix} a & e & b \\ d & f & c \end{Bmatrix}_q^{RW} = \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q^{RW}$$

3. $$\begin{Bmatrix} a & c & f \\ d & b & e \end{Bmatrix}_q^{RW} = \begin{Bmatrix} a & e & b \\ d & f & c \end{Bmatrix}_q^{RW}$$

4. q-Regge symmetry. Let

$$s_1 = \frac{b+c+e+f}{2}, \quad s_2 = \frac{a+d+e+f}{2}, \quad s_3 = \frac{a+b+c+d}{2},$$

then the following equalities hold

$$\begin{Bmatrix} a & s_1-c & s_1-f \\ d & s_1-b & s_1-e \end{Bmatrix}_q^{RW} = \begin{Bmatrix} a & e & b \\ d & f & c \end{Bmatrix}_q^{RW}$$

$$\begin{Bmatrix} s_2-d & b & s_2-f \\ s_2-a & c & s_2-e \end{Bmatrix}_q^{RW} = \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q^{RW}$$

$$\begin{Bmatrix} s_3-d & s_3-c & e \\ s_3-a & s_3-b & f \end{Bmatrix}_q^{RW} = \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q^{RW}$$

$$\begin{Bmatrix} s_2-d & s_3-c & s_1-f \\ s_2-a & s_3-b & s_1-e \end{Bmatrix}_q^{RW} = \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q^{RW}$$

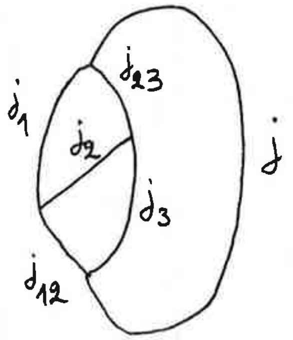
5.

$$\lim_{N\to\infty} \begin{Bmatrix} N-m, & b, & N-m_2 \\ d, & N, & f \end{Bmatrix}_q^{RW} = \left\{ \frac{1}{[2f+1]} \right\}^{1/2} \begin{bmatrix} b, & d, & f \\ m_1 & m_2 & m \end{bmatrix}_q,$$

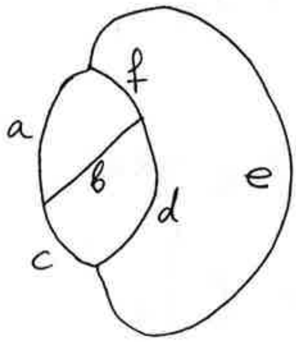
$m_1 + m_2 = m$.

The symmetries 1-3 of $q-6j$ -symbols have a simple graphical interpretation. To describe it let us write $q-6j$ -symbol as the following trace:

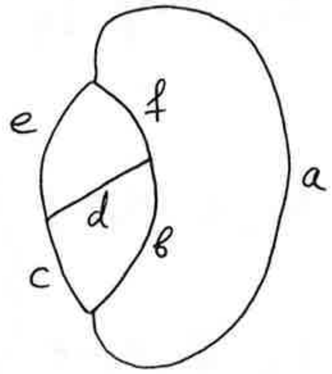
$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}_q = \frac{1}{[2j+1]} \qquad (5.8)$$

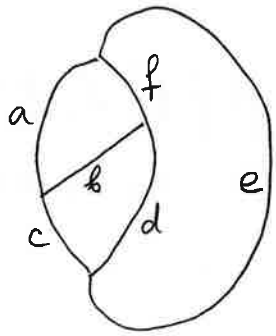


The symmetries 1-3 of $q-6j$ -symbols correspond to rotating of thetraedrom formed by the edges (j_1 , j_2 , j_3 , j_{12} , j_{23} , j) in accordance with rules (4.6)-(4.10). For example we have:

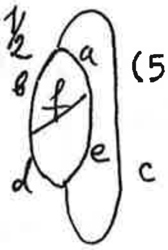


$$= \frac{[2e+1]}{[2a+1]} \qquad (5.9)$$

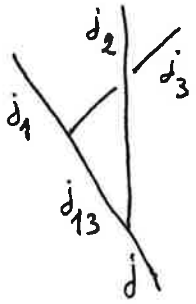




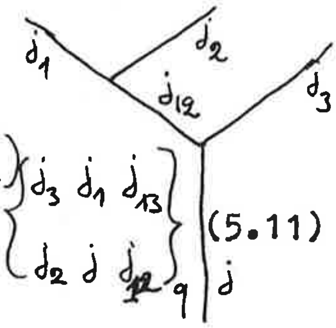
$$= \frac{[2e+1]}{[2c+1]} (-1)^{a+f-c-d} \left\{ \frac{[2a+1][2f+1]}{[2c+1][2d+1]} \right\}^{1/2} \qquad (5.10)$$

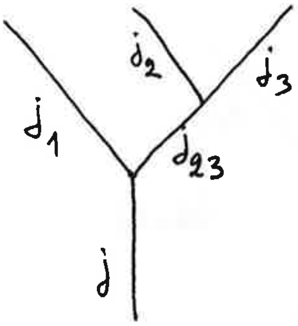


To conclude this section we give two important formulae with $q-6j$ -symbols

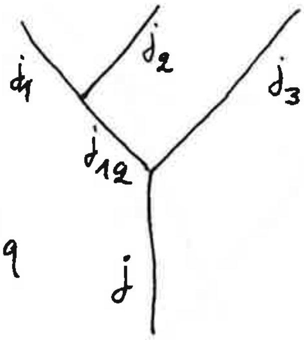


$$= \sum_{j_{12}} (-1)^{j_{12}+j_{13}-j-j_1} q^{\frac{1}{2}(c_j + c_{j_1} - c_{j_{13}} - c_{j_{12}})} \begin{Bmatrix} j_3 & j_1 & j_{13} \\ j_2 & j & j_{12} \end{Bmatrix}_q \tag{5.11}$$



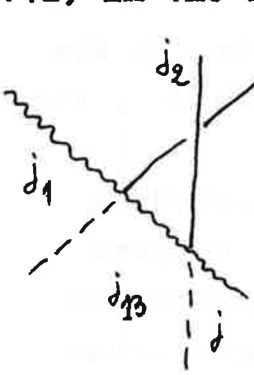


$$= \sum_{j_{12}} \begin{Bmatrix} j_3 & j_2 & j_{23} \\ j_1 & j & j_{12} \end{Bmatrix}_q \tag{5.12}$$

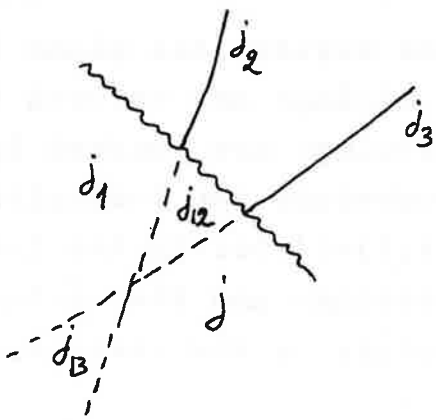


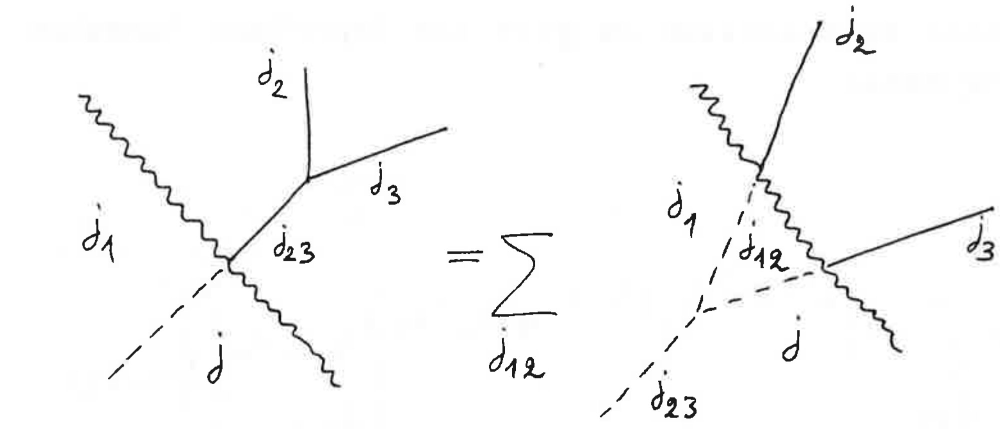
6. Graphical Representation Of $q-6j$-Symbols

To give a graphical tehcnique for representing $q-6j$-symbols let us rewrite the relations (5.5), (5.11) and (5.12) in the following form:

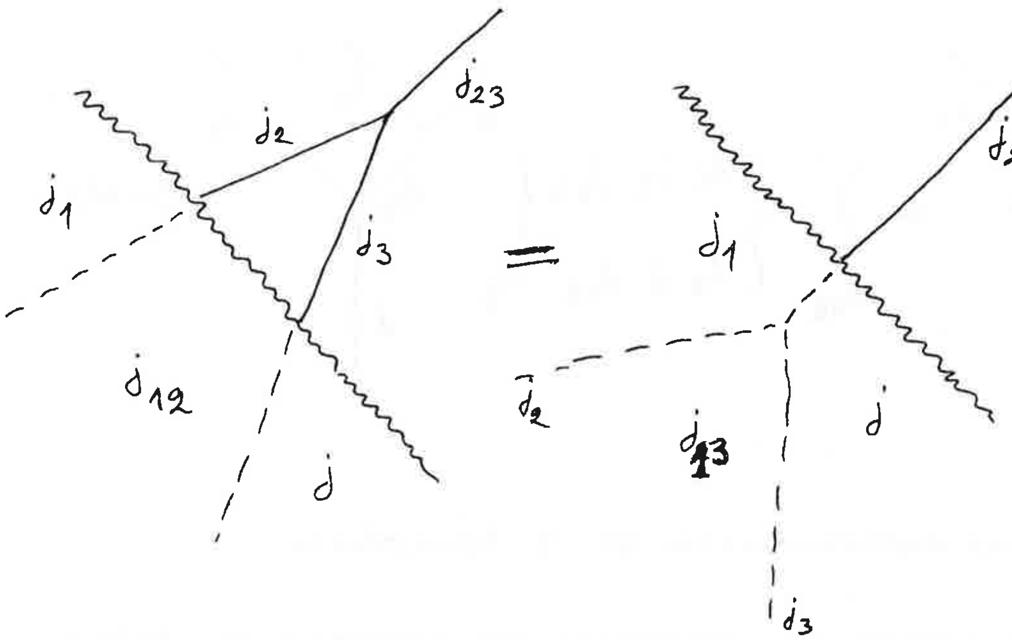


$$= \sum_{j_{12}} \tag{6.1}$$



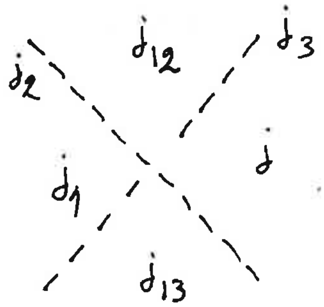


(6.2)

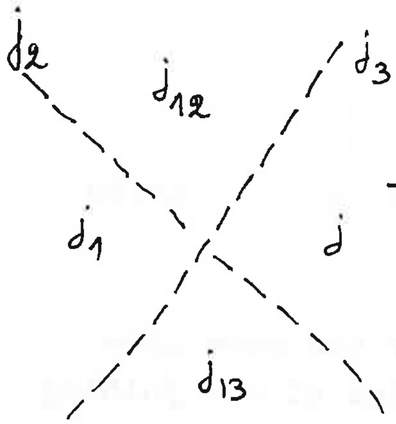


(6.3)

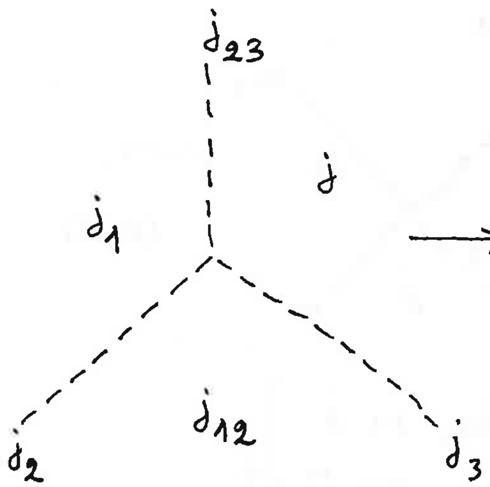
Here a wave line divides the plane into two parts. In the upper part the strings are colored by the numbers j , the ends of these strings are marked by the states $\{m\}$ and vertices represent the R-matrices and CGC according to the rules (4.1)-(4.5). In the lower part the numbers j color the strings and the sectors placed between the strings. The colors of the strings are not changed after crossing the wave line. The points of intersections of two strings and the triple vertices correspond to the following $q-6j$-symbols:



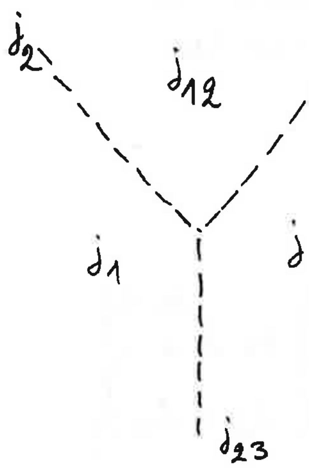
$$\rightarrow (-1)^{j_{13}+j_{12}-j_1-j}\, q^{\frac{1}{2}(c_j+c_{j_1}-c_{j_{12}}-c_{j_{13}})} \begin{Bmatrix} j_3 & j_1 & j_{13} \\ j_2 & j & j_{12} \end{Bmatrix}_q \tag{6.4}$$



$$\rightarrow (-1)^{j+j_1-j_{12}-j_{13}}\, q^{\frac{1}{2}(c_{j_{13}}+c_{j_{12}}-c_j-c_{j_1})} \begin{Bmatrix} j_3 & j_1 & j_{13} \\ j_2 & j & j_{12} \end{Bmatrix}_q \tag{6.5}$$

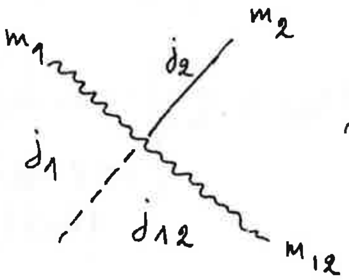


$$\longrightarrow \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}_q \tag{6.6}$$



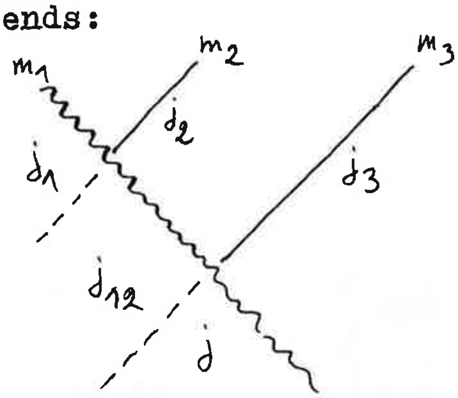
$$\longrightarrow \begin{Bmatrix} j_3 & j_2 & j_{23} \\ j_1 & j & j_{12} \end{Bmatrix}_q . \qquad (6.7)$$

The points of intersections of strings with the wave line correspond to CGC:

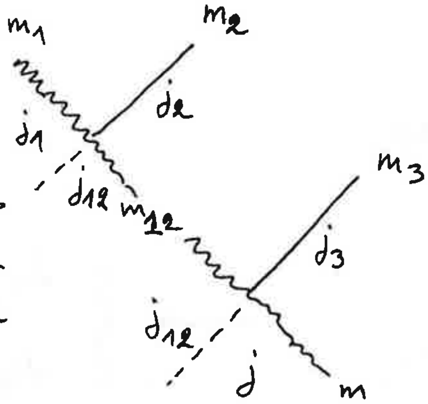


$$\rightsquigarrow \begin{bmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{bmatrix}_q . \qquad (6.8)$$

The joining of the fragments (6.8) by the wave line correspond to the summation over the states of the joining ends:



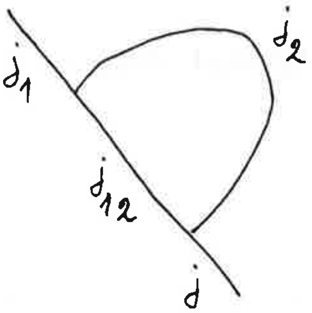
$$= \sum_{m_{12}}$$



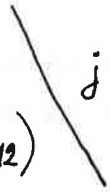
$$\rightsquigarrow \qquad (6.9)$$

$$\rightsquigarrow \sum_{m_{12}} \begin{bmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{bmatrix}_q \begin{bmatrix} j_{12} & j_3 & j \\ m_{12} & m_3 & m \end{bmatrix}_q .$$

Let us call the lower part side of the plane "the shadows world". The rules (6.4)-(6.7) represent $q-6j$ - sybols in the shadows world. To find the weights, corresponding to the extremal fragments consider the relations which follow from (4.9), (4.11), (4.12)

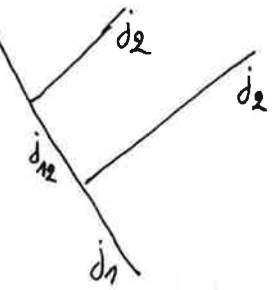


$$= \delta_{j_1 j} (-1)^{j_1+j_2-j_{12}} \left(\frac{[2j_{12}+1]}{[2j_1+1]}\right)^{1/2} \delta(j_1 j_2 j_{12})$$

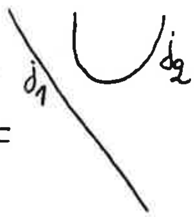


(6.10)

$$\sum_{j_{12}}$$

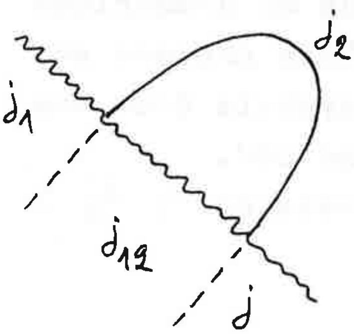


$$(-1)^{j_2-j_1-j_{12}} \delta(j_1 j_2 j_{12}) \left(\frac{[2j_{12}+1]}{[2j_1+1]}\right)^{1/2} =$$

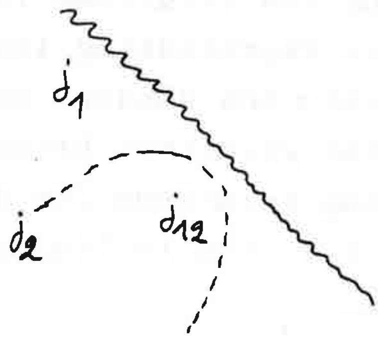


(6.11)

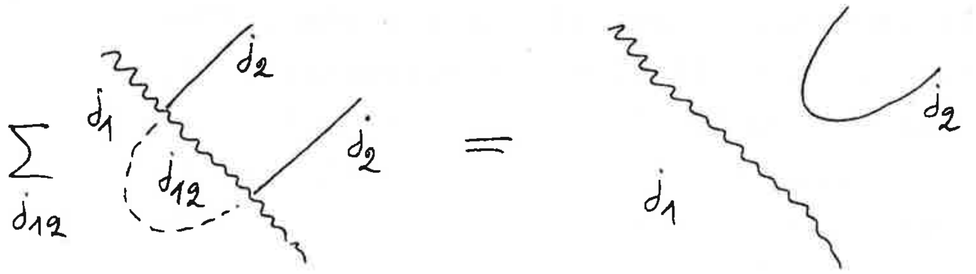
We see that one can rewrite these relations in the form similar to (6.1)-(6.3)



$$= \delta_{j_1 j_2}$$

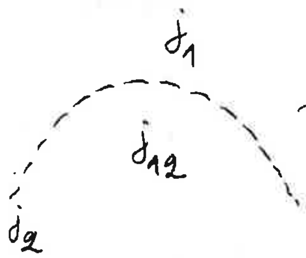


(6.12)

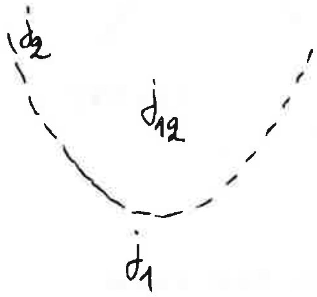


, (6.13)

if we associate the following weights with extremal fragments in the shadows world



$$\rightsquigarrow (-1)^{j_1+j_2-j_{12}} \left(\frac{[2j_{12}+1]}{[2j_1+1]} \right)^{1/2} \delta(j_1 j_2 j_{12}) \tag{6.14}$$



$$\rightsquigarrow (-1)^{j_1+j_2-j_{12}} \left(\frac{[2j_{12}+1]}{[2j_1+1]} \right)^{1/2} \delta(j_1 j_2 j_{12}) \tag{6.15}$$

So, using the relations (6.1)-(6.15) we can transpose every picture representing the combinations of R-matrices and CGC in into the shadows world. Using this process we can obtain the relations between $q-6j$ -symbols from the corresponding relations for R-matrices and CGC.

THEOREM 6.1. The following relations between $q-6j$ -bymbols hold:

$$\sum_{j} \begin{Bmatrix} j_2 & j_1 & j \\ j_3 & j_5 & j_4 \end{Bmatrix}_q \begin{Bmatrix} j_3 & j_1 & j_6 \\ j_2 & j_5 & j \end{Bmatrix}_q = \delta_{j_4 j_6} \,. \tag{6.16}$$

$$\sum_{j_{13}} (-1)^{j_{13}} q^{-\frac{1}{2} c_{j_{13}}} \begin{Bmatrix} j_3 & j_1 & j_{13} \\ j_2 & j & j_{12} \end{Bmatrix}_q \begin{Bmatrix} j_3 & j_2 & j_{23} \\ j_1 & j & j_{13} \end{Bmatrix}_q =$$

$$= (-1)^{j+j_1+j_2+j_3-j_{12}-j_{23}} q^{\frac{1}{2}\left(c_{j_{23}}+c_{j_{12}}-c_{j_2}-c_{j_3}-c_{j_1}-c_j\right)} \times \tag{6.17}$$

$$\times \begin{Bmatrix} j_3 & j_2 & j_{23} \\ j_1 & j & j_{12} \end{Bmatrix}_q .$$

$$\sum_{d} \begin{Bmatrix} j_2 & a & d \\ j_1 & c & b \end{Bmatrix}_q \begin{Bmatrix} j_3 & d & e \\ j_1 & f & c \end{Bmatrix}_q \begin{Bmatrix} j_3 & j_2 & j_{23} \\ a & e & d \end{Bmatrix}_q =$$

$$= \begin{Bmatrix} j_{23} & a & e \\ j_1 & f & b \end{Bmatrix}_q \begin{Bmatrix} j_3 & j_2 & j_{23} \\ b & f & c \end{Bmatrix}_q . \tag{6.18}$$

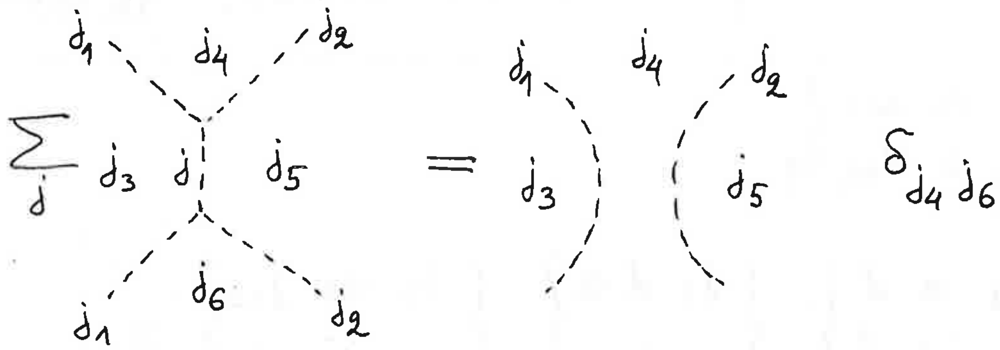
$$\sum_{g} (-1)^{a-b-g-f} q^{\frac{1}{2}(c_a-c_b-c_g-c_f)} \begin{Bmatrix} j_2 & a & g \\ j_1 & c & b \end{Bmatrix}_q \begin{Bmatrix} j_3 & g & e \\ j_1 & d & c \end{Bmatrix}_q \begin{Bmatrix} j_3 & a & f \\ j_2 & e & g \end{Bmatrix}_q =$$

$$= \sum_{g} (-1)^{d-c-g-e} q^{\frac{1}{2}(c_d-c_c-c_g-c_e)} \begin{Bmatrix} j_3 & b & g \\ j_2 & d & c \end{Bmatrix}_q \begin{Bmatrix} j_3 & a & f \\ j_1 & g & b \end{Bmatrix}_q \begin{Bmatrix} j_2 & f & e \\ j_1 & d & g \end{Bmatrix}_q . \tag{6.19}$$

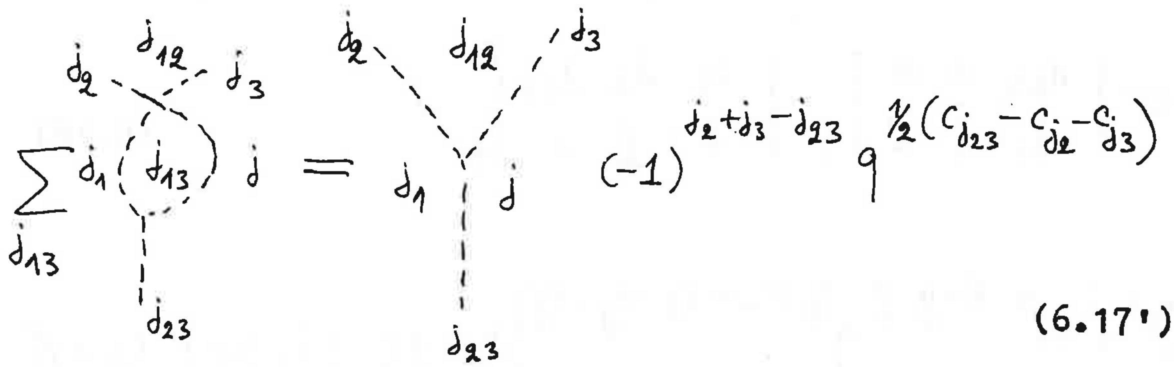
$$\sum_{c} (-1)^{c} q^{-\frac{c(c+1)}{2}} \begin{Bmatrix} j & c & b \\ j & a & b \end{Bmatrix} \frac{[2c+1]}{[2b+1]} = \tag{6.20}$$
$$= (-1)^{2j+2b-a} \, q^{-c_j + c_b - \frac{1}{2} c_a} .$$

The relation (6.16) is called the orthogonality relation between $q-6j$ -symbols. The relation (6.17) is the q-analog of the Racah identity, the relation (6.18) is the q-analog of the Bidenharn-Elliot identity.

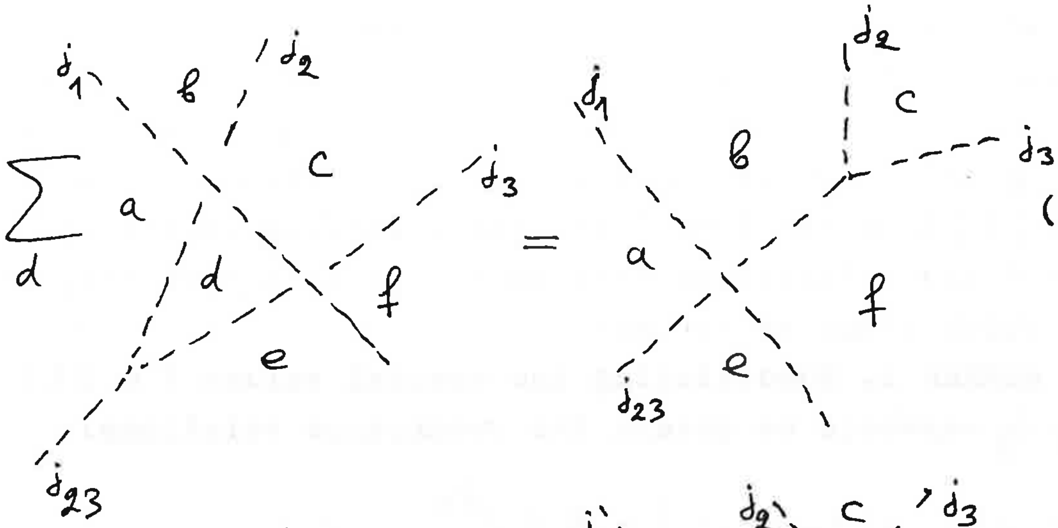
PROOF. Let us rewrite the relations (6.16)-(6.20) in the graphical representation:



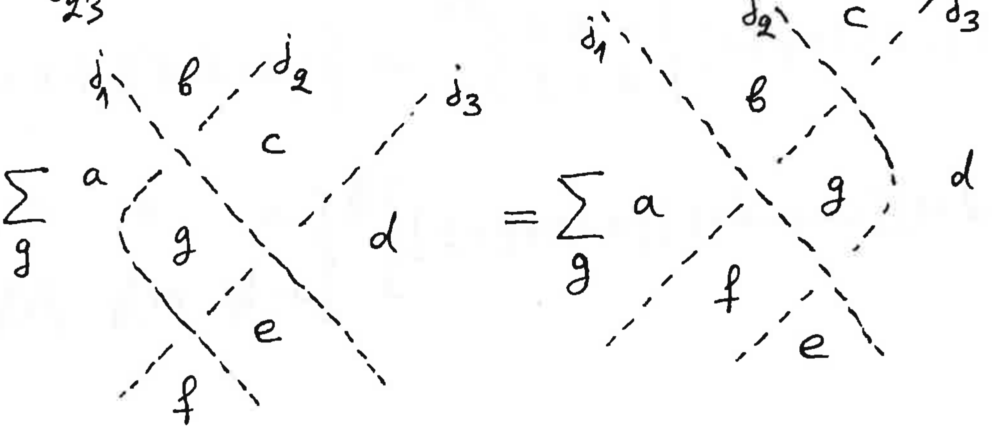
(6.16')



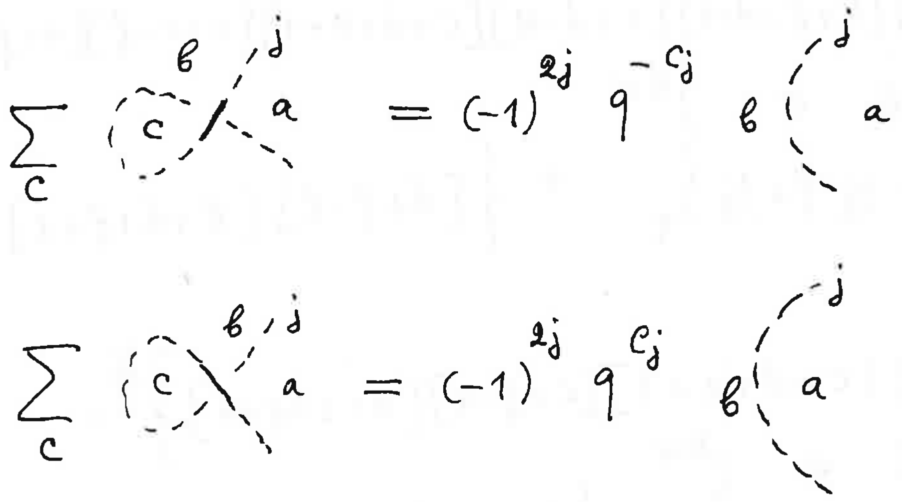
(6.17')



(6.18')



(6.19')



(6.20')

Now we see that these relations follow from (4.11), (4.8), (4.7), (4.13) and (4.6) respectively if we transpose the latter into the shadows world. It seems that this is the simplest proof of the identities (6.18)-(6.20).

REMARK 1. The relations (6.16) are equivalent to the orthogonality relations for the Racah-Wilson polynomials (see [1] formulae (4.1)). The relations (6.17) and (6.18) give identities between the Racah-Wigner polynomials which seems to be new.

REMARK 2. Substituting the special values (6.21) of $q-6j$-symbols we obtain the recurrence relations:

$$
\begin{aligned}
&[2c+1][2d][2f+1]\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q^{RW} = \Big\{[b+d-f][d+e-c] \\
&[b+f-d+1][c+e-d+1][a+c+f+2]\Big\}^{1/2}\begin{Bmatrix} a & b & e \\ d-1/2 & c+1/2 & f+1/2 \end{Bmatrix}_q^{RW} + \\
&+\Big\{[b+d-f][b+f-d+1][c+d-e][c+d+e+1][a+c-f][a+f-c+1]\Big\}^{1/2} \\
&\times\begin{Bmatrix} a & b & e \\ d-1/2 & c-1/2 & f+1/2 \end{Bmatrix}_q^{RW} + \Big\{[d+f-b][b+d+f+1] \\
&[c+d-e][c+d+e+1][c+f-a][a+c+f+1]\Big\}^{1/2}\times \\
&\begin{Bmatrix} a & b & e \\ d-1/2 & c-1/2 & f-1/2 \end{Bmatrix}_q^{RW} + \Big\{[d+f-b][b+d+f+1][d+e-c] \\
&[c+e-d+1][a+f-c][a+c-f+1]\Big\}^{1/2}\begin{Bmatrix} a & b & e \\ d-1/2 & c+1/2 & f-1/2 \end{Bmatrix}_q^{RW},
\end{aligned}
$$

$$\left\{\begin{matrix} a & b & e \\ 1/2, & e+1/2, & b+1/2 \end{matrix}\right\}_q^{RW} = (-1)^{a+b+e+1}\left\{\frac{[a+b+e+2][b+e-a+1]}{[2e+2]\,[2b+1]}\right\}^{1/2},$$

$$\left\{\begin{matrix} a & b & e \\ 1/2 & e+1/2, & b-1/2 \end{matrix}\right\}_q^{RW} = (-1)^{a+b+e}\left\{\frac{[a+b-e][a+e-b+1]}{[2e+2]\,[2b+1]}\right\}^{1/2}, \qquad (6.21)$$

$$\left\{\begin{matrix} a & b & e \\ 1/2 & e-1/2, & b+1/2 \end{matrix}\right\}_q^{RW} = (-1)^{a+b+e}\left\{\frac{[a+e-b]\,[a+b-e+1]}{[2e]\,[2b+1]}\right\}^{1/2},$$

$$\left\{\begin{matrix} a & b & e \\ 1/2 & e-1/2 & b-1/2 \end{matrix}\right\}_q^{RW} = (-1)^{a+b+e}\left\{\frac{[a+b+e+1][e+b-a]}{[2e]\,[2b+1]}\right\}^{1/2}.$$

From (5.7), (5.7'), (5.7") it follows that these relations are equivalent to the recurrence relation for W_n given in [1](formulae (4.6)).

REMARK 3. Identity (6.19) is the face form of Yang-Baxter equation [20] for constant R-matrices.

REMARK 4. In a similar way one can define $q-9j$ -symbols. These symbols are connected with q -ortogonal polynomials depending on two variables. The details will be given in separate publication.

7. The Invariants Of Links Associated With $U_q(su(2))$

Using the q-analog of $6j$ -symbols we give here a new model for the invariants corresponding to higher representations of $U_q(su(2))$, [2, 17, 18]. This model is based on the graphical representation (6.4), (6.5) for $q-6j$ -symbols and is obtained from the model based on R-matrices [2] by transposing the latter into the shadow world in accordance with the rules of the section 6.

DEFINITION 7.1. Let $\mathcal{D}_L$ be the diagram of the link L.

a) to each component of L we associate a number j_α (α numerates the components of L), which we call the colour of the component.

b) let us paint the plane on which the diagram is located into different colours (numerated by $j \in \frac{1}{2}\mathbb{Z}_+$) following the rules described below:

- to the extremal part of the plane we associate the number $j=0$.
- to those parts of the plane which can be reached from the extremal part by crossing only one string, we associate the colour of this string (these are parts neighbouring to the extremal one).
- other internal parts of the diagram are pointed according to the following inductive rule: let $\mathcal{O}_K$ be a part which can be reached from the exteriour by crossing a minimum of K strings and let $\mathcal{O}^{\alpha}_{K-1}$ be the neighbours of this part which can be reached by crossing $(K-1)$ -strings; let be the colour of $\mathcal{O}^{\alpha}_{K-1}$ parts, then the colour j_K of the part $\mathcal{O}_K$ must satisfy the inequalities $|j^{\alpha}_{K-1} - \ell^{\alpha}_{K,K-1}| \leq$ $\leq j_K \leq j^{\alpha}_K + \ell^{\alpha}_{K,K-1}$ for any α ; here $\ell^{\alpha}_{K,K-1}$ is the colour of the string dividing $\mathcal{O}_K$ and $\mathcal{O}_{K-1}$. We shall call each set of colours satisfying these inequalities state on the diagram.

c) let the diagram $\mathcal{D}_L$ be in a general position.

d) for each state on $\mathcal{D}_L$ let us associate a weight to each intersecting and each extremal fragment following the rules (6.4), (6.5), (6.14), (6.15). Then we multiply all of these weights and sum up the product over all possible states.

The obtained functional is denoted by $Z_{j_1 \cdots j_K}(\mathcal{D}_L)$ where K is the number of the components of L and j_1, $j_2, \cdots, j_K$ are the colours of these components.

<u>An example of the calculation of the functional</u> $Z_{j_1 \cdots j_K}(\mathcal{D}_L)$.

Let $\mathcal{D}_L$ be the diagram given in Fig.1. The numbers j_1 and j_2 are the colours of the components of L. The states on $\mathcal{D}_L$ are given in Fig.1. For colours K and j

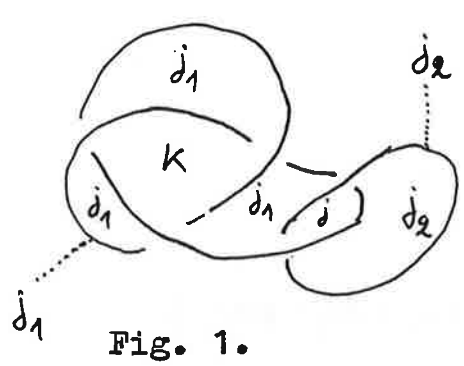


Fig. 1.

we have the following restrictions: $0 \le k \le 2j_1$, $|j_1 - j_2| \le j \le j_1 + j_2$. In accordance with Definition .1 for the functional $\mathcal{D}_{j_1 j_2}(\mathcal{D}_L)$ we obtain the following expression:

$$Z_{j_1 j_2}(\mathcal{D}_L) = \sum_{\substack{0 \le k \le 2j_1 \\ |j_1 - j_2| \le j \le j_1 + j_2}} (-1)^{k+2j_1} \cdot$$

$$q^{\frac{-c_k + 2c_{j_1}}{2}} \left\{ \begin{matrix} j_1 & 0 & j_1 \\ j_1 & k & j_1 \end{matrix} \right\}_q (-1)^{k-2j_1} q^{\frac{2c_{j_1} - c_k}{2}} \left\{ \begin{matrix} j_1 & k & j_1 \\ j_1 & 0 & j_1 \end{matrix} \right\}_q$$

$$(-1)^{k-2j_1} q^{\frac{-2c_{j_1} + c_k}{2}} \left\{ \begin{matrix} j_1 & j_1 & 0 \\ j_1 & j_1 & k \end{matrix} \right\}_q (-1)^{j_1 + j_2 - j} q^{\frac{c_j - c_{j_1} - c_{j_2}}{2}} \cdot$$

$$\left\{ \begin{matrix} j_2 & j_1 & j \\ j_1 & j_2 & 0 \end{matrix} \right\}_q (-1)^{j_1 + j_2 - j} q^{\frac{c_j - c_{j_1} - c_{j_2}}{2}} \left\{ \begin{matrix} j_1 & j_1 & 0 \\ j_2 & j_2 & j \end{matrix} \right\}_q \cdot$$

$$(-1)^{k-2j_1} \left\{ \frac{[2k+1]}{[2j_1+1]} \right\}^{1/2} [2j_1+1]^{3/2} [2j_2+1] =$$

$$= \sum_{k,j} q^{1/2(4c_{j_1} - 3c_k + 2c_j - 2c_{j_2})} \left\{ \begin{matrix} j_1 & 0 & j_1 \\ j_1 & k & j_1 \end{matrix} \right\} .$$

$$\left\{\begin{matrix} j_1 & k & j_1 \\ j_1 & 0 & j_1 \end{matrix}\right\}_q \left\{\begin{matrix} j_1 & j_1 & 0 \\ j_1 & j_1 & k \end{matrix}\right\}_q \left\{\begin{matrix} j_2 & j_1 & j \\ j_1 & j_2 & 0 \end{matrix}\right\}_q \left\{\begin{matrix} j_1 & j_1 & 0 \\ j_2 & j_2 & j \end{matrix}\right\}_q \cdot$$

$$\cdot [2k+1]^{1/2} [2j_1+1][2j_2+1] =$$

$$= \frac{1}{[2j_1+1]} \sum_{\substack{0 \le k \le 2j_1 \\ |j_1-j_2| \le j \le j_1+j_2}} q^{1/2(4c_{j_1} - 3c_k + 2c_j - 2c_{j_2})} \cdot$$

$$\cdot (-1)^{2j_1-k} [2k+1][2j+1] \quad \in \mathbb{Z}[q, q^{-1}].$$

In [2] with the help of the matrices (2.4) and (2.6) a functional, analogous to the one described was defined. Let us remind the reader this definition.

DEFINITION 7.2.

a) let the diagram be in the general position

b) to each component of L we associate the number $j_\alpha \in \frac{1}{2}\mathbb{Z}_+$ (colour of the components)

c) divide the diagram into elementary fragments

d) to each edge connecting elementary fragments we associate the states $|m| \le j_\alpha$, $2m \equiv 2j_\alpha \pmod 2$ were j_α is the colour of the component.

e) to each elementary fragments we associate the weights using the rules (4.1), (4.2) (4.4), (4.5) of the section 4.

g) multiplying the matrix elements corresponding to the elementary fragment over all fragments and taking the sum of resulting product over all states on $\mathcal{D}_L$ we obtain the functional $\widetilde{Z}_{j_1 \dots j_K}(\mathcal{D}_L)$.

<u>An example of calculation of the functional</u> $\widetilde{Z}(\mathcal{D}_L)$

Let $\mathcal{D}_L$ is the diagram given in Fig.1. In accordance with the definition of $\widetilde{Z}_{j_1 j_2}(\mathcal{D}_L)$ we have:

$$\widetilde{Z}_{j_1 j_2}(\mathcal{D}_L) = \sum_{\{m\}} w^{j_1}_{m_1 m_2} \left(\left(\mathbb{R}^{j_1 j_1} \right)^{-1} \right)^{m_1' m_3'}_{m_1 m_3} w^{j_2}_{m_3 m_4}$$

$$\left(\left(\mathbb{R}^{j_1 j_1} \right)^{-1} \right)^{m_4' m_2'}_{m_4 m_2} \left(\mathbb{R}^{j_1 j_1} \right)^{m_7 m_5}_{m_1' m_2'} w^{j_1}_{m_6 m_5} w^{j_1}_{m_3' m_7}$$

$$\left(\left(\mathbb{R}^{j_1 j_2} \right)^{-1} \right)^{m_6' m'}_{m_1' m} \left(\left(\mathbb{R}^{j_2 j_1} \right)^{-1} \right)^{m'' m_6}_{m' m_6'} w^{j_2}_{m_8 m} w^{j_2}_{m'' m_8}$$

$$q^{2c_{j_1} + c_{j_2}}$$

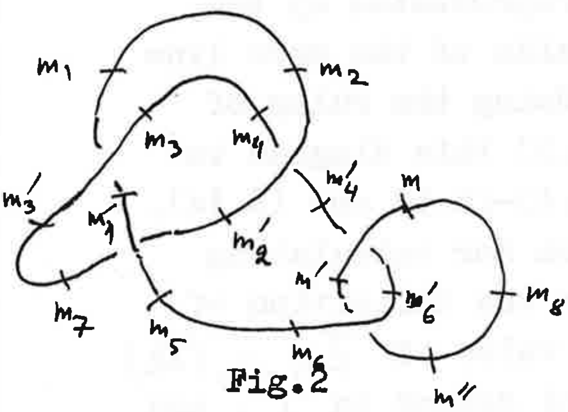


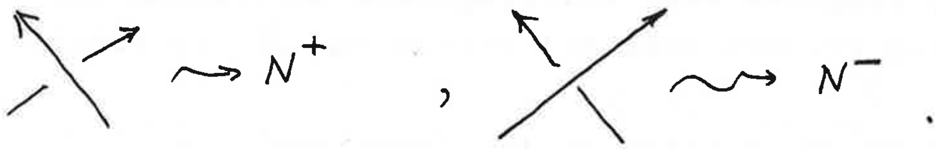
Fig.2

The states $\{m\}$ on the diagram $\mathcal{D}_L$ are given in Fig.2.

Let us introduce an orientation on L and define the numbers

$$w_\alpha(\mathcal{D}_L) = N^+_\alpha(\mathcal{D}_L) - N^-_\alpha(\mathcal{D}_L)$$

where $N^{\pm}_\alpha(\mathcal{D}_L)$ are the numbers of positive (N^+) and negative (N^-) selfintersections on the component



It is easy to see that the numbers $w_\alpha(\mathcal{D}_L)$ do not depend on the orientation of the component α.

THEOREM 7.1. i) The functionals $Z_{j_1\cdots j_K}(\mathcal{D}_L)$ and $\widetilde{Z}_{j_1\cdots j_K}(\mathcal{D}_L)$ coincide,

ii) the functional

$$\varphi_{j_1\cdots j_K}(\mathcal{D}_L)=\prod_{\alpha=1}^{K}\left(q^{c_{j_\alpha}}(-1)^{2j_\alpha}\right)^{w_\alpha(\mathcal{D}_L)} Z_{j_1\cdots j_K}(\mathcal{D}_L) \quad (7.1)$$

is the invariant of the link L.

PROOF. i) Consider the functional $\widetilde{Z}_{j_1\cdots j_K}(\mathcal{D}_L)$ as the product of the matrices corresponding to the elementary fragments and the matrix $I^j\,\widetilde{Z}_{j_1\cdots j_K}(\mathcal{D}_L)$ acting in V^j (I^j is a unit matrix). In accordance with the graphical rules this matrix is represented by the diagram $\mathcal{D}_L$ situated in the upper side of the wave line which correspond to the space V^j. Using the rules of section 6 we can transpose (see Fig.3) this diagram to the shadow world. From the rules (6.4)-(6.7) and (6.14), (6.15) we obtain a new representation for calculating the functional $\widetilde{Z}_{j_1\cdots j_K}(\mathcal{D}_L)$. From the definition of this procedure it follows that wthe value of $\widetilde{Z}_{j_1\cdots j_K}(\mathcal{D}_L)$ calculated in the shadow world do not depend on j, and therefore we can put $j=0$ after that we obtain the rules for calculating $Z_{j_1\cdots j_K}(\mathcal{D}_L)$.

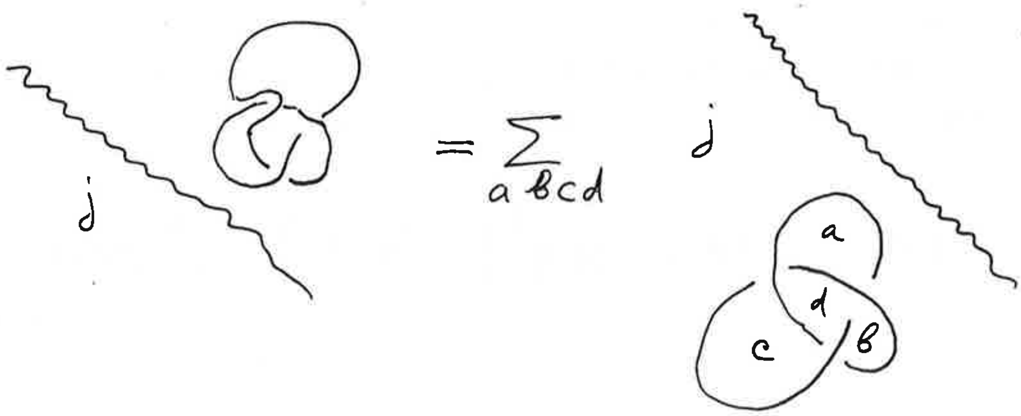


Fig. 3

The part ii) of the theorem was proved in [2] (see also [18, 29]). The main idea is that the invariance of $\widetilde{Z}(\mathcal{D}_L)$ under regular isotopies follows from the relations (4.6), (4.10) and the invariance under singular isotopies follows from (4.12) and from the structure of the multiplier before in (7.1).

Let us give a representation for the invariant φ in the case when L is given in the closed braid form. Consider the case $j_1 = j_2 = \cdots = j_K = j$.

We should remind that the braid group B_N [35] is generated by the elements s_i, $i = 1, \ldots, N-1$ with the following relations

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} ,$$

$$s_i s_j = s_j s_i , \quad |i-j| > 1 .$$

One can define a representation π of the braid group in the space $(V^j)^{\otimes N}$

$$\pi(s_i) := g_i = 1 \otimes \cdots \otimes P_{i\,i+1} R^{jj}_{i\,i+1} \otimes \cdots \otimes 1 \tag{7.2}$$

where $P_{i\,i+1}$ is the permutation matrix.

It is well known that for $j=1/2$ this representation is the representation of the braid group in the Temperley-Lieb-Jones algebra. Indeed, in this case

$$\pi(s_i) = 1\otimes \cdots \otimes q^{-\frac{1}{2}}\left(e_i(q+q^{-1}) - q^{-\frac{1}{2}}\right)\otimes \cdots \otimes 1 \tag{7.3}$$

where the elements e_i form T.L.J. algebra:

$$e_i e_{i+1} e_i = \tau e_i\,, \quad e_i^2 = e_i$$

$$e_{i+1} e_i e_{i+1} = \tau e_{i+1} \tag{7.4}$$

$$\tau = (q+q^{-1})^{-2}$$

Representations (7.2) for $j>\frac{1}{2}$ can be obtained from one for $j=\frac{1}{2}$ by the fusion-procedure [2]. Let us define the following matrices acting in $(V^{1/2})^{\otimes 2j N}$:

$$\mathcal{P}_j^{(n+1,\dots,n+2j)} = \prod_{\ell=1}^{2j}\left(q^{2j-2\ell+1} - q^{-(2j-2\ell+1)}\right)^{-1}\cdot$$

$$\sum_{\varepsilon_1\cdots\varepsilon_{2j}=\pm 1}\ \prod_{\ell=1}^{2j}\varepsilon_\ell\, q^{(2j+2-\ell)\varepsilon_\ell/2}\left(R_{n+1}^{\varepsilon_1}\cdots R_{n+2j}^{\varepsilon_{2j}}\right)$$

$$\left(R_{n+1}^{\varepsilon_2}\cdots R_{n+2j-1}^{\varepsilon_2}\right)\cdots R_{n+1}^{\varepsilon_{2j}}$$

$$\mathbb{R}_n^{(j)} = \left(R_{n+2j} \cdots R_{n+4j-1}\right)\left(R_{n+2j-1} \cdots \right.$$

$$\left. \cdots R_{n+4j-2}\right) \cdots \left(R_{n+1} \cdots R_{n+2j}\right) \tag{7.6}$$

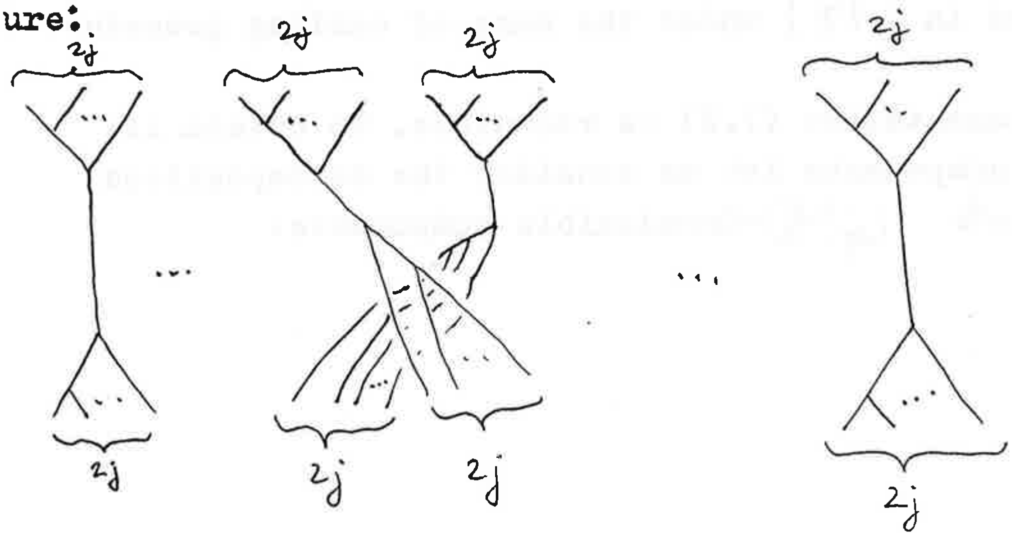
$$R_i = 1 \otimes \cdots \otimes P_{ii+1} R_{ii+1}^{\frac{1}{2}\frac{1}{2}} \otimes \cdots \otimes 1$$

THEOREM 7.2. The map $\widetilde{\pi} : B_N \longrightarrow End\left(\left(V^{1/2}\right)^{\otimes 2jN}\right)$. defined by the following formula

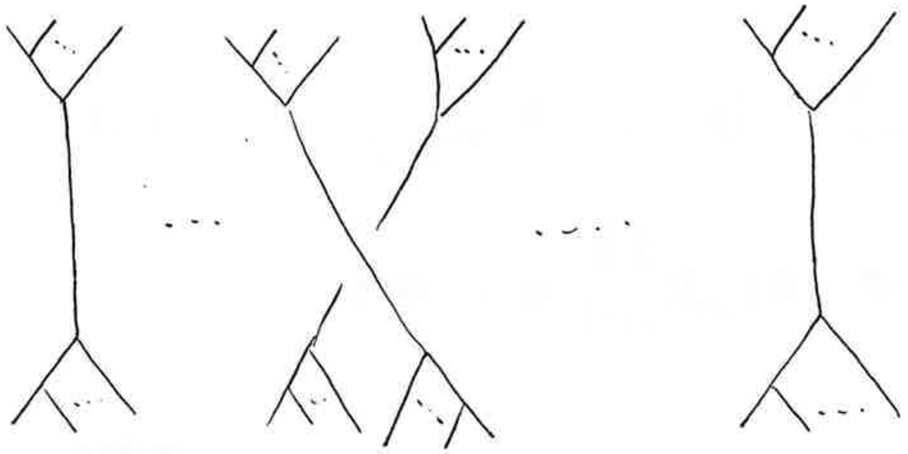
$$\widetilde{\pi}(g_i) = R_i^{(j)} = \varphi_j^{(2ji+1,\ldots,2j(i+1))} \varphi_j^{(2j(i+1)+1,\ldots,2j(i+2))} \mathbb{R}_{2ji}^{(j)} \tag{7.7}$$

is the representation of the braid-group B_N and this representation is isomorphic to the representation (7.2).

PROOF. Let us give graphical prove of this theorem. The representation $\widetilde{\pi}$ can be written as the following picture:



Using the relations (4.6) and (4.7) we can transform this picture in the form



This picture represent the action (7.2) of braid group. So, the theorem is proved.

The fusion-procedure can be applied to the construction of links invariants corresponding to higher representations starting from the invariants that correspond to the vector representation. This procedure was independently considered in [17] under the name of cabling procedure.

The representation (7.2) is reducible. To obtain its irreducible components let us consider the decomposition of $(V^j)^{\otimes N}$ into $U_q(sl_2)$-irreducible components:

$$(V^j)^{\otimes N} = \sum_{\ell}^{\oplus} (W_\ell \otimes V^\ell)$$

where $W_\ell \otimes V^\ell$ are prime components, $\dim W_\ell$ = multiplicity of V^ℓ in $(V^j)^{\otimes N}$. From (1.6) it follows that the matrices $R_{i\,i+1}$ commute with the action of $U_q(s\ell_2)$ in $(V^j)^{\otimes N}$. Therefore they act in the spaces W_ℓ.

Let us choose the following basis in the space W_ℓ:

$$E^\ell(a) \otimes e^\ell_m = \sum_{n_1 \dots n_N} \begin{bmatrix} a_{N-1} & j & \ell \\ n_{N-1} & n_N & m \end{bmatrix}_q \dots$$

$$\dots \begin{bmatrix} a_1 & j & a_2 \\ n_1 & n_3 & n_2 \end{bmatrix}_q \begin{bmatrix} j & j & a_1 \\ n_1 & n_2 & n_1 \end{bmatrix}_q e^j_{n_1} \otimes \dots \otimes e^j_{n_N} \qquad (7.8)$$

The elements of this basis are numerated by the sequences $(a) \equiv (j, a_2, \dots, a_{N-1}, \ell)$ of numbers $a_i \in \frac{1}{2}\mathbb{Z}_+$ satisfying the following conditions $0 \le a_2 \le 2j_1, \dots,$ $|a_k - j| \le a_{k+1} \le a_k + j, \dots, a_N = \ell$.

Let us define the following action of the braid group in the space W_ℓ:

$$\pi_j(s_i)\left(E^\ell(a) \otimes e^\ell_m\right) = \left(\pi_j^\ell(s_i) E^\ell(a)\right) \otimes e^\ell_m \qquad (7.9)$$

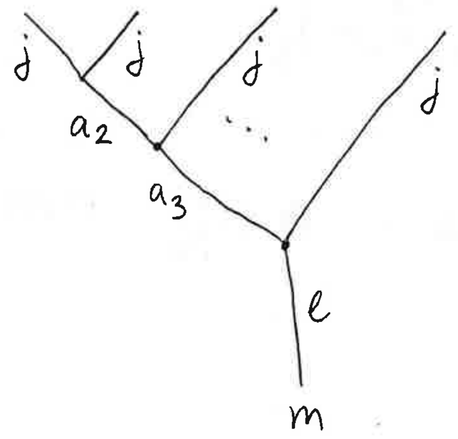
PROPOSITION.

$$\pi_j^\ell(s_i) E^\ell(a) = \sum_{a_i'} (-1)^{a_i' + a_i - a_{i+1} - a_{i-1}} \begin{Bmatrix} j & a_{i-1} & a_i \\ j & a_{i+1} & a_i' \end{Bmatrix}_q$$

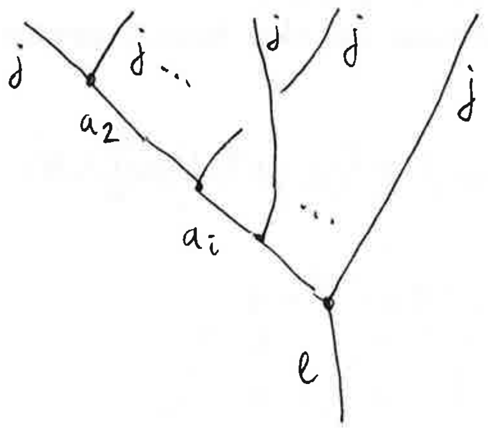
$$q^{\pm \frac{c_{a_{i-1}} + c_{a_{i+1}} - c_{a_i} - c_{a_i'}}{2}} E^\ell(j, a_2, \dots, a_i', \dots, a_{N-1}, \ell)$$

$$q^{\pm \frac{c_{a_{i-1}} + c_{a_{i+1}} - c_{a_i'} - c_{a_i}}{2}} E^{\ell}(j, a_2, \dots, a_i', \dots, a_{N-1}, \ell) \tag{7.10}$$

PROOF. According to the graphical rules of section 4 the vector $E^{\ell}(a) \otimes e^{\ell}_m$ is represented by the following picture:



The left hand side of (7.9) is written as



The formula (7.9) follows immediately from (5.11).

Now let the link L be a closure of the braid α. In this case from the definition $\tilde{Z}(L)$ and $Z(L)$ we have:

$$\tilde{Z}_j(\alpha) = tr_{(V^j)^{\otimes N}}\left(\left(q^{H/2}\otimes\cdots\otimes q^{H/2}\right)\pi_j(\alpha)\right) =$$
$$= \sum_{\ell\in j^{\otimes N}} [2\ell+1]_q \, tr_{W_\ell}\left(\pi_j^\ell(\alpha)\right), \tag{7.11}$$

where $tr_V(\cdot)$ means the matrix trace over the space V.

CONCLUSION

In the present work we considered only the case of the algebra $\mathfrak{g} = s\ell(2)$. $q-6j$ symbols in the general case may be defined in a similar way. The main idea can be extracted from [2]. The investigation of $q-6j$ -analogous of CGC for $U_q(s\ell(n))$ will be given in a separate publication.

Here we don't consider the dual algebra $U_q(s\ell(2))^* = \mathbb{C}_q(SL(2))$. The algebra $\mathbb{C}_q(SL(2))$ appeared in different contexts in [9, 25]. The algebra $\mathbb{C}_q(SU(2))$ appeared in the theory C^* algebras in the works of Woronovich [26], who constructed some elements of harmonic analysis for this algebra. The corepresentations of $\mathbb{C}_q(SU(2))$ and q-analogue of spherical functions on $SU(2)$ were studied in [27][37]. Using these results one can obtain the representation for CGC by integrating over $\mathbb{C}_q(SU(2))$ the product of three quantum spherical functions. A similar representation can also be given for $q-6j$ -symbols.

As in the case of $q=1$ the relations between $q-6j$-symbols can be organized in the Wigner-Recah algebra [23].

We do not consider in details the case when q is a root of unity. In this case one can introduce the calculus of restricted $q-6j$ -symbols. These $q-6j$ -symbols are defined by the formulas of § 5, but the values of their arguments are restricted by the additional condition $a, b, c, d, e, f \le r-2$. As in the case of general q the arguments of restricted $q-6j$ -symbols satisfy also standard inequalities $|b-e| \le a \le e+c$, $|c-f| \le a \le c+f$, $|d-f| \le b \le d+f$, $|e-c| \le d \le e+c$. From the symmetries of $q-6j$ -symbols it follows that for $q^{\frac{r}{2}} = 1$

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q = (-1)^{b+f-c-e+1+2a} \begin{Bmatrix} \bar{a} & b & e \\ d & c & f \end{Bmatrix}_q =$$

$$= \begin{Bmatrix} \bar{a} & b & e \\ \bar{d} & c & f \end{Bmatrix}_q ,$$

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q = (-1)^{f-b-d} \begin{Bmatrix} a & \bar{b} & e \\ \bar{d} & c & \bar{f} \end{Bmatrix}_q$$

where $\bar{a} = \frac{r}{2} - 1 - a$.

It is easy to check that restricted $q-6j$ -symbols satisfy all relations (6.16)-(6.19).

There is an interesting application of restricted $q-6j$ -symbols in conformal field theory. Moore and Seiberg found the polynomial equations describing operator algebras of conformal field theories. It is not hard to check that the relations (6.16)-(6.19) for restricted $q-6j$ -symbols coincide with polynomial equation of Moore and Seiberg, and therefore restricted $q-6j$ -symbols define some operator algebra. It follows from [32] that it is the operator algebra of the Wess-Zumino model of level $r-2$ with central charge

$$c = \frac{3(r-2)}{r}$$

and with the anomaleous dimensions $h_j = \frac{j(j+1)}{r}$,
$j = 0, 1/2, \dots, \frac{r-2}{2}$.

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