USSR ACADEMY OF SCI
STEKLOV MATHEMATICAL IN
LENINGRAD DEPARTME

LOMI PREPRI

E-9-88

A.N.Kirillov, N.Yu.R

REPRESENTATIONS OF THE ALGEBRA $U_q(sl(2))$, q-ORTHOGONAL POLYNOMIALS AND INVARIANTS OF LINKS

Recommended for publication
by the Scientific Council of
Steklov Mathematical Institute,
Leningrad Department (LOMI)

25, April, 1988

Leningrad

1988

Bibliograpical reference:
A.N.Kirillov, N.Yu.Reshetikhin,LOMI,preprint E-9-88,
Leningrad 1988

Leningrad Department
Steklov Mathematical Institute

INTRODUCTION

This work shows how quantized universal enveloping algebras are connected with other areas of mathematics, using algebra $sl(2)$ as an example. It is shown that in the representation theory of the algebra $U_q(sl(2))$ the q-analogues of $6j$-symbols which were introduced by Askey and Wilson [1] in connection with q-orthogonal polynomials, appear naturally. The connection between the quantized universal algebras and the theory of invariants of links, discovered in [2], is considered in more detail. With the help of q-analogues of $6j$-symbols we propose a new representation for the invariants of links, related to $U_q(sl(2))$, which is to a great extend similar to SOS models of statistical physics. The representation theory of algebra $U_q(sl(2))$ is closely connected with Femperly-Lieb-Jones algebra, which emerged in statistical mechanics 3 and in the theory of John-Neumann algebras 4. It happens that the matrix elements of generators in irreducible representations of Jones algebras are special values of q-analogues of $6j$-symbols.

Let us make some historical comments. Quantum universal enveloping algebras appeared as a result of research into on algebraic aspects of quantum integrable systems [5, 6]. The first example of such an algebra was the algebra $U_q(sl(2))$ found by Kulish and Reshetikhin [7]. The structure of Hopf algebra on $U_q(sl(2))$ was discovered independently by Drinfeld [9] Jimbo [10], Sklyanin [8], who built the q-deformation of the universal enveloping algebra for any simple Lie algebra $\mathfrak{g}$ [9,10]. A new approach to $U_q($ algebras, which reflects most adequatly their connection with quantum integrable systems, was proposed by Faddeev, Reshetikhin and Takhtadjan [11], (see also [12]). Finite dimensional representations of $U_q(sl(2))$ are described in [ ] first substantial results in the representation theory of algebras $U_q(\mathfrak{g})$ were obtained by Lusztig [13] and Rosso [14] for simple $\mathfrak{g}$. The q-analogues of the Rakah-Fock formulae for $3j$-symbols were obtained by Vaksman [15]. Other representations of the q-analogues of $3j$-symbols, as well as the properties of their symmetry were found by Kirillov [16].

The connection between universal enveloping algebras with the link theory was established in [9]. We should also mention that the invariants of links, connected with tensor representations of $U_q(\mathfrak{g})$ for classical Lie algebras, can be obtained with the help of cabling (Murakami [17]). Cabling invariants correspond in the terminology of [2] to invariants parametrized by the tensor products of the vector representations of $U_q(\mathfrak{g})$. In particular, the invariants corresponding to finite dimensional representations of $U_q(s\ell(2))$ are built by Akutsu and Wadatti [18] with help of braid representation using the results from vertex models of statistical mechanics. The main results obtained in this direction, are represented in the review by Jo nes [29].

When studying classical orthogonal polynomials Askey and Wilson [1], proposed q-analogues of $6j$-symbols. As it turns out the q-analogues of $6j$-symbols, that occur in the representation theory of $U_q(s\ell(2))$ are proportional to the ones defined in [1]. The orthogonality of these polynomials as well as the recurrent relations for them follows from equalities for q-$6j$-symbols in the representation theory of the algebra $U_q(s\ell(2))$.

Let us briefly consider the content of this paper. Section [1] contains the description of the algebra $U_q(s\ell(2))$ and of the corresponding universal R-matrix and also presents some useful formulae. The irreducible representations of $U_q(s\ell(2))$ and the q-analog of the Weyl element are described in Section 2. In the same section an extension of algebra $U_q(s\ell(2))$ by the q-analog of the Weyl element is introduced.

The decomposition of tensor product of two irreducible finite-dimensional representation of $U_q(s\ell(2))$ is given in Section 3. It also contains the relations between R-matrices and Clebsh-Gordan coefficients (CGC). We prove that the extension of $U_q(s\ell(2))$ by the Weyl element is a Hopf algebra. In Section 4, following [2], a graphical representation of relations between R-matrices and CGC is proposed. In Section 5 the q-analogues of $6j$-symbols are described. It is shown that they are defined by q-hypergeometrical function ${}_4\Phi_3$ and the symmetries of them are found. Note that the q-analogues of CGC and $6j$-symbols correspond to the functions ${}_3\Phi_2$ and ${}_4\Phi_3$ for such values of arguments when these functions become polynomials. Graphical representations for q-$6j$-symbols are introduced in Section 6. Relations between q-$6j$ symbols, particularly their orthogonality, are easily obtained with their help. Section 7 focuses on the structure of the centraliser C^j_N of the algebra $U_q(s\ell(2))$ in $(\pi^j)^{\otimes N}$, where π^j is an irreducible representation of $U_q(s\ell(2))$. It is shown that q-$6j$ symbols are matrix elements of the generators of algebra C^j_N in the irreducible representations of this algebra. In Section 8 a new representation for the invariants of links connected with $U_q(s\ell(2))$ is built with the help of q-$6j$-symbols. This representation is an exact alanogue of SOS models in statistical mechanics [19]. The relation of models in statistical mechanics to q-analogue of $6j$-symbols was recently found also by Pasquier [31]. The calculation of q-$6j$ symbols is given in Appendix A. An Appendix B we remind the definition of quasi-triangle Hopf algebras.

1. ALGEBRA $U_q(s\ell(2))$

The algebra $U_q(s\ell(2))$ [7-10] is generated by elements $H, X^{\pm}$ with the commutation relations

$$[X^{\pm}, H] = \mp 2X^{\pm}, \quad [X^{+}, X^{-}] = \frac{q^{\frac{H}{2}} - q^{-\frac{H}{2}}}{q^{1/2} - q^{-1/2}}, \tag{1.1}$$

As a linear algebra $U_q(s\ell(2))$ space consists of convergent power series in H and of polynomials in $X^{\pm}$. The following formulae for the comultiplication, the antipode and counit on the generators define the structure of a Hopf algebra [ ] on $U_q(s\ell(2))$:

$$\Delta(X^{\pm}) = X^{\pm} \otimes q^{\frac{H}{4}} + q^{-\frac{H}{4}} \otimes X^{\pm}, \quad \Delta(H) = H \otimes 1 + 1 \otimes H, \tag{1.2}$$

$$S(X^{\pm}) = -q^{\pm\frac{1}{2}} X^{\pm}, \quad S(H) = -H, \tag{1.3}$$

$$\varepsilon(H) = \varepsilon(X^{\pm}) = 0 \tag{1.4}$$

The maps $\Delta' = \sigma \circ \Delta$, $S' = S^{-1}$ where σ is the permutation in $U_q(s\ell(2))^{\otimes 2}$, $\sigma(a \otimes b) = b \otimes a$ also define the structure of Hopf algebra on $U_q(s\ell(2))$. Let us denote this Hopf algebra as by $U_q(s\ell(2))'$. It is evident from (1.2), (1.3) that

$$U_q(s\ell(2))' = U_{q^{-1}}(s\ell(2)) \tag{1.5}$$

Comultiplications Δ and Δ' are connected in $U_q(s\ell(2))^{\otimes 2}$ by the following automorphism [9]:

$$\Delta'(a) = R\, \Delta(a)\, R^{-1} \tag{1.6}$$

where $R \in U_q(s\ell(2))^{\otimes 2}$ and

$$R = \exp\left(\frac{h}{4} H \otimes H\right) \sum_{n \geq 0} \frac{(1-q^{-1})^n}{[n]!} e^n \otimes f^n, \quad q = e^{h} \tag{1.7}$$

Here $[n] = (q^{\frac{n}{2}} - q^{-\frac{n}{2}})/(q^{\frac{1}{2}} - q^{-\frac{1}{2}})$, $e = \exp(\frac{h}{4} H) X^{+}$, $f = \exp(-\frac{h}{4} H) X^{-}$.

The element R is called the universal R-matrix. It satisfies the relations:

$$(\Delta \otimes id) R = R_{13} R_{23} \tag{1.8}$$

$$(id \otimes \Delta) R = R_{13} R_{12} \tag{1.9}$$

$$(S \otimes id) R = R^{-1} \tag{1.10}$$

where the indices show the embeddings of R into $U_q(s\ell(2))^{\otimes 3}$.

Formula (1.8) (or (1.9)) imply the Yang-Baxter equation for R:

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \tag{1.11}$$

The center of the algebra $U_q(s\ell(2))$ is generated by the q-analog of Casimir's elements [8, 10]:

$$C = \left(\frac{q^{\frac{H+1}{2}} - q^{-\frac{H+1}{2}}}{q^{1/2} - q^{-1/2}} \right)^2 + X^{-} X^{+} \tag{1.12}$$

For real q one can introduce $*$-antiinvolution

$$(X^{\pm})^{*} = X^{\mp}, \quad H^{*} = H \tag{1.13}$$

The real form of $U_q(s\ell(2))$ corresponding to this $*$-antiinvolution is denoted by $U_q(su(2))$. The element (1.12) is invariant under the action of involution (1.13) and therefore all finite-dimensional representations of $U_q(su(2))$ are completely reducible.

Let us give now some useful formulae:

$$\Delta^{(N)}\left((X^{\pm})^m\right) = \sum_{0 \le a_1 \le \cdots \le a_{N-1} \le m} \begin{bmatrix} m \\ a_1, a_2 - a_1, \dots, a_N - a_{N-1} \end{bmatrix}_q$$

$$\bigotimes_{i=1}^{N} (X^{\pm})^{a_i - a_{i-1}} \, q^{-(m - a_i - a_{i-1})\frac{H}{4}} \tag{1.14}$$

$$\left[(X^+)^n, (X^-)^m\right] = (X^-)^{m-n} \frac{[m]!}{[m-n]!} \prod_{j=1}^{n} \frac{q^{\frac{m-j}{2}} q^{\frac{H}{2}} - q^{-\frac{m-j}{2}} q^{-\frac{H}{2}}}{q^{1/2} - q^{-1/2}} \tag{1.15}$$

where $\Delta^{(N)} : U_q(s\ell(2)) \to U_q(s\ell(2))^{\otimes N}$ is the composition of comultiplications and

$$\begin{bmatrix} m \\ \ell_1 \ell_2 \dots \ell_N \end{bmatrix} = \frac{[m]!}{[\ell_1]! \cdots [\ell_N]!} \tag{1.16}$$

2. IRREDUCIBLE REPRESENTATIONS OF $U_q(su(2))$ AND FINITE DIMENSIONAL R-MATRICES

All irreducible representations of $U_q(su(2))$ are finite dimensional. It is useful to describe them in the weight basis. In this basis the element H is a diagonal matrix. The irreducible representations of $U_q(su(2))$ are parametrized by integer (or halfinteger) numbers j. The representation π^j have dimension $2j+1$. The generators $H, X^{\pm}$ act in weight basis e_m^j in the following way:

$$\pi^j(X^{\pm}) e_m^j = \left([j \mp m][j \pm m + 1]\right)^{1/2} e_{m \pm 1}^j \tag{2.1}$$

$$\pi^j(H) e_m^j = 2m e_m^j \tag{2.2}$$

where $-j \le m \le j$, $2m = 2j (\mathrm{mod}\, 2)$. We shall denote by V^j the representation space of the representation π^j. From (1.1) and (1.2) at $m = j$ we find the value of the central element (1.12)in the irreducible representation:

$$C V^j = \left(\frac{q^{\frac{2j+1}{4}} - q^{-\frac{2j+1}{4}}}{q^{1/2} - q^{-1/2}}\right)^2 V^j = [j + \tfrac{1}{2}]^2 V^j \tag{2.3}$$

The following important formula holds

$$\pi^j\left((X^{\pm})^a\right) e_n^j = \left(\frac{[j \mp n]!\,[j \pm n + a]!}{[j \mp n - a]!\,[j \pm n]!}\right)^{1/2} e_{n \pm a}^j \tag{2.4}$$

Here the r.h.s. is nonzero only for $|n \pm a| \le j$. Let us consider the matrix $R^{j_1 j_2} = (\pi^{j_1} \otimes \pi^{j_2}) R$ acting in $V^{j_1} \otimes V^{j_2}$. It is not difficult find the matrix elements of $R^{j_1 j_2}$:

$$\left(\mathbb{R}^{j_1 j_2}\right)^{n_1 n_2}_{n_1+n,\, n_2-n} = \frac{(1-q^{-1})^n}{[n]!}\left(\frac{[j_1-n_1]!\,[j_1+n_1+n]!}{[j_1-n_1-n]!\,[j_1+n_1]!}\right)^{1/2} \tag{2.5}$$

$$\left(\frac{[j_2+n_2]!\,[j_2+n-n_2]!}{[j_2+n_2-n]!\,[j_2-n_2]!}\right)^{1/2} q^{\frac{n_1-n_2+2n}{2}}$$

define the matrix w^j acting $\overbrace{\text{with}}^{\text{in } V^j}$ the elements:

$$w^j_{mm'} = q^{-\frac{c_j}{4}} (-1)^{j-m} \delta_{m,-m'}\, q^{-\frac{m}{2}} \tag{2.6}$$

where $c_j = j(j+1)$ and consider the algebra $\widetilde{U_q}(su(2))$ which is the extension of $U_q(su(2))$ by the element with the value (2.6) in any representation. Let τ be the linear antiautomorphism of $U_q(su(2))$ which is the transposition in

$$\tau(X^{\pm}) = X^{\mp}, \quad \tau(H) = H \tag{2.7}$$

PROPOSITION 2.1. In $\widetilde{U_q}(su(2))$ we have:

$$w\, a\, w^{-1} = \tau S(a), \quad \forall a \in U_q(su(2)) \tag{2.8}$$

To prove (2.8) it is sufficient to check it in any irreducible $U_q(su(2))$ -module.

From (1.10) ~~and (2.8) we~~ obtain the crossing-symmetry of universal $\mathcal{R}$ ~~-matrix~~:

$$(\tau \otimes id)\mathcal{R}^{-1} = (w\otimes 1)\, \mathcal{R}\, (w^{-1}\otimes 1) \tag{2.9}$$

and finite dimensional R -matrices $\mathbb{R}^{j_1 j_2}$,

$$\left(\left(\mathbb{R}^{j_1 j_2}\right)^{-1}\right)^{t_1} = w_1\, \mathbb{R}^{j_1 j_2}\, w_1^{-1} \tag{2.10}$$

Here t_1 is the transposition in the first space in $V^{j_1}\otimes V^{j_2}$ and $w_1 = w^{j_1}\otimes 1$.

3. q -ANALOG OF CLEBSCH-GORDAN COEFFICIENTS

Let us consider the tensor product $\pi^{j_1}\otimes\pi^{j_2}$ of two irreducible representations of $U_q(su(2))$. In accordance with the complete reducibility of representations of $U_q(su(2))$ $\pi^{j_1}\otimes\pi^{j_2}$ is decomposed into the following sum of irreducible components [10]:

$$V^{j_1}\otimes V^{j_2} = \sum^{\oplus}_{\substack{|j_1-j_2|\le j\le j_1+j_2 \\ 2j = 2j_1+2j_2 \ (\mathrm{mod}\ 2)}} V^j \tag{3.1}$$

Let $e^j_m(j_1 j_2)$ be the weight basis in the irreducible component V^j. The coordinates of the vectors $e^j_m(j_1 j_2)$ in the basis $e^{j_1}_{m_1}\otimes e^{j_2}_{m_2}$, by analogy with $q=1$ case, will are called the Clebsch-Gordan coefficients (CGC):

$$e^j_m(j_1 j_2) = \sum_{m_1 m_2} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q e^{j_1}_{m_1}\otimes e^{j_2}_{m_2} \tag{3.2}$$

Since π^{j_1} are $*$ -representations of $U_q(su(2))$ we have also

$$e^{j_1}_{m_1} \otimes e^{j_2}_{m_2} = \sum_{j,m} \begin{bmatrix} j_1 j_2 j \\ m_1 m_2 m \end{bmatrix}_q e^{j}_{m}(j_1 j_2)\, \delta(j_1 j_2 j) \qquad (3.2')$$

Here $\delta(j_1 j_2 j) = 1$ if $|j_1 - j_2| \le j \le j_1 + j_2$, $2j = 2j_1 + 2j_2$ and $\delta(j_1 j_2 j) = 0$ in other cases.

The coefficients in (3.2), (3.2') can be considered as q-analog of CGC for $SU(2)$. The coefficients $\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q$ can be calculated using the formulae of Section 2. We shall omit the calculations and present only the final formulae in the following special cases:

q-analog of the Majumdar formula [16]

$$\begin{bmatrix} j_1 j_2 j \\ m_1 m_2 m \end{bmatrix}_q = \delta_{m_1+m_2,m}\, q^{-\frac{1}{4}(j_1+j_2-j)(j_1-j_2+j+1) - j\frac{m_2}{2} - j_2\frac{m}{2}}$$

$$(-1)^{j_1-m_1} \left\{ \frac{[j-m]!\,[j+m]!\,[j_1+m_1]!\,[j_2+m_2]!\,[j_1+j_2-j]!\,[2j+1]}{[j_1-m_1]!\,[j_2-m_2]!\,[j_1-j_2+j]!\,[j_2-j_1+j]!\,[j_1+j_2+j+1]!} \right\}^{1/2} \qquad (3.3)$$

$$\sum_{\tau \ge 0} (-1)^{\tau} q^{\frac{\tau(m-j-1)}{2}} \frac{[2j_2-\tau]!\,[j_1-j_2+j+\tau]!}{[\tau]!\,[j_2+m_2-\tau]!\,[j-j_2+m_1+\tau]!\,[j_1+j_2-j-\tau]!}$$

q-analog of the Rakah-Fock formula [15]

$$\begin{bmatrix} j_1 j_2 j \\ m_1 m_2 m \end{bmatrix}_q = \delta_{m_1+m_2,m} (-1)^{j_1-m_1} q^{\frac{1}{4}(j_2(j_2+1) - j_1(j_1+1) - j(j+1)) + \frac{m_1(m_1+1)}{2}}$$

$$\left\{ \frac{[j+m]!\,[j-m]!\,[j_1-m_1]!\,[j_2-m_2]!\,[j_1+j_2-j]!\,[2j+1]}{[j_1+m_1]!\,[j_2+m_2]!\,[j_1-j_2+j]!\,[j_2-j_1+j]!\,[j_1+j_2+j+1]!} \right\}^{1/2} \qquad (3.4)$$

$$\sum_{\tau \ge 0} (-1)^{\tau} q^{\frac{1}{2}\tau(m+j+1)} \frac{[j_1+m_1+\tau]!\,[j_2+j-m_1-\tau]!}{[\tau]!\,[j-m-\tau]!\,[j_1-m_1-\tau]!\,[j_2-j+m_1+\tau]!}$$

q-analog of the Van der Waerder formula [16]

$$\begin{bmatrix} j_1 j_2 j \\ m_1 m_2 m \end{bmatrix}_q = \delta_{m_1+m_2,m}\, \Delta(j_1 j_2 j)\, q^{\frac{1}{4}(j_1+j_2-j)(j_1+j_2+j+1) + \frac{j_1 m_2 - j_2 m_1}{2}}$$

$$\{ [j_1+m_1]!\,[j_1-m_1]!\,[j_2+m_2]!\,[j_2-m_2]!\,[j+m]!$$

$$[j-m]!\,[2j+1] \}^{1/2} \sum_{\tau \ge 0} (-1)^{\tau} q^{-\frac{1}{2}\tau(j_1+j_2+j+1)} \qquad (3.5)$$

$$\frac{1}{[\tau]!\,[j_1+j_2-j-\tau]!\,[j_1-m_1-\tau]!\,[j_2+m_2-\tau]!\,[j-j_2+m_1+\tau]!\,[j-j_1+\tau-m_2]!}$$

where

$$\Delta(abc) = \left(\frac{[-a+b+c]!\,[a-b+c]!\,[a+b-c]!}{[a+b+c+1]!} \right)^{1/2} \qquad (3.6)$$

The following relations between CGC reflect the completeness and the orthogonality of the basis $e^{j}_{m}(j_1 j_2)$ in $V^{j_1} \otimes V^{j_2}$:

$$\sum_{m_1 m_2} \begin{bmatrix} j_1 j_2 j \\ m_1 m_2 m \end{bmatrix}_q \begin{bmatrix} j_1 j_2 j' \\ m_1 m_2 m' \end{bmatrix}_q = \delta_{jj'}\, \delta_{mm'}\, \delta(j_1 j_2 j) \qquad (3.7)$$

$$\sum_{\substack{|j_1-j_2| \le j \le j_1+j_2 \\ |m| \le j}} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q \begin{bmatrix} j_1 & j_2 & j \\ m_1' & m_2' & m \end{bmatrix}_q = \delta_{m_1 m_1'} \delta_{m_2 m_2'} \tag{3.8}$$

THEOREM 3.1. There are the following relations between CGC:

$$\sum_{m_1' m_2'} \left(R^{j_1 j_2}\right)^{m_1' m_2'}_{m_1 m_2} \begin{bmatrix} j_1 & j_2 & j \\ m_1' & m_2' & m \end{bmatrix}_q = (-1)^{j_1+j_2-j} q^{c_j - c_{j_1} - c_{j_2}} \begin{bmatrix} j_2 & j_1 & j \\ m_2 & m_1 & m \end{bmatrix}_q \tag{3.9}$$

$$\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q = (-1)^{j_1+j_2-j} \begin{bmatrix} j_2 & j_1 & j \\ m_2 & m_1 & m \end{bmatrix}_{q^{-1}} \tag{3.10}$$

$$\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q = (-1)^{j_1-m_1} q^{-\frac{m_1}{2}} (-1)^{j_1+j-j_2} \left(\frac{[2j+1]}{[2j_2+1]}\right)^{\frac{1}{2}} \begin{bmatrix} j_1 & j & j_2 \\ -m_1 & m & m_2 \end{bmatrix}_q \tag{3.11}$$

where the numbers c_j and $[n]$ defined in (2.6), (1.7).

The proof of this theorem follows from the representations (3.3) - (3.5) (see also [2]).

THEOREM 3.2.

$$\sum_{m'} \left(R^{j j_3}\right)^{m m_3'}_{m' m_3} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m' \end{bmatrix}_q = \sum_{m_1' m_2' m_3''} \begin{bmatrix} j_1 & j_2 & j \\ m_1' & m_2' & m \end{bmatrix}_q \left(R^{j_1 j_3}\right)^{m_1' m_3''}_{m_1 m_3} \left(R^{j_2 j_3}\right)^{m_2' m_3'}_{m_2 m_3''} \tag{3.12}$$

To prove this formula it is sufficient to consider relation (1.8) in the representation $\pi^{j_3} \otimes \pi^{j_1} \otimes \pi^{j_2}$ (for more details see [2]).

From (1.11) it follows that the matrices $R^{j_1 j_2}$ satisfy the Yang-Baxter relation:

$$R^{j_1 j_2}_{12} R^{j_1 j_3}_{13} R^{j_2 j_3}_{23} = R^{j_2 j_3}_{23} R^{j_1 j_3}_{13} R^{j_1 j_2}_{12} \tag{3.13}$$

Here all matrices act in $V^{j_1} \otimes V^{j_2} \otimes V^{j_3}$ and the indices show how these matrices are acting in the product space.

To conclud this section let us prove that the extension of $U_q(su(2))$ by the Weyl element w is a Hopf algebra.

THEOREM 3.3. Formulas

$$\Delta(w) = R^{-1}(w \otimes w), \quad \varepsilon(w) = 1 \tag{3.14}$$

define the structure of a Hopf algebra on $\widetilde{U_q}(su(2))$

PROOF. Let us check (3.14) in all irreducible representations. In accordance with the definition of CGC we have:

$$\left(\pi^{j_1} \otimes \pi^{j_2}\right) \Delta(w)^{m_1 m_2}_{m_1' m_2'} = \sum_{j, m, m'} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q w^{j}_{m m'} \begin{bmatrix} j_1 & j_2 & j \\ m_1' & m_2' & m' \end{bmatrix}_q \tag{3.15}$$

The symmetries of CGC imply the identity

$$\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q = \sum_{m_1' m_2'} \left(\left(R^{j_1 j_2} \right)^{-1} \right)^{m_1 m_2}_{m_1' m_2'} q^{c_j - c_{j_1} - c_{j_2}} (-1)^{j_1 + j_2 - j} \begin{bmatrix} j_1 & j_2 & j \\ -m_1' & -m_2' & -m \end{bmatrix}_q$$

Comparing this identity with (3.15) we obtain the equality (3.14) in the representation

$$(\pi^{j_1} \otimes \pi^{j_2}) \Delta w = \left(R^{j_1 j_2} \right)^{-1} w^{j_1} \otimes w^{j_2}$$

Formulae (1.8) and (1.9) imply thecoassociativity of the action (3.14).

Let $\Delta(a) = \sum_i a^i \otimes a_i$. The Hopf axiom for the antipode is $S(a^i) a_i = a^i S(a_i) = \varepsilon(a) 1$. To define the action of the antipode on w we need the following lemma.

LEMMA. Let A be the quasitriangle Hopf algebra (see Appendix B) and $R = e_i \otimes e^i$ is the universal R-matrix, then

1. The element $u = \sum s(e^i) e_i$ isinvertible and
$u^{-1} = \sum_j e^j s^2(e_j)$
2. $s^2(a) = u a u^{-1}$ for any $a \in A$.

We do not give here the proof of this lemma, because it would have demanded the description of many auxiliary constructions. The proof of this lemma for arbitrary quasitriangle Hopf algebra was given by V.G.Drinfeld (private communication) and for quasitriangle algebras of special structure (for dubles of Hopf algebras) by one of the authors (N.R. unpublished). Let us check now the property of the antipode in $\widetilde{U}_q(su(2))$. From (3.14) we obtain $(S \otimes id) \Delta w = S(w) S^2(e^i) \otimes e_i w$ and therefore we must have

$$S(w) S^2(e_i) e^i w = \varepsilon(w) \cdot 1 \qquad (3.16)$$

Lemma 1 implies that $S(u^{-1}) = S^2(e_i) e^i$. Comparing with (3.16) we get

$$S(w) w = \varepsilon(w) u$$

Further, the equalities

$$\varepsilon(w) S(u) u = S(w) u w = S(w)^2 w^2 = S(w) w \, w S(w)$$

$$S(u) u = u S(u)$$

prove the relation

$$w S(w) = \varepsilon(w) S(u)$$

Using this relation we have:

$$e_i w S(w) e^i = \varepsilon(w) e_i S(u) e^i = \varepsilon(w) S(u) S(u^{-1}) = \varepsilon(w) \cdot 1$$

So, the conditions of the axiom for an antipode in $\widetilde{U}_q(su(2))$ are satisfied. To calculate $\varepsilon(w)$ we substitute w into (2.8). Since $\tau^2 = id$ we obtain

$$S(w) = \tau(w)$$

Therefore in every irreducible representation we have

$$(w^j)^t w^j = \varepsilon(w) \pi^j(u), \quad w^j (w^j)^t = \varepsilon(w) \pi^j(s(u))$$

Comparing this formula with (2.6) we get

$$\varepsilon(w) = 1\,,\quad \pi^{j}(u) = q^{-2c_j + H}\,;\quad \pi^{j}(S(u)) = q^{-2c_j - H}$$

4. GRAPHICAL REPRESENTATION OF R-MATRICES AND q-CLEBSCH-GORDAN COEFFICIENTS

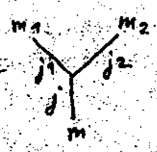
The relations (3.7) - (3.13) between CGC and R-matrices can be represented graphically [2].

Let us represent the R-matrices and CGC by graphs with strings colored by numbers j and with states $\{m_i\}$ on the end of strings:

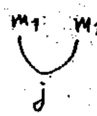
$$\longrightarrow \left(R^{j^2 j^1}\right)^{m_2' m_1'}_{m_1 m_2} \tag{4.1}$$

$$\longrightarrow \left(\left(R^{j^2 j^1}\right)^{-1}\right)^{m_1' m_2'}_{m_1 m_2} \tag{4.2}$$

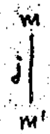
$$\longrightarrow \begin{bmatrix} j^1 & j^2 & j \\ m_1 & m_2 & m \end{bmatrix}_q \longleftarrow \tag{4.3}$$



$$\longrightarrow \delta_{m_1,-m_2}(-1)^{-j+m_2}\, q^{\frac{m_2}{2}}\,, \tag{4.4}$$

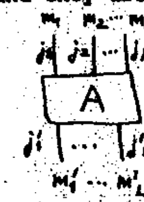


$$\longrightarrow \delta_{m_1,-m_2}(-1)^{j-m_1}\, q^{-\frac{m_1}{2}}\,,$$

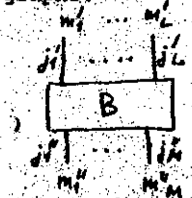


$$\longrightarrow \delta_{mm'} \tag{4.5}$$

The multiplication of matrices correspond to the joining of graphs of these matrices together. For example if the matrix A acts from $V^{j_1'} \otimes \ldots \otimes V^{j_L'}$ into $V^{j^1} \otimes \ldots \otimes V^{j^N}$ the matrix B acts from $V^{j_1''} \otimes \ldots \otimes V^{j_M''}$ to $V^{j_1'} \otimes \ldots \otimes V^{j_L'}$ and they are represented by graphs:

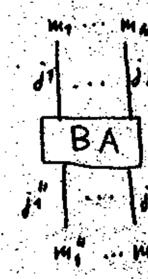


$$\longrightarrow A^{m_1' \cdots m_L'}_{m_1 \cdots m_N}\,,$$

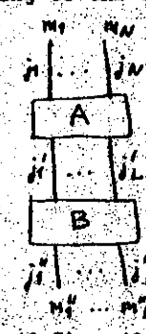


$$\longrightarrow B^{m_1'' \cdots m_M''}_{m_1' \cdots m_L'}$$

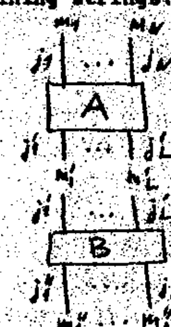
then the product BA is represented by the joining of strings $j_1' \cdots j_L'$ connected with B with strings $j_1' \cdots j_L'$ connected with A. The joining means the summation over the states corresponding to the ends of the joining strings:



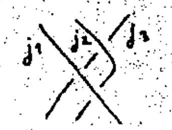
$$\overset{def}{\equiv}$$



$$\equiv \sum_{m_1' \cdots m_L'}$$

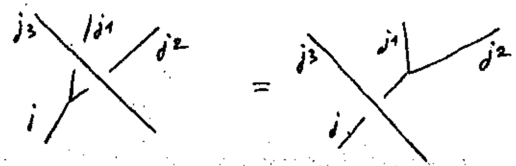


The relations (3.7) - (3.13) and (2.10) are represented by the following graphical equalities:

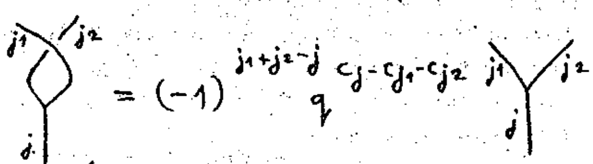


$$=$$

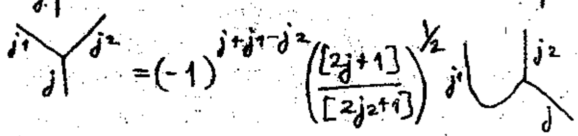
(4.6)



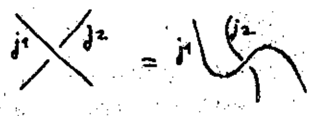
(4.7)



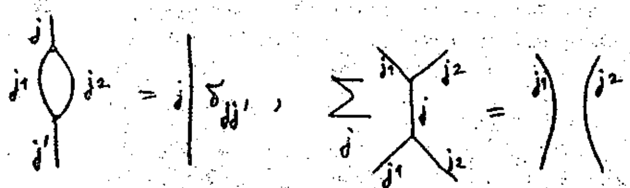
(4.8)



(4.9)



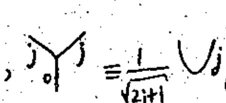
(4.10)



(4.11)

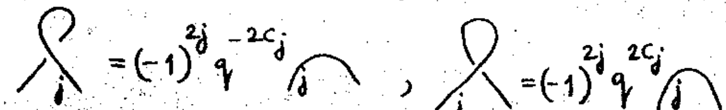
The relations (4.11), (4.8), (4.9), (4.7), (4.6) represent the formulae (3.7) - (3.13) respectively and (4.10) represents the crossing-symmetry (2.10) of R -matrices. In the relation (4.8) the value $j = 0$ is very important for the further application to links theory. In this case:

$$\begin{bmatrix} j & j & 0 \\ m_1 & m_2 & 0 \end{bmatrix} = \frac{w^j_{m_1 m_2}}{\sqrt{[2j+1]}} q^{c_j} ,$$



(4.12)

and therefore from (4.8) with $j = 0$ it follows that



(4.13)

5. THE q -ANALOGS OF THE WIGNER.RAKAH $6j$ -SYMBOLS

Let us consider now the q -analog of $6j$ -symbols and the properties of these $q-6j$ -symbols. For this purpose let us consider tensor product $V^{j_1} \otimes V^{j_2} \otimes V^{j_3}$ of three irreducible representations of $U_q(su(2))$. There are two symplest ways to obtain irreducible components in this representation. One is to decompose first $V^{j_1} \otimes V^{j_2} = \sum\limits_{j_{12}}^{\oplus} V^{j_{12}}$ and then to take irreducible submodules in $V^{j_{12}} \otimes V^{j_3}$. The other is to decompose first $V^{j_2} \otimes V^{j_3} = \bigoplus\limits_{j_{23}} V^{j_{23}}$ and then $V^{j_1} \otimes V^{j_{23}}$. These two ways give two complete orthogonal bases in $V^{j_1} \otimes V^{j_2} \otimes V^{j_3}$,

$$e^{j_{12} j}_{m}(j_1 j_2 | j_3) = \sum_{m_1 m_2 m_3} \begin{bmatrix} j_{12} & j_3 & j \\ m_{12} & m_3 & m \end{bmatrix}_q \begin{bmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{bmatrix}_q e^{j_1}_{m_1} \otimes e^{j_2}_{m_2} \otimes e^{j_3}_{m_3} \tag{5.1}$$

$$e^{j_{23} j}_{m}(j_1 | j_2 j_3) = \sum_{m_1 m_2 m_3} \begin{bmatrix} j_1 & j_{23} & j \\ m_1 & m_{23} & m \end{bmatrix}_q \begin{bmatrix} j_2 & j_3 & j_{23} \\ m_2 & m_3 & m_{23} \end{bmatrix}_q e^{j_1}_{m_1} \otimes e^{j_2}_{m_2} \otimes e^{j_3}_{m_3} \tag{5.2}$$

The matrix elements of the matrix, connecting these bases will be called $q-6j$ -symbols

$$e^{j_{12}j}_{m}(j_1j_2|j_3) = \sum_{j_{23}} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}_q e^{j_{23}j}_{m}(j_1|j_2j_3) \quad (5.3)$$

In the case $q=1$ we have

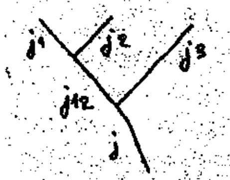
$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}_1 = \left([2j_{12}+1]\,[2j_{23}+1]\right)^{-\frac{1}{2}} (-1)^{j_1+j_2-j-j_3-2j_{12}} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}$$

where $\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}$ is Racah-Wigner $6j$ -symbol.

For $q \neq 1$ we shall also use Racah-Wigner normalization:

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}^{RW}_q = \left([2j_{12}+1][2j_{23}+1]\right)^{\frac{1}{2}} (-1)^{j_1+j_2-j-j_3-2j_{12}} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}_q$$

If we use the graphical technique of the previous section the definition (5.3) of $q-6j$ -symbols will have the form

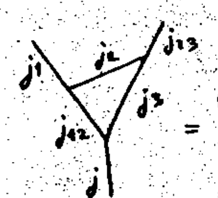


$$= \sum_{j^{23}} \begin{Bmatrix} j^1 & j^2 & j^{12} \\ j^3 & j & j^{23} \end{Bmatrix}_q \quad (5.4)$$

Using the orthogonality (4.11) of CGC we obtain an expression of $q-6j$ -symbols in terms of CGC:

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}_q = \begin{bmatrix} j_1 & j_{23} & j \\ m_1 & m_{23} & m \end{bmatrix}_q^{-1} \sum_{m_2+m_3=m-m_1} \begin{bmatrix} j_2 & j_3 & j_{23} \\ m_2 & m_3 & m_{23} \end{bmatrix}_q \begin{bmatrix} j_{12} & j_3 & j \\ m_{12} & m_3 & m \end{bmatrix}_q \begin{bmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{bmatrix}_q \quad (5.5)$$

or



$$= \begin{Bmatrix} j^1 & j^2 & j^{12} \\ j^3 & j & j^{23} \end{Bmatrix}_q \quad (5.6)$$

THEOREM 5.1.

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}^{RW}_q = \Delta(abe)\Delta(acf)\Delta(ce\overset{d}{\cancel{f}})\Delta(dbf)$$

$$\sum_z (-1)^z [z+1]! \{ [z-a-b-e]! [z-a-c-f]! \quad (5.7)$$
$$[z-b-d-f]! [z-d-c-e]! [a+b+c+d-z]!$$
$$[a+d+e+f-z]! [b+c+e+f-z]! \}^{-1}$$

Here the sum is taken only over z with nonnegative arguments in square brackets, $[0]! \equiv 1$.

The proof of this theorem is given in Appendix A.

REMARK. The sum (5.7) can be expressed through the generalized hypergeometric function ${}_4\Phi_3$ (see [1]):

$$\frac{[a+c+f+1]!}{[b+e-a]!\,[b+d-f]!\,[d+e-c]!\,[a-b-d-c]!\,[f-d-e+a]!\,[c+f-b-e]!}$$

$$ {}_4\Phi_3\left(\begin{matrix} -b+a-e,\ -b-d+f,\ -d-e+c,\ a+c+f+2 \\ a-b-d+c+1,\ f-d-e+a+1,\ c+f-b-e+1 \end{matrix};q;q\right) $$

The symmetries of $q-6j$ -symbols follow from (A.18)

1. $\begin{Bmatrix} b & a & e \\ c & d & f \end{Bmatrix}_q^{RW} = \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q^{RW}$

2. $\begin{Bmatrix} a & e & b \\ d & f & c \end{Bmatrix}_q^{RW} = \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q^{RW}$

3. $\begin{Bmatrix} a & c & f \\ d & b & e \end{Bmatrix}_q^{RW} = \begin{Bmatrix} a & e & b \\ d & f & c \end{Bmatrix}_q^{RW}$

4. $q-$ Regge symmetry. Let

$$ s_1 = \frac{b+c+e+f}{2},\quad s_2 = \frac{a+d+e+f}{2},\quad s_3 = \frac{a+b+c+d}{2} $$

then the following equalities hold

$$ \begin{Bmatrix} a & s_1-a & s_1-f \\ d & s_1-b & s_1-e \end{Bmatrix}_q^{RW} = \begin{Bmatrix} a & e & b \\ d & f & c \end{Bmatrix}_q^{RW} $$

$$ \begin{Bmatrix} s_2-d & b & s_2-f \\ s_2-a & c & s_2-e \end{Bmatrix}_q^{RW} = \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q^{RW} $$

$$ \begin{Bmatrix} s_3-d & s_3-c & e \\ s_3-a & s_3-b & f \end{Bmatrix}_q^{RW} = \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q^{RW}, $$

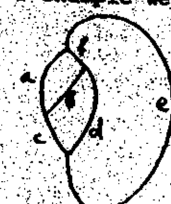
$$ \begin{Bmatrix} s_2-d & s_3-c & s_1-f \\ s_2-a & s_3-b & s_1-e \end{Bmatrix}_q^{RW} = \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q^{RW} $$

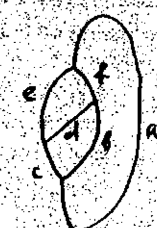
The symmetries 1 - 3 of $q-6j$ -symbols have a simple graphical interpretation. To describe it let us write $q-6j$-symbol as the following trace:

$$ \begin{Bmatrix} j^1 & j^2 & j^{12} \\ j^3 & j & j^{23} \end{Bmatrix}_q = \frac{1}{[2j+1]} $$ 

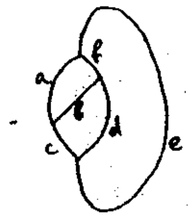
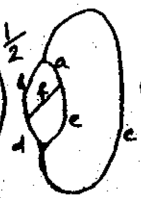
(5.8)

The symmetries 1 - 3 of $q-6j$ -symbols correspond to rotating of thetraedron formed by the edges $(j^1, j^2, j^3, j^{12}, j^{23}, j)$ in accordance with rules (4.6) - (4.10). For example we have:

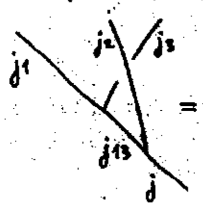
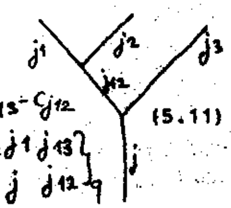


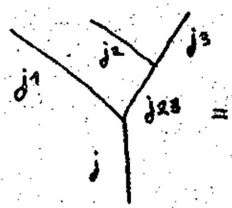
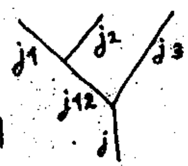
$$ = \frac{[2e+1]}{[2a+1]} $$


(5.9)


$$= \frac{[2e+1]}{[2c+1]}(-1)^{a+f-c-d}\left(\frac{[2a+1][2f+1]}{[2c+1][2d+1]}\right)^{\frac{1}{2}}$$

(5.10)

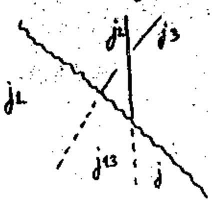
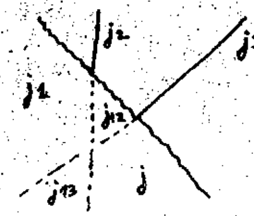
To conclude this section we give two important formulae with $q-6j$ -symbols

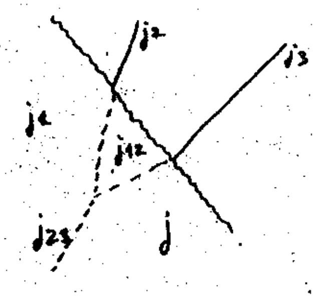

$$= \sum_{j_{12}} (-1)^{j_{13}+j_{12}-j-j_1} q^{c_j+c_{j_1}-c_{j_{13}}-c_{j_{12}}} \begin{Bmatrix} j_3 & j_1 & j_{13} \\ j_2 & j & j_{12} \end{Bmatrix}_q$$

(5.11)

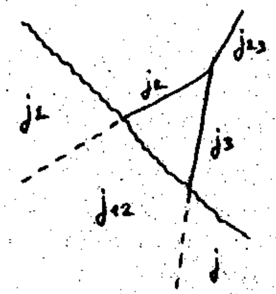

$$= \sum_{j_{12}} \begin{Bmatrix} j_3 & j_2 & j_{23} \\ j_1 & j & j_{12} \end{Bmatrix}_q$$

(5.12)

6. GRAPHICAL REPRESENTATION OF $q-6j$ -SYMBOLS

To give a graphical technique for representing $q-6j$ -symbols let us rewrite the relations (5.5), (5.11) and (5.12) in the following form:

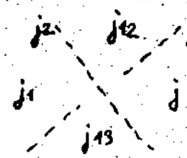

$$= \sum_{j_{12}}$$

(6.1)

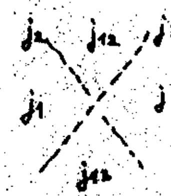

$$= \sum_{j_{12}}$$

(6.2)


$$=$$

(6.3)

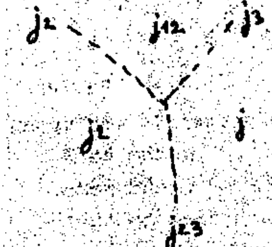
Here a wave line divides the plane into two parts. In the upper part the strings are colored by the numbers j ; the ends of these strings are marked by the states $\{m\}$ and vertices represent the R -matrices and CGC according to the rules (4.1) - (4.5). In the lower part the numbers j color the strings and the sectors placed between the strings. The colors of the strings are not changed after crossing the wave line. The points of intersections of two strings and the triple vertices correspond to the following $q-6j$ -symbols:


$$\rightarrow (-1)^{j_{13}+j_{12}-j_1-j} q^{c_j+c_{j_1}-c_{j_{12}}-c_{j_{13}}} \begin{Bmatrix} j_3 & j_1 & j_{13} \\ j_2 & j & j_{12} \end{Bmatrix}_q$$
(6.4)



$$\rightarrow (-1)^{j+j_1-j_{12}-j_{13}}\, q^{-c_j-c_{j_1}+c_{j_{13}}+c_{j_{12}}} \begin{Bmatrix} j_3 & j_1 & j_{13} \\ j_2 & j & j_{12} \end{Bmatrix} \qquad (6.5)$$

$$\rightarrow \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}_q \qquad (6.6)$$

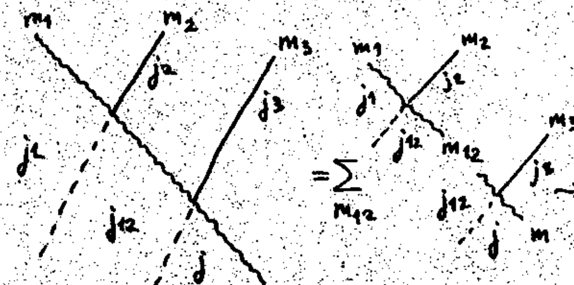


$$\rightarrow \begin{Bmatrix} j_3 & j_2 & j_{23} \\ j_1 & j & j_{12} \end{Bmatrix}_q \qquad (6.7)$$

The points of intersections of strings with the wave line correspond to CGC:

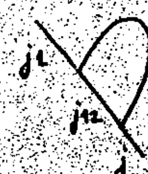
$$\rightarrow \begin{bmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{bmatrix}_q \qquad (6.8)$$

The joining of the fragments (6.8) by the wave line correspond to the summation over the states on the joining ends:

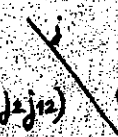


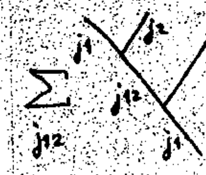
$$\rightarrow \sum_{m_{12}} \begin{bmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{bmatrix}_q \cdot \begin{bmatrix} j_{12} & j_3 & j \\ m_{12} & m_3 & m \end{bmatrix}_q \qquad (6.9)$$

Let us call the lower part side of the plane "the shadows world". The rules (6.4) - (6.7) represent $q-6j$ -symbols in the shadows world. To find the weights, corresponding to the extremal fragments consider the relations which follow from (4.9), (4.11), (4.12)

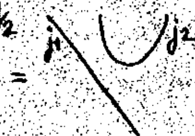


$$= \delta_{j_1 j_2} (-1)^{j_1+j_2-j_{12}} \left(\frac{[2j_{12}+1]}{[2j_1+1]}\right)^{1/2} \delta(j_1 j_2 j_{12}) \qquad (6.10)$$

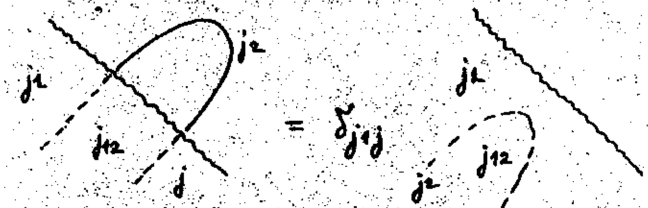




$$\sum_{j_{12}} \ldots (-1)^{j_2-j_1-j_{12}}\, \delta(j_1 j_2 j_{12}) \left(\frac{[2j_{12}+1]}{[2j_1+1]}\right)^{1/2} = \qquad (6.11)$$



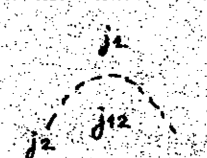
We see that one can rewrite these relations in the form similar to (6.1) - (6.3)



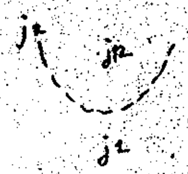
(6.12)

(6.13)

if we associate the following weights with extremal fragments in the shadows world



$$\to (-1)^{j_1+j_2-j_{12}} \left(\frac{[2j_{12}+1]}{[2j_1+1]}\right)^{1/2} \delta(j_1 j_2 j_{12}) \tag{6.14}$$



$$\to (-1)^{j_1+j_2-j_{12}} \left(\frac{[2j_{12}+1]}{[2j_1+1]}\right)^{1/2} \delta(j_1 j_2 j_{12}) \tag{6.15}$$

So, using the relations (6.1) - (6.15) we can transpose every picture representing the combinations of R -matrices and CGC in into the shadows world. Using this process we can obtain the relations between $q-6j$ -symbols from the corresponding relations for R -matrices and CGC.

THEOREM 6.1. The following relations between $q-6j$ -symbols hold:

$$\sum_{j} \begin{Bmatrix} j_2 & j_1 & j \\ j_3 & j_5 & j_4 \end{Bmatrix}_q \begin{Bmatrix} j_3 & j_1 & j_6 \\ j_2 & j_5 & j \end{Bmatrix}_q = \delta_{jj} \tag{6.16}$$

$$\sum_{j_{13}} (-1)^{j_{13}} q^{-c_{j_{13}}} \begin{Bmatrix} j_3 & j_1 & j_{13} \\ j_2 & j & j_{12} \end{Bmatrix}_q \begin{Bmatrix} j_3 & j_2 & j_{23} \\ j_1 & j & j_{13} \end{Bmatrix}_q =$$

$$= (-1)^{j+j_1+j_2+j_3-j_{12}-j_{23}} q^{c_{j_{23}}+c_{j_{12}}-c_{j_2}-c_{j_3}-c_{j_1}-c_j} \begin{Bmatrix} j_3 & j_2 & j_{23} \\ j_1 & j & j_{12} \end{Bmatrix}_q \tag{6.17}$$

$$\sum_{d} \begin{Bmatrix} j_2 & a & d \\ j_1 & c & b \end{Bmatrix}_q \begin{Bmatrix} j_3 & d & e \\ j_1 & f & c \end{Bmatrix}_q \begin{Bmatrix} j_3 & j_2 & j_{23} \\ a & e & d \end{Bmatrix}_q =$$

$$= \begin{Bmatrix} j_{23} & a & e \\ j_1 & f & b \end{Bmatrix}_q \begin{Bmatrix} j_3 & j_2 & j_{23} \\ b & f & c \end{Bmatrix}_q \tag{6.18}$$

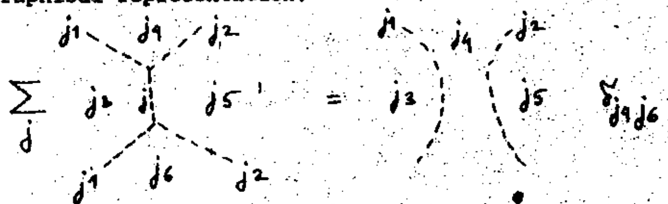
$$\sum_{g} (-1)^{a-b-g-f} q^{c_a-c_b-c_g-c_f} \begin{Bmatrix} j_2 & a & g \\ j_1 & c & b \end{Bmatrix}_q \begin{Bmatrix} j_3 & g & e \\ j_1 & d & c \end{Bmatrix}_q \begin{Bmatrix} j_3 & a & f \\ j_2 & e & g \end{Bmatrix}_q =$$

$$= \sum_{g} (-1)^{d-c-g-e} q^{c_d-c_c-c_g-c_e} \begin{Bmatrix} j_3 & b & g \\ j_2 & d & c \end{Bmatrix}_q \begin{Bmatrix} j_3 & a & f \\ j_1 & g & b \end{Bmatrix}_q \begin{Bmatrix} j_2 & f & e \\ j_1 & d & g \end{Bmatrix}_q \tag{6.19}$$

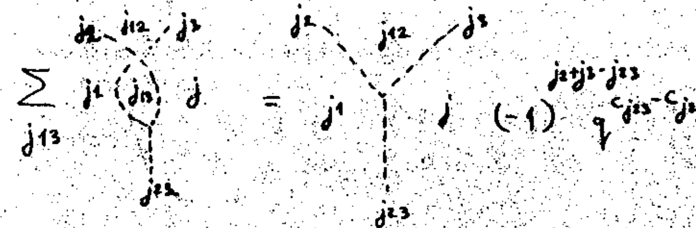
$$\sum_c (-1)^c q^{-c(c+1)} \begin{Bmatrix} j & c & b \\ j & a & b \end{Bmatrix} \frac{[2c+1]}{[2b+1]} = (-1)^{2j+2b-a} q^{-2C_j+2C_b-C_a} \quad (6.20)$$

The relation (6.16) is called the orthogonality relation between $q-6j$ -symbols. The relation (6.17) is the q-analog of the Racah identity, the relation (6.18) is the q-analog of the Bidenharn-Elliot identity.

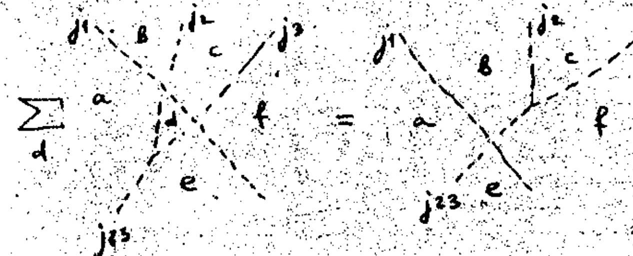
PROOF. Let us rewrite the relations (6.16) - (6.20) in the graphical representation:



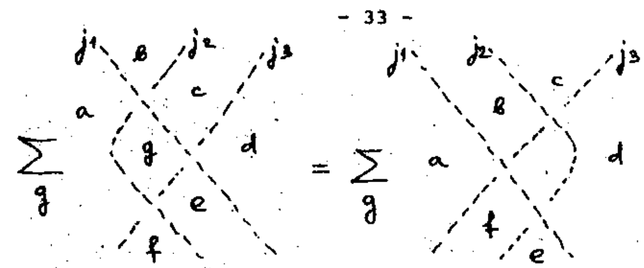
(6.16')



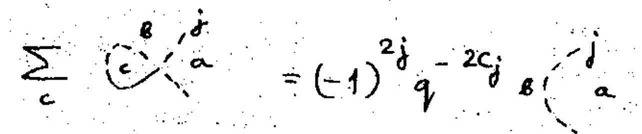
(6.17')



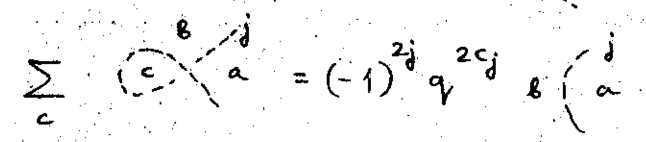
(6.18')



(6.19')



(6.20')



Now we see that these relations follow from (4.11), (4.8), (4.7), (4.13) and (4.6) respectively if we transpose the latter into the shadows world. It seems that this is the simplest proof of the identities (6.16) - (6.20).

7. CENTRALIZER OF $U_q(su(2))$ IN $(V^j)^{\otimes N}$

Let us consider the representation $(\pi^j)^{\otimes N}$ of $U_q(su(2))$ acting in $\mathcal{H}_N^j = (V^j)^{\otimes N}$. The centralizer of $U_q(su(2))$ in this representation will be denoted by C_N^j :

$$C_N^j = \{ y : \mathcal{H}_N^j \to \mathcal{H}_N^j \mid y(\pi^j \otimes \cdots \otimes \pi^j)(a) = (\pi^j \otimes \cdots \otimes \pi^j)(a) y ,\ \forall a \in U_q(su(2)) \}$$

this is an associative finite dimensional algebra.

THEOREM 7.1. The algebra C^j_N is generated by the unit matrix in $\mathcal{H}^j_N$ and by the elements:

$$g_i = 1 \otimes \cdots \otimes R^{jj} \otimes \cdots \otimes 1$$

where $R^{jj} = P^{jj} \mathbb{R}^{jj}$ and P^{jj} is the permutation in $(V^j)^{\otimes 2}$ $P(a \otimes b) = b \otimes a$.

The proof of this theorem will be given later.

The algebra $C^{1/2}_N$ is well known. Consider the elements

$$e_i = \frac{q^{1/2} g_i + q^{-1/2}}{q + q^{-1}}$$

in $C^{1/2}_N$. From the theorem 7.1 it follows that these elements and the unit matrix generate the algebra $C^{1/2}_N$. Formula (2.5) for the matrix elements of $\mathbb{R}^{1/2\,1/2}$ implies relations

$$e_i e_{i+1} e_i = \tau e_i \ , \quad e_{i+1} e_i e_{i+1} = \tau e_{i+1}$$

$$e_i^2 = e_i \ , \qquad \tau = \frac{1}{(q+q^{-1})^2}$$

for the elements e_i. So, we have the theorem.

THEOREM 7.2. $C^{1/2}_N$ is isomorphic to Temperly-Lieb-Jones algebra [4].

Consider the following elements in $C^{1/2}_{2jN}$:

$$\varphi_j^{(n+1,\ldots,n+2j)} = \prod_{l=1}^{2j} \left(q^{2j-2l+1} - q^{-(2j-2l+1)}\right)^{-1}$$

$$\sum_{\varepsilon_1 \cdots \varepsilon_{2j} = \pm 1} \prod_{l=1}^{2j} \varepsilon_l q^{\frac{2j+2-l}{2}\varepsilon_l} \left(R^{\varepsilon_1}_{n+1} \cdots R^{\varepsilon_{2j}}_{n+2j}\right) \tag{7.4}$$

$$\left(R^{\varepsilon_2}_{n+1} \cdots R^{\varepsilon_2}_{n+2j-1}\right) \cdots R^{\varepsilon_{2j}}_{n+1}$$

$$\mathbb{R}^{(j)}_n = \left(R_{n+2j} \cdots R_{n+4j-1}\right)\left(R_{n+2j-1} \cdots R_{n+4j-2}\right) \tag{7.5}$$

$$\cdots \left(R_{n+1} \cdots R_{n+2j}\right)$$

where

$$R_i = 1 \otimes \cdots \otimes R^{\frac{1}{2}\frac{1}{2}}_{i\,i+1} \otimes \cdots \otimes 1 \tag{7.6}$$

THEOREM 7.3. The subalgebra $\tilde{C}^j_N$ of $C^{1/2}_{2jN}$, generated by the unit matrix and by the elements

$$R^{(j)}_i = \varphi_j^{(2ji+1,\ldots,2j(i+1))} \varphi_j^{(2j(i+1)+1,\ldots,2j(i+2))} \mathbb{R}^{(j)}_{2ji} \tag{7.7}$$

is isomorphic to the algebra C^j_N. The elements $R^{(j)}_i$ in $\tilde{C}^j_N$ correspond to the elements g_i in C^j_N.

To prove this theorem let us embed $(V^j)^{\otimes N}$ into $(V^{1/2})^{\otimes 2jN}$ by dividing the latter into N multipliers $(V^{1/2})^{\otimes 2j}$ and then by embedding V^j into $(V^{\frac{1}{2}})^{\otimes 2j}$. After this procedure C^j_N will be embeded into $C^{1/2}_{2jN}$ and the elements $R^{(j)}_i$ will be the images of the elements

$g_i \in C^j_N$. Now consider the decomposition of the space $\mathcal{H}^j_N$ into irreducible components

$$\mathcal{H}^j_N = (V^j)^{\otimes N} = \sum_{0 \le \ell \le 2jN} W^j_\ell \otimes V^\ell . \tag{7.8}$$

Here $W^j_\ell \otimes V^\ell$ are the prime components of the representation $(\pi^j)^{\otimes N}$. The multiplicity of V^ℓ in $(V^j)^{\otimes N}$ is equal to the dimension of W^j_ℓ.

It is obvious that the spaces W^j_ℓ form the complete family of irreducible C^j_N-modules.

Let us choose the basis in W^j_ℓ in accordance with the ordered (from left to right) of the spaces V^j in $(V^j)^{\otimes N}$. The elements of this basis are numerated by the sequences $(j, a_2, \dots, a_{N-1}, \ell)$ of numbers $a_i \in \frac{1}{2}\mathbb{Z}_+$ satisfying the following conditions $0 \le a_2 \le 2j, \dots, |a_k - j| \le a_{k+1} \le a_k + j, \dots, a_N = \ell$. Let us denote these sequences by $(a)^\ell_N = \{j, a_2, \dots, a_{N-1}, \ell\}$. Let $E((a)^\ell_N)$ be the basis vector in W^j_ℓ corresponding to the sequence $(a)^\ell_N$. The vector $E((a)^\ell_N) \otimes e^\ell_m$ has the following coordinates in the $e^j_{n_1} \otimes \dots \otimes e^j_{n_N}$ basis:

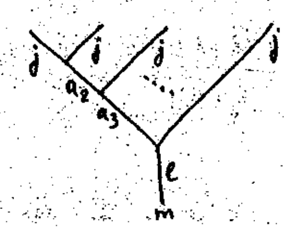
$$E((a)^\ell_N) \otimes e^\ell_m = \sum_{n_1 \dots n_N} \begin{bmatrix} a_{N-1} & j & \ell \\ n_{N-1} & n_N & m \end{bmatrix}_q \dots$$

$$\dots \begin{bmatrix} a_1 & j & a_2 \\ n_1 & n_3 & n_2 \end{bmatrix}_q \begin{bmatrix} j & j & a_1 \\ n_1 & n_2 & n_1 \end{bmatrix}_q e^j_{n_1} \otimes \dots \otimes e^j_{n_N} \tag{7.9}$$

THEOREM 7.4. The generators $g_i^{\pm 1}$ have the following matrix elements in the basis $E((a)^\ell_N)$ of the representation W^j_ℓ:

$$\rho_\ell(g_i^{\pm 1}) E((a)^\ell_N) = \sum_{a_i'} (-1)^{a_i' + a_i - a_{i+1} - a_{i-1}} \begin{Bmatrix} j & a_{i-1} & a_i \\ j & a_{i+1} & a_i' \end{Bmatrix}_q \tag{7.10}$$
$$q^{\pm(c_{a_{i-1}} + c_{a_{i+1}} - c_{a_i} - c_{a_i'})} E((j, a_2, \dots, a_i', \dots, a_{N-1}, \ell))$$

PROOF. According to the graphical rules of section 4 the vector $E((a)^\ell_N) \otimes e^\ell_m$ is represented by the following picture:



In accordance with the definition of ρ_ℓ we have $g_i(E((a)^\ell_N) \otimes e^\ell_m) = (\rho_\ell(g_i) E((a)^\ell_N)) \otimes e^\ell_m$.
The left hand side of this equality is written as

The equality (7.10) follows immediately from (5.11).

Ocneanu [34] introduced the trace on Hecke algebra. There is a natural analog of this trace for the algebra $C^j_\infty = \bigsqcup_{n \ge 1} C^j_n$. Define $tr(a)$ for $a \in C^j_N \subset C^j_\infty$

by the following formula:

$$tr(a) = \frac{1}{[2j+1]^N}\, tr_{\mathcal{K}_N^j}\left(\left(q^{\frac{H}{2}}\otimes \dots \otimes q^{\frac{H}{2}}\right)a\right) \quad (7.11)$$

In accordance with the decomposition (7.8) we have:

$$tr(a) = \sum_{0\le \ell \le 2jN,} tr_{W_\ell^j}\left(\rho_\ell(a)\right) \frac{[2\ell+1]}{[2j+1]^N} \quad (7.12)$$

THEOREM 7.5. The functional $tr: C_\infty^j \to \mathbb{C}$ has the following properties:

$$tr(1) = 1 \quad (7.13)$$

$$tr(ab) = tr(ba) \quad (7.14)$$

$$tr(ab) = tr(a)\,tr(b)\,\frac{1}{[2j+1]} \quad (7.15)$$

Here in (7.15) $a \in C_N^j \subset C_\infty^j$ and b is the product of the generators g_i with $i \geq N$.

PROOF. The relation (7.13) is obvious. The property (7.14) follows from the relation

$$\left(q^{\frac{H}{2}}\otimes q^{\frac{H}{2}}\right)R^{jj} = R^{jj}\left(q^{\frac{H}{2}}\otimes q^{\frac{H}{2}}\right)$$

To prove (7.15) let us note that the operator

$$\rho(b) = (id \otimes \underbrace{tr_{V^j}\otimes \dots \otimes tr_{V^j}}_{M-1})\left((1\otimes \underbrace{q^{\frac{H}{2}}\otimes \dots \otimes q^{\frac{H}{2}}}_{M-1})\, b\right)$$

$$b \in C_M^j$$

acting (in V^j) commutes with the action of $U_q(su(2))$ in V^j and therefore

$$\rho(b) = I \cdot c(b) \quad (7.16)$$

where I is the unit matrix in V^j and $c(b)$ is some functional on C_M^j. Comparing this with the definition (7.11) of the trace we have

$$c(b) = [2j+1]^{M-1}\, tr\,(b) \quad (7.17)$$

The relation (7.15) follows immediately from (7.16) and (7.17).

So, we have a set of algebras C_N^j with all irreducible representations described and with the trace defined on $C_\infty^j = \bigsqcup_{n\geq 0} C_n^j$. Therefore we can introduce the basis of matrix units in C_N^j. This basis in also the basis in C_N^j as in a linear space. Let us define the elements E_{ab}^ℓ in the following way

$$\rho_m(E_{ab}^\ell) = \delta_{\ell m}\; E((a)_N^\ell)\otimes E^*((b)_N^\ell) \quad (7.18)$$

where $\{E^*((b)_M^\ell)\}$ is the dual basis in $(W_\ell^j)^*$. The next theorem is immediately implied by this definition.

THEOREM 7.6. The elements E_{ab}^ℓ form the basis in

C_N^j regarded as a linear space and satisfy the following multiplication law:

$$E^{\ell}_{ab} E^{m}_{cd} = \delta_{\ell m} \delta_{bc} E^{\ell}_{ad} \qquad (7.19)$$

So, we have a family of finite dimensional semisimple algebras $\dots C_N^j \subset C_{N+1}^j \subset \cdots$. In the theory of such algebras the decomposition of the C_N^j-module W_ℓ^j considered as a C_{N-1}^j-module plays an important role [ ] . For our case we have:

$$W_\ell^j(N)\Big\downarrow_{C^j_{N-1}} = \bigoplus_{\substack{|j-\ell| \le \ell' \le j+\ell \\ 2\ell' = 2j+2\ell \,(\mathrm{mod}\, 2)}} W^j_{\ell'}(N-1) \qquad (7.20)$$

To conclude this section let us note that the case corresponds to the Birmann-Wenzl algebra with special values of parameters.

THEOREM 7.7. $C_N^1 \simeq BW_N(-iq, i(q^{\frac12}-q^{-\frac12}))$, where $BW_N(\ell, m)$ is the Birmann-Wenzl algebra with the generators G_j and with relations

$$G_i G_j = G_j G_i \ , \qquad |i-j| > 1$$

$$G_i G_{i+1} G_i = G_{i+1} G_i G_{i+1} \ ,$$

$$G_i E_i = E_i G_i = \ell^{-1} E_i$$

where E_i is defined by the relation

$$G_i + G_i^{-1} = m(1+E_i)$$

$$G_j \approx i g_j$$

Other relations are given in [ ] .

This theorem follows immediately from the matrix structure of $R^{1,1}$ (see also [2]).

8. THE INVARIANTS OF LINKS ASSOCIATED WITH $U_q(su(2))$

Using the q-analog of $6j$-symbols we give here a new model for the invariants corresponding to higher representations of $U_q(su(2))$ [2, 17, 18] . This model is based on the graphical representation (6.4), (6.5) for q-$6j$-symbols and is obtained from the model based on R-matrices [2] by transposing the latter into the shadow world in accordance with the rules of the section 6.

DEFINITION 8.1. Let $\mathcal{D}_L$ be the diagram of the link L

a) to each component of we associate a number (α numerates the components of L), which we call the colour of the component.

b) let us paint the plane on which the diagram is located into different colours (numerated by $j \in \frac12 \mathbb{Z}_+$) following the rules described below:

- to the extremal part of the plane we associate the number $j = 0$.
- to those parts of the plane which can be reached from the extremal part by crossing only one string, we associate the colour of this string (these are parts neighbouring to the extremal one).
- other internal parts of the diagram are pointed according to the following inductive rule: let O_K be a part which can be reached from the exteriour by crossing a minimum of K strings

and let $\mathcal{O}^{\alpha}_{k-1}$ be the neighbours of this part which can be reached by crossing $(k-1)$ -strings; let be the colour of $\mathcal{O}^{\alpha}_{k-1}$ parts, then the colour j_k of the part $\mathcal{O}_k$ must satisfy the inequalities $|j^{\alpha}_{k-1} - \ell^{\alpha}_{k,k-1}| \le j_k \le j^{\alpha}_{k} + \ell^{\alpha}_{k,k-1}$ for any α ; here $\ell^{\alpha}_{k,k-1}$ is the colour of the string dividing $\mathcal{O}_k$ and $\mathcal{O}_{k-1}$. We shall call each set of colours satisfying these inequalities state on the diagram.

c) let the diagram $\mathcal{D}_L$ be in a general position.

d) for each state on $\mathcal{D}_L$ let us associate a weight to each intersecting and each extremal fragment following the rules (6.4), (6.5), (6.14), (6.15). Then we multiply all of these weights and sum up the product over all possible states.

The obtained functional is denoted by $Z_{j_1 \cdots j_k}(\mathcal{D}_L)$ where k is the number of the components of L and $j_1, \dots, j_k$ are the colours of these components.

<u>An example of the calculation of the functional</u> $Z_{j_1 \cdots j_k}(\mathcal{D}_L)$

Let $\mathcal{D}_L$ be the diagram given in Fig. 1. The numbers j_1 and j_2 are the colours of the components of L . The states on $\mathcal{D}_L$ are given in Fig. 1. For colours k and j we have the following restrictions: $0 \le k \le 2j_1$, $|j_1 - j_2| \le j \le j_1 + j_2$. In accordance with Definition 8.1 for the functional $Z_{j_1 j_2}(\mathcal{D}_L)$ we obtain the following expression.

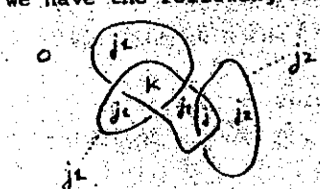


Fig. 1.

$$Z_{j_1 j_2}(\mathcal{D}_L) = \sum_{\substack{0 \le k \le 2j_1 \\ |j_1 - j_2| \le j \le j_1 + j_2}} (-1)^{-k+2j_1}$$

$$q^{-c_k + 2c_{j_1}} \begin{Bmatrix} j_1 & 0 & j_1 \\ j_1 & k & j_1 \end{Bmatrix}_q (-1)^{k-2j_1} q^{2c_{j_1} - c_k} \begin{Bmatrix} j_1 & k & j_1 \\ j_1 & 0 & j_1 \end{Bmatrix}_q (-1)^{k-2j_1}$$

$$q^{-2c_{j_1} + c_k} \begin{Bmatrix} j_1 & j_1 & 0 \\ j_1 & j_1 & k \end{Bmatrix}_q (-1)^{j_1 + j_2 - j} q^{c_j - c_{j_1} - c_{j_2}} \begin{Bmatrix} j_2 & j_1 & j \\ j_1 & j_2 & 0 \end{Bmatrix}_q$$

$$(-1)^{j_1 + j_2 - j} q^{c_j - c_{j_1} - c_{j_2}} \begin{Bmatrix} j_1 & j_1 & 0 \\ j_2 & j_2 & j \end{Bmatrix}_q (-1)^{k-2j_1} \left(\frac{[2k+1]}{[2j_1+1]} \right)^{1/2} \cdot$$

$$[2j_1+1]^{3/2} \cdot [2j_2+1] = \sum_{k,j} q^{4c_{j_1} - 3c_k + 2c_j - 2c_{j_2}} \begin{Bmatrix} j_1 & 0 & j_1 \\ j_1 & k & j_1 \end{Bmatrix}_q$$

$$\begin{Bmatrix} j_1 & k & j_1 \\ j_1 & 0 & j_1 \end{Bmatrix}_q \begin{Bmatrix} j_1 & j_1 & 0 \\ j_1 & j_1 & k \end{Bmatrix}_q \begin{Bmatrix} j_2 & j_1 & j \\ j_1 & j_2 & 0 \end{Bmatrix}_q \begin{Bmatrix} j_1 & j_1 & 0 \\ j_2 & j_2 & j \end{Bmatrix}_q [2k+1]^{1/2} [2j_1+1]$$

$$[2j_2+1] =$$

$$= \frac{1}{[2j_1+1]} \sum_{\substack{0 \le k \le 2j_1 \\ |j_1 - j_2| \le j \le j_1 + j_2}} [2k+1][2j+1](-1)^{2j_1 - k} q^{-3c_k + 4c_{j_1} + 2c_j - 2c_{j_2}}$$

In [2] with the help of the matrices (2.4) and (2.6) a functional, analogous to the one described was defined. Let us remind the reader this definition.

DEFINITION 8.2.

a) let the diagram be in the general position

b) to each component of L we associate the number $j_\alpha \in \frac{1}{2}N$ (colour of the components)

c) divide the diagram into elementary fragments

d) to each edge connecting elementary fragments we associate the states $|m| \le j_\alpha$, $2m = 2j_\alpha \pmod 2$ were j_α is the colour of the component.

e) to each elementary fragments we associate the weights using the rules (4.1), (4.2), (4.4), (4.5) of the section 4.

g) multiplying the matrix elements corresponding to the elementary fragment over all fragments and taking the sum of resulting product over all states on $\mathcal{D}_L$ we obtain the functional $\widetilde{Z}_{j_1 \cdots j_K}(\mathcal{D}_L)$.

An example of calculation of the functional $\widetilde{Z}(\mathcal{D}_L)$

Let $\mathcal{D}_L$ is the diagram given in Fig. 1. In accordance with the definition of $\widetilde{Z}_{j_1 j_2}(\mathcal{D}_L)$ we have:

$$\widetilde{Z}_{j_1 j_2}(\mathcal{D}_L) = \sum_{\{m\}} w^{j_1}_{m_1 m_2} \left(\left(R^{j_1 j_1}\right)^{-1}\right)^{m_1' m_3'}_{m_1 m_3} w^{j_2}_{m_3 m_4} \left(\left(R^{j_1 j_1}\right)^{-1}\right)^{m_4' m_2'}_{m_4 m_2}$$

$$\left(R^{j_1 j_1}\right)^{m_7 m_5}_{m_1' m_2'} w^{j_1}_{m_6 m_5} w^{j_1}_{m_3' m_7} \left(\left(R^{j_1 j_2}\right)^{-1}\right)^{m_6' m'}_{m_4' m}$$

$$\left(\left(R^{j_2 j_1}\right)^{-1}\right)^{m'' m_6}_{m' m_6'} w^{j_2}_{m m_8} w^{j_2}_{m'' m_8} q^{4c_{j_1} + 2c_{j_2}}$$

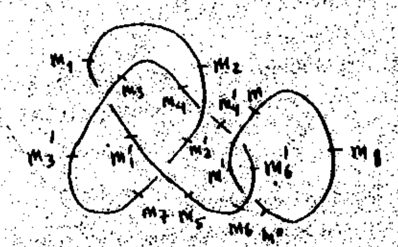


Fig. 2.

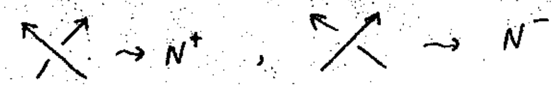
The states $\{m\}$ on the diagram $\mathcal{D}_L$ are given in Fig. 2.

Let us introduce an orientation on L and define the

numbers $w_\alpha(\mathcal{D}_L)$

$$w_\alpha(\mathcal{D}_L) = N^+_\alpha(\mathcal{D}_L) - N^-_\alpha(\mathcal{D}_L)$$

where $N^\pm_\alpha(\mathcal{D}_L)$ are the numbers of positive (N^+) and negative (N^-) selfintersections of the component



It is easy to see that the numbers $w_\alpha(\mathcal{D}_L)$ do not depend on the orientation of the component α.

THEOREM 8.1. i) The functionals $Z_{j_1 \cdots j_K}(\mathcal{D}_L)$ and $\widetilde{Z}_{j_1 \cdots j_K}(\mathcal{D}_L)$ coincide,

ii) the functional

$$\varphi_{j_1 \cdots j_K}(\mathcal{D}_L) = \prod_{\alpha=1}^{K} \left(q^{2c_{j_\alpha}} (-1)^{2j_\alpha} \right)^{w_\alpha(\mathcal{D}_L)} Z_{j_1 \cdots j_K}(\mathcal{D}_L) \quad (8.1)$$

is the invariant of the link L.

PROOF. i) Consider the functional $\widetilde{Z}_{j_1 \cdots j_K}(\mathcal{D}_L)$ as the product of the matrices corresponding to the elementary fragments and the matrix $I^j \widetilde{Z}_{j_1 \cdots j_K}(\mathcal{D}_L)$ acting in V^j (I^j is a unit matrix). In accordance with the graphical rules this matrix is represented by the diagram $\mathcal{D}_L$ situated in the upper side of the wave line which correspond to the space V^j. Using the rules of section 6 we can transpose (see Fig.3) this diagram to the shadow world. From the rules (6.4) - (6.7) and (6.14), (6.15) we obtain a new representation for calculating the functional $\widetilde{Z}_{j_1 \cdots j_K}(\mathcal{D}_L)$. From the definition of this procedure it follows that the value of $\widetilde{Z}_{j_1 \cdots j_K}(\mathcal{D}_L)$

calculated in the shadow world do not depend on j, and therefore we can put $j = 0$ after that we obtain the rules for calculating $Z_{j_1 \cdots j_k}(D_L)$.

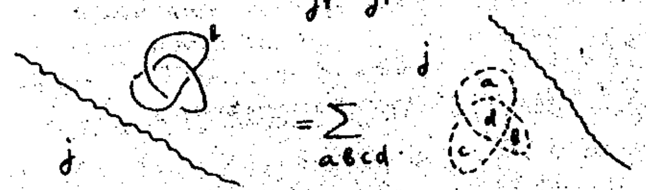


Fig. 3.

The part ii) of the theorem was proved in [2] (see also [18,29]). The main idea is that the invariance of $\tilde{Z}(D_L)$ under regular isotopies follows from the relations (4.6), (4.10) and the invariance under singular isotopies follows from (4.12) and from the structure of the multiplier before in (8.1).

Let us give a representation for the invariant φ in the case when L is given in the closed braid form. Consider the case $j_1 = j_2 = \cdots = j_k = j$.

We should remind that the braid group B_N [35] is generated vy the elements s_i, $i = 1, \ldots, N-1$ with the following relations

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

$$s_i s_j = s_j s_i \qquad |i-j| > 1$$

One can define a natural representation π of the braid group in the algebra

$$\pi(s_i) = g_i = 1 \otimes \cdots \otimes R^{jj}_{i,i+1} \otimes \cdots \otimes 1 \qquad (8.3)$$

The irreducible components of this representation

$$\sigma_\ell(s_i) = \rho_\ell(g_i) \qquad (8.4)$$

act in the spaces W^j_ℓ.

The matrix elements of g_i in the representations (8.4) are given by (7.10). Some of the representations (8.4) were obtained from SOS model of statistical physics in [33].

THEOREM 8.2. Let L be the closure of the braid α, $L = \hat{\alpha}$. In this case

$$\tilde{Z}_j(\alpha) = tr(\pi(\alpha))$$

where $\pi(\alpha)$ is the representation (8.3) and tr is the trace (7.11).

This theorem follows from the definition of $\tilde{Z}_{j_1 \cdots j_k}(D_L)$ and from special form of the link obtained as the closure of the braid α.

APPENDIX A

The calculation of $q-6j$ -symbols

We use here the version of the Racah method [23] for calculating $q-6j$ -symbols. From (5.6) we obtain the following expression for $q-6j$ -symbol in the Racah-Wigner normalization:

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}^{RW}_q = \begin{bmatrix} d & e & c \\ m_2 & m_4 & m_3 \end{bmatrix}^{-1}_q \sum_{m_4 m_5 m_6} \begin{bmatrix} b & a & e \\ m_5 & m_4 & m_1 \end{bmatrix}_q$$

$$\begin{bmatrix} d & b & f \\ m_2 & m_5 & m_6 \end{bmatrix}_q \begin{bmatrix} f & a & c \\ m_6 & m_4 & m_3 \end{bmatrix}_q \left([2e+1][2f+1]\right)^{-1/2} (-1)^{a+b-c-d-2e} \qquad (A.1)$$

Let us put $m_1 = e$, $m_3 = c$, $m_4 = \alpha$, then we have $m_2 = c - e$, $m_5 = e - \alpha$, $m_6 = c - \alpha$. Using the following formula for special values of the arguments in CGC

$$\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & j \end{bmatrix}_q = (-1)^{j_1 - m_1} \left\{ \frac{[j_1+m_1]!\,[j_2+m_2]!\,[2j+1]!}{[j_1-m_1]!\,[j_2-m_2]!\,[j_1+j-j_2]!} \right. \quad (A.2)$$

$$\left. \frac{[j_1+j_2-j]!}{[j_2+j-j_1]!\,[j_1+j_2+j+1]!} \right\}^{1/2} q^{j_2(j_2+1)-j_1(j_1+1)+(j+1)(m_1-m_2)}$$

and the q -analog of Van der Waerden representation (3.5) we obtain:

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q^{RW} = (-1)^{a+c+f} \Delta(abe)\Delta(acf)\Delta(ced)\Delta(dbf)$$

$$\frac{[c+d+e+1]!}{[a-b+e]!\,[b-a+e]!\,[f+c-a]!\,[a+c-f]!}\, q^{-\frac{1}{2}(a(a+1)+d(d+1)-f(f+1))}$$

$$\sum_{\alpha, z} q^{-2\alpha(e+c+d+2)-\frac{1}{2}(b(d-c)+e(b+c+d+1))} \frac{[a+\alpha]!\,[b+e-\alpha]!\,[f+c-\alpha]!}{[f+c-b-e+z]!\,[a-\alpha]!\,[f-d-e+\alpha+z]!} \quad (A.3)$$

$$(-1)^z q^{-2z(d+b+f+1)} \frac{1}{[z]!\,[d+b-f-z]!\,[d+e-c-z]!\,[b+e-\alpha-z]!}$$

To perform the double summation in (A.3) we use the Racah method based on the following lemma.

LEMMA.

$$\sum_c q^{\pm[2(a-b)c-2(c-b)]} \frac{(-1)^c}{[a-c]!\,[c-b]!} = (-1)^a \delta_{ab} \quad (A.4)$$

$$\sum_s q^{\pm 2as} \frac{1}{[s]!\,[b-s]!\,[c-s]!\,[a-b-c+s]!} = $$

$$= q^{\pm 2bc} \frac{[a]!}{[b]!\,[c]!\,[a-b]!\,[a-c]!} \quad (A.5)$$

$$\sum_s q^{\pm 2(a+b-c+2)s} \frac{[a-s]!\,[b+s]!}{[s]!\,[c-s]!} =$$

$$= q^{\pm 2(b+1)c} \frac{[a-c]!\,[b]!\,[a+b+1]!}{[c]!\,[a+b-c+1]!} \quad (A.6)$$

$$\sum_s (-1)^s q^{\pm 2(a-b-c+1)s} \frac{[a-s]!}{[s]!\,[b-s]!\,[c-s]!} =$$

$$= q^{\mp 2bc} \frac{[a-b]!\,[a-c]!}{[b]!\,[c]!\,[a-b+c]!}, \quad a \geq b, c \quad (A.7)$$

Using the identity (A.4) we can rewrite the sum over α in (A.3) in the following form

$$\sum_\alpha q^{-2\alpha(e+c+d+2)} \frac{[a+\alpha]!\,[b+e-\alpha]!\,[f+c-\alpha]!}{[a-\alpha]!\,[b+e-\alpha-z]!\,[f-d-e+\alpha+z]!} =$$

$$= \sum_{s,t,\alpha} (-1)^{s-t} q^{-2\alpha(e+c+d+2)+2(t-\alpha)(a+b+e-c-f-z-1)-2s(t-\alpha)+2(s-\alpha)} \frac{[a+\alpha]!\,[b+e-\alpha]!\,[c+f-t]!}{[f-d-e+z+\alpha]!\,[s-\alpha]!\,[t-s]!\,[a-t]!\,[b+e-z-t]!} \quad (A.8)$$

In this expression we can perform the summation over using the identity (A.6)

$$\sum_\alpha \frac{[a+\alpha]!\,[b+e-\alpha]!}{[f-d-e+z+\alpha]!\,[s-\alpha]!}\, q^{2(a+b+d+2e-f-z-s+2)(s-\alpha)} =$$

$$= q^{2(b+e-s+1)(f-d-e+z-s)} \frac{[a+b+e+1]!\,[b+e-s]!\,[a+b+d+e-f-z]!}{[f-d-e+z+s]!\,[a+b+d+2e-f-z-s+1]!} \quad \text{(A.9)}$$

and the summation over t using the identity (A.7)

$$\sum_t (-1)^t \frac{[c+f-t]!}{[t-s]!\,[a-t]!\,[b+e-z-t]!} q^{-2(c+f-a-b-e+z+s+1)(t-s)} =$$

$$= q^{2(a-s)(b+e-z-s)} (-1)^s \frac{[c+f-a]!\,[c+f-b-e+z]!}{[a-s]!\,[b+e-z-s]!\,[c+f-a-e+z+s]!} \quad \text{(A.10)}$$

Substituting (A.8) in (A.3) and using the identities (A.9) and (A.10) for double summation over s and z we obtain the following expression

$$q^{2(b+e+1)(f-d-e)+2a(b+e)} [a+b+e+1]!\,[c+f-a]!$$

$$\sum_{s,z} (-1)^z q^{-2s(a+c+f+1)-2z(a+d+f-e)} [b+e-s]!\,[a+d+e-f-z]!$$

$$\{ [z]!\,[d+b-f-z]!\,[d+e-c-z]!\,[f-d-e+z+s]!\,[a+b+d+2e-f-z-s+1]!\,[a-s]!\,[b+e-z-s]!\,[c+f-a-b-e+z+s]! \}^{-1} \quad \text{(A.11)}$$

Let us now put $s = u - z$ and let us take the sum over z using (A.4)

$$\sum_z q^{2z(c-d+e+1)} \frac{(-1)^z [b+e-u+z]!\,[a+d+e-f-z]!}{[z]!\,[d+b-f-z]!\,[d+e-c-z]!\,[a-u+z]!} =$$

$$= \sum_{z,t,r} q^{2z(c-d+e+1)+2(t-1)r-2(r-z)-2(z-t)(a-b-d+c+1)} (-1)^{z+r+t} \quad \text{(A.12)}$$

$$\frac{[b+e+z-u]!\,[a+d+e-f-t]!}{[z]!\,[a-u+z]!\,[r-z]!\,[t-r]!\,[b+d-f-t]!\,[d+e-c-t]!}$$

The summation over t and z is performed with the help of the identity (A.7)

$$\sum_t (-1)^t \frac{[a+d+e-f-t]!\; q^{2(a-b-d+c+r+1)(t-r)}}{[t-r]!\,[b+d-f-t]!\,[d+e-c-t]!} =$$

$$= (-1)^r q^{-2(b+d-f-r)(d+e-c-r)} \frac{[a+e-b]!\,[a+c-f]!}{[b+d-f-r]!\,[d+e-c-r]!\,[a-b-d+c+r]!} \quad \text{(A.13)}$$

$$\sum_z (-1)^z \frac{[b+e+z-u]!}{[z]!\,[r-z]!\,[a-u+z]!} q^{-2(b+e-a-r+1)(r-z)} =$$

$$= (-1)^r q^{2r(a-u+r)} \frac{[b+e-u]!\,[b+e-a]!}{[r]!\,[a-u+r]!\,[b+e-a-r]!} \quad \text{(A.14)}$$

Therefore the summation over z and t in the right hand side of (A.12) yields the expression

$$q^{-2(b+d-f)(d+e-c)} [b+e-a]!\,[a+e-b]!\,[a+c-f]!$$

$$\sum_{z} (-1)^{z}\, q^{2z(a+b+d+2e-f-u+1)}$$

$$\frac{[b+e-u]!}{[z]!\,[a-u+z]!\,[b+e-a-z]!\,[b+d-f-z]!\,[d+e-c-z]!\,[a-}$$

$$\frac{1}{-b-d+c+z]!}$$

So, after the summation over Z and u in (A.11) we get the following expression ($Z + s = u$)

$$q^{-2(b+d-f)(d+e-c)} [b+e-a]!\,[a+e-b]!\,[a+c-f]!$$

$$\sum_{z} (-1)^{z}\, q^{2z(a+b+d+2e-f-u+1)} \quad [b+e-u]! \qquad \text{(A.15)}$$

$$\frac{1}{[z]!\,[a-u+z]!\,[b+e-a-z]!\,[b+d-f-z]!\,[d+e-c-z]!\,[a-b-d+c+z]!}$$

The summation over u can be performed using the identity (A.5)

$$\sum_{u} q^{2(a+c+f+z+1)(a+z-u)} \frac{1}{[a-u+z]!\,[f-d-e+u]!\,[a+b+d+2e-f-u+1]!} \qquad \text{(A.16)}$$

$$\frac{1}{[c+f-a-b-e+u]!} =$$

$$= q^{2(f-d-e+a+z)(c+f-b-e+z)} \frac{[a+c+f+z+1]!}{[f-d-e+a+z]!\,[c+f+z-b-e]!\,[c+e+d+1]!\,[a+b+e+1]!}$$

Substituting (A.16) in (A.15) we obtain the following expression for (A.3)

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q^{RW} = (-1)^{a+c+f} \Delta(abe)\,\Delta(acf)\,\Delta(ced)\,\Delta(dbf)$$

$$\sum_{z} (-1)^{z} \frac{[a+c+f+z+1]!}{[z]!\,[b+e-a-z]!\,[b+d-f-z]!} \qquad \text{(A.17)}$$

$$\frac{1}{[d+e-c-z]!\,[a-b-d+c+z]!\,[f-d-e+a+z]!\,[c+f-b-e+z]!}$$

Substituting $Z = a+c+f+z$ we obtain the expression

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q^{RW} = \Delta(abe)\,\Delta(acf)\,\Delta(ced)\,\Delta(dbf)$$

$$\sum_{Z} (-1)^{Z} \frac{[Z+1]!}{[Z-a-b-e]!\,[Z-a-c-f]!\,[Z-c-d-e]!} \qquad \text{(A.18)}$$

$$\frac{1}{[Z-b-d-f]!\,[a+b+c+d-Z]!\,[a+d+e+f-Z]!\,[b+c+e+f-Z]!}$$

This formula is a q-analog of Racah representation for $q-6j$ -symbol. Similarly, one can found a q-analog of other representations (see, for example [30], p.249-250).

Let us give now some special values of $q-6j$ -symbols.

$$\begin{Bmatrix} a & b & a+b \\ d & c & f \end{Bmatrix}^{RW}_q = (-1)^{a+b+c+d} \left\{ \frac{[2a]!\,[2b]!}{[2a+2b+1]!\,[c+d-a-b]!} \cdot \frac{[a+b+c+d+1]!\,[a+b+c-d]!\,[a+b+d-c]!\,[c+f-a]!\,[d+f-b]!}{[a+c-b]!\,[b+d+f+1]!\,[a+c+f+1]!\,[b+d-f]!\,[b+f-d]!} \right\}^{1/2} \quad (A.19)$$

$$\begin{Bmatrix} a & b & e \\ d & c & 0 \end{Bmatrix}^{RW}_q = (-1)^{a+b+c} \frac{\delta_{ac}\,\delta_{bd}\,\delta(abe)}{([2a+1][2b+1])^{1/2}} \quad (A.20)$$

From Elliot-Bidenharn identity and from formulae (A.22) we obtain the following recursive relations for $q-6j$-symbols:

$$[2c+1][2d][2f+1]\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}^{RW}_q = \{[b+d-f][b+f-d+1]$$
$$[d+e-c][c+e-d+1][c+f-a+1][a+c+f+2]\}^{1/2} \begin{Bmatrix} a & b & e \\ d-\frac12 & c+\frac12 & f+\frac12 \end{Bmatrix}_q^{RW}$$
$$-\{[b+d-f][b+f-d+1][c+d-e][c+d+e+1][a+c-f][a+f-c+1]\}^{1/2}$$
$$\begin{Bmatrix} a & b & e \\ d-\frac12 & c-\frac12 & f+\frac12 \end{Bmatrix}^{RW}_q + \{[d+f-b][b+d+f+1][c+d-e][c+d+e$$
$$+1][c+f-a][a+c+f+1]\}^{1/2} \begin{Bmatrix} a & b & e \\ d-\frac12 & c-\frac12 & f-\frac12 \end{Bmatrix}^{RW}_q + \quad (A.21)$$
$$+\{[d+f-b][b+d+f+1][d+e-c][c+e-d+1][a+f-c][a+$$
$$+c+f+1]\}^{1/2},$$

$$\begin{Bmatrix} a & b & e \\ \frac12 & e+\frac12 & b+\frac12 \end{Bmatrix}^{RW}_q = (-1)^{a+b+e+1} \left(\frac{[a+b+e+2][b+e-a+1]}{[2e+2][2b+1]} \right)^{1/2} \quad (A.22)$$

$$\begin{Bmatrix} a & b & e \\ \frac12 & e+\frac12 & b-\frac12 \end{Bmatrix}^{RW}_q = (-1)^{a+b+e} \left(\frac{[a+b-e][a+e-b+1]}{[2e+2][2b+1]} \right)^{1/2}, \quad \begin{Bmatrix} a & b & e \\ \frac12, & e-\frac12, & b+\frac12 \end{Bmatrix}_q =$$
$$=(-1)^{a+b+e} \left(\frac{[a+e-b][a+b-e+1]}{[2e][2b+1]} \right)^{1/2}, \quad \begin{Bmatrix} a & b & e \\ \frac12 & e-\frac12 & b-\frac12 \end{Bmatrix}^{RW}_q = (-1)^{a+b+e} \left(\frac{[a+b+e+1][e+b-a]}{[2e][2b+1]} \right)^{1/2}$$

CONCLUSION

In the present work we considered only the case of the algebra $\mathfrak{g} = sl(2)$. $q-6j$ symbols in the general case may be defined in a similar way. The main idea can be extracted from [2]. The investigation of $q-6j$-analogous of CGC for $U_q(sl(2))$ will be given in a separate publication.

Here we don't consider the dual algebra $U_q(sl(2))^* \equiv \mathbb{C}_q(SL(2))$. The algebra $\mathbb{C}_q(SL(2))$ appeared in different contexts in [9, 25]. The algebra $\mathbb{C}_q(SU(2))$ appeared in the theory C^* algebras in the works of Woronovich [26] who constructed some elements of harmonic analysis for this algebra. The corepresentations of $\mathbb{C}_q(SU(2))$ and q-analogues of spherical functions on $SU(2)$ were studied in [27]. Let us mention also the work "Algebra of regular functions on quantum group $SU(2)$" by Soybelman and Vaksman of two years old, part of which is being published now in "Funct. Analysis and Its Applications". This work contains $*$-representations of $\mathbb{C}_q(SU(2))$, a definition of the invariant integral on $SU(2)$, a definition of the Weyl element as a CNS state on $\mathbb{C}_q(SU(2))$ and q-analogs od spherical functions on $SU(2)$. Using these results one can obtain the representation for CGC by integrating over $\mathbb{C}_q(SU(2))$ the product of three quantum spherical functions. A similar representation can also be given for $q-6j$-symbols.

As in the case of $q = 1$ the relations between q-$6j$-symbols can be organized in the Wigner-Racah algebra [23].

Notice that for $q^N = 1$ objects similar to q-$6j$-symbols and the representation (8.4) of the braid group were considered by Kanie and Tsuchiya [32] concerning the correlation functions in conformal field theory.

We thank L.Faddeev, V.Bazhanov, M.Semenov-Tian-Shansky, J.Soybelman, L.Takhtajan and L.Vaksman for interesting discussions.

References

1. R.Askey, J.Wilson, SIAM J.Math.Anal., v.10, N 5, 1979.
2. N.Yu.Reshetikhin, Quantized universal enveloping algebras, the Yang-Baxter equation and invariants of links. I. II. LOMI-preprints, E-4-87, E-17-87, Leningrad, 1988.
3. H.N.V.Temperly, E.Lieb, Prob.Roy.Soc. (London), (1971), 252-280.
4. V.Jones, Invent.Math., 72 (1983), 1-25.
5. L.D.Faddeev, Integrable models in 1+1-Dimensional Quantum Field Theory.In Les Houches Lectures 1982, Elsevier, Amsterdam, 1984.
6. P.P.Kulish, E.K.Sklyanin, Lecture Notes in Physics, 1982, v.151, p.61.
7. П.П.Кулиш, Н.Ю.Решетихин, Зап.Науч.Семин.ЛОМИ, 1980, т.101, с.112.
8. Е.К.Склянин, Усп.Мат.Наук, 1985, 40, № 2, р.214.
9. В.Г.Дринфельд. ДАН СССР, 1985, т.283, № 5, с.1060-1064. В.Г.Дринфельд, Доклад на ИМУ-86, Беркли 1986.
10. M.Jimbo, Lett.Math.Phys., 1985, 10, 63-69.
11. L.D.Faddeev, N.Yu.Reshetikhin, L.A.Takhtajan, LOMI - preprint, E- -87, 1987.
12. N.Yu.Reshetikhin, M.Semenov-Tian-Shansky. Quantum R-matrices and factorization problem in quantum groups. Imperial College-preprint, April 1988.
13. G.Lustig, Quantum deformations of certain simple modules over enveloping algebra n.I.T.preprint, December 1987.
14. M.Rosso. C.R.Acad.Sci.Paris t.305, Série I, p.587-590, 1987.
15. Л.Ваксман, Функц.анализ и его прил. 1988.
16. А.Н.Кириллов. Зап.Науч.Семин.ЛОМИ, т.170, 1988.
17. J.Murakami, The parallel version of polinomial invariants of links. Osaka University preprint 1988.
18. Y.Akutsu, M.Wadati, J.Phys.Soc.Jpn. 56 (1987) 839; 56 (1987) 3039; 59 (1987) 3464;
19. E.Date, M.Jimbo, T.Miwa, M.Okado, Solvable lattice models.

RIMS - 590 - preprint, 1987.
20. M.Jimbo, T.Miwa, M.Okado, An $A_n^{(1)}$ family of solvable lattice models. RIMS - 579 - preprint, 1987.
21. E.Abe, Hopf algebras. Cambridge tracts in mathematics 74. Cambridge University Press, 1980.
22. H.Wenzl, Representations of Hecke algebras and subfactors. Ph.D.Thesis. University of Pennsilvania (1985).
23. L.C.Biedenharn, J.D.Louck, Angular momentum in Quantum Physics, Encidopedia of Math.and Appl. volume 8, Addison-Wesley Publ.Comp.1981.
24. Д.А.Варшалович, А.Н.Москалёв, В.К.Херсонский, Квантовая теория углового момента. - Ленинград, Наука, 1975.
25. L.Faddeev, L.Takhtajan, Lect.Notes Phys. 246 (1986), 166-179.
26. S.Woronowich Publ.RIMS, Kyoto Univ., 23, 1987, 117-181.
27. T.Masuda, K.Mimachi, Y.Nakagami, M.Noumi, K.Ueno. Representations of quantum groups and -analoge of orthogonal polynomials, RIMS 613, preprint, 1988.
28. P.Podles, Lett.Math.Phys., 304 (1987)323-326.
29. V.Jones. On knot invariants related to some statistical mechanical models. California University preprint. Berkley 1988.
30. А.П.Юцис, А.А.Банзайтис, Теория момента количества движения в квантовой механике. Вильнюс, Минтис, 1965.
31. V.Pasquier. Etiology of IRF models. Sacley-preprint 1988.
32. Y.Kanie, A.Tsuchiya, Advanced Studies in Pure Mathematics. Vol.16, p.297-372, Kinokuniay, Tokyo Japan.
33. T.Deguchi, M.Wadati, Y.Akutsu. Link Polinomials constructed from Solvable Models in Statistical Mechanics. Tokio University preprint, 1988.
34. Freyd P., Yetter D., Lickorish W.B.R., Millett K., Ocneanu A., Hoste J., Bull.Math.Soc., 1985, v.12, N 2, p.239--246.
35. Turaev V.G. The Yang-Baxter equation and invariants of links. - LOMI-preprint E-3-87, 1987.

РТП ЛИЯФ, зак.705, тир.220, уч-изд.л.2,6; I5/07-88г.
Бесплатно