QUANTIZED UNIVERSAL ENVELOPING ALGEBRAS,
THE YANG-BAXTER EQUATION AND INVARIANTS OF LINKS, II

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8. The solutions of the Yang-Baxter equation, representations of a braid group and invariants of links

Now, we will show how to construct representations of the braid group and link invariants which corresponds to the $R$-matrices described above.

A detailed definition of the braid group is given in [9]. Some elementary facts are given in Appendix B.

There is a natural homomorphism of the braid group $B_N$ onto the symmetric group $S_N$: $s_i \rightarrow s_i$, where $s_i$ are the generators of $B_N$ and $s_i^N$ are the generators (elementary transpositions) of $S_N$. Let us fix the Schur subgroup $S_N^{k_1, \ldots, k_t} = S_{k_1} \times \ldots \times S_{k_t}$ in $S_N$ $(k_1 + \ldots + k_t = N)$. A subgroup $B_N^{k_1, \ldots, k_t} \subset B_N$ corresponding to $S_N^{k_1, \ldots, k_t}$ is called a partial coloured braid group.

Let $(\omega_1, \ldots, \omega_N)$ be an ordered set of symbols: $\omega_i = 1, \ldots, a_{k_i}, a_{k_i+1}, \ldots, a_N = \lambda_i, \ldots, a_{k_i} = \lambda_{k_i-1}, \ldots, a_1$. The set $(\omega_1, \ldots, \omega_N)$ is invariant under the action of $S_N$.

Let us consider the spaces $\mathcal{V}(\omega_1, \ldots, \omega_N) = \mathcal{V}(\omega_1) \otimes \ldots \otimes \mathcal{V}(\omega_N)$, where $(\omega_1, \ldots, \omega_N)$ is some permutation of the set $(\omega_1, \ldots, \omega_N)$ and $\mathcal{V}(\omega_i) = \mathcal{V}^{\lambda_{k_i}}$ if $\omega_i = a_i$ and $\mathcal{V}(\omega_i, \ldots, \omega_N) = \mathcal{V}(\omega_1) \otimes \ldots \otimes \mathcal{V}(\omega_N)$.

Let us assume that the matrices $M^{(\ell, c)}: \mathcal{V}(\ell) \rightarrow \mathcal{V}(\ell)$ and $R^{(\ell, c)}: \mathcal{V}(\ell) \otimes \mathcal{V}(c) \rightarrow \mathcal{V}(\ell) \otimes \mathcal{V}(c)$ satisfy the relations:

\[(R^{(a,b)} \otimes 1)(1 \otimes R^{(c,b)}) = (1 \otimes R^{(c,a)})(R^{(a,b)} \otimes 1)(1 \otimes R^{(a,b)}) \quad (8.1)\]

\[(M^{(c)} \otimes M^{(a)})R^{(a,b)} = R^{(c,b)}(M^{(a)} \otimes M^{(c)}) \quad (8.2)\]

\[\frac{t a_i^{-1}}{t a_i}((1 \otimes M^{(a)}) (R^{(a,a)} \pm 1)) = a_i^{\pm 1} \mathcal{I} \quad (8.3)\]
Here the trace in (8.3) is taken over the second multiplier in $\nu^{(a)} \otimes \nu^{(a')}$.

Let us define the matrices $R_i(b): \mathcal{K}(\cdots \otimes \nu_{k_i}^{(a_i)}) \rightarrow \mathcal{K}(\cdots \otimes \nu_{k_i}^{(a_i')})$.

\[
R_i(b) = (I \otimes \cdots \otimes R_i(\nu_{k_i}^{(a_i)}) \otimes \cdots \otimes I) \tag{8.4}
\]

where $b = (b_1, \ldots, b_N)$.

**Theorem 8.1.** If $a \in B_{N_1}^{k_1} \cdots k_c$ and $a = S_{i_1}^{k_1} \cdots S_{i_c}^{k_c}$ is a decomposition of $a$ in a product of generators of $B_N$ then the map $\pi$

\[
\pi(a) = R_{i_1}^{k_1}(a_{i_1}^{(1)} \cdots a_{i_1}^{(1)}) \cdots R_{i_c}^{k_c}(a_{i_c}^{(1)} \cdots a_{i_c}^{(1)}) \tag{8.5}
\]

where $a_{i_j}^{(1)} = (a_{i_j}^{(1)}, \ldots, a_{i_j}^{(1)})$, $\sigma \in S_{i_j}$ define a representation of $B_{N_1}^{k_1} \cdots k_c$ in $\text{Mat}(\nu^{(a_1)} \otimes \cdots \otimes \nu^{(a_n)})$.

To prove this theorem it is sufficient to use the relation (8.2).

Let us define the character of this representation by setting:

\[
\chi^{k_1 \cdots k_c}_{\lambda_1 \cdots \lambda_c}(a) = \prod_{j=1}^{c} \chi^{k_j}_{\lambda_j}(a_j) \tag{8.6}
\]

where $\chi^{k_j}_{\lambda_j}(a_j) = N_{\lambda_j}^{(\pm)}(a_j) - N_{\lambda_j}^{(\pm)}(a_j)$ and $N_{\lambda_j}^{(\pm)}(a_j)$ is the number of the generators $s_{i_j}^{(\pm)} \in B_{k_j}$ in the representation of the element $a_j$ through the generators of $B_{k_j}$.

Using the relation (8.1)-(8.3) the following theorem is easily proved.

**Theorem 8.2.** The character (8.6) satisfies the relations:

\[
\chi^{k_1 \cdots k_c}_{\lambda_1 \cdots \lambda_c}(a) = \chi^{k_1 \cdots k_c}_{\lambda_1 \cdots \lambda_c}(a), \quad a \in B_{N_1}^{k_1} \cdots k_c \tag{8.7}
\]
\begin{align}
X^{k_1 \ldots k_l}_{\lambda_1 \ldots \lambda_l} (\beta \alpha \beta^{-1}) &= X^{k_1 \ldots k_l}_{\lambda_1 \ldots \lambda_l} (\alpha), \quad \alpha, \beta \in B^{k_1 \ldots k_l}_N, \tag{8.8} \\
X^{k_1 \ldots k_l}_{\lambda_1 \ldots \lambda_l} (\alpha S_N^1) &= X^{k_1 \ldots k_l}_{\lambda_1 \ldots \lambda_l} (\alpha), \quad \alpha \in B^{k_1 \ldots k_l}_N \tag{8.9} \\
X^{k_1 \ldots k_l}_{\lambda_1 \ldots \lambda_l} (\beta \alpha \beta^{-1}) &= \tilde{X} (\alpha), \quad \beta \alpha \beta^{-1} \in B^{k_1 \ldots k_l}_N, \beta \in B_N \tag{8.10} 
\end{align}

where \( X_{\lambda} = \text{g}^{\lambda} (M^L) \) and \( \tilde{X} \) is the character (8.6) 
with \( \alpha \mapsto \sigma \alpha \), where \( \sigma \) is the element of \( S_N \) corresponding to \( \beta \in B_N \).

**Corollary 8.2.1.** The character \( X^{k_1 \ldots k_l}_{\lambda_1 \ldots \lambda_l} (\alpha) \) is an invariant of the link \( L = A \), where \( A \) is the closure of the braid \( \alpha \in B^{k_1 \ldots k_l}_N \).

Let us specify the general construction described above for the \( R \)-matrices connected with the algebras \( U_q (g) \). The symbols \( \lambda_i \) are now the highest weights of the finite-dimensional irreducible representations of \( U_q (g) \).

From the results of sections 1-3 we obtain:

**Proposition 8.1.** Let \( V^{\lambda_1}, \ldots, V^{\lambda_l} \) be the ICR of \( U_q (g) \) and \( R^{\lambda_i \lambda_j} \) be the corresponding \( R \)-matrices. Then the matrix \( M^\lambda \) satisfying (8.2), (8.3) exists and is given by

\[
M^\lambda = q^{-\rho} |_{V^\lambda}, \quad \rho = q^{-\frac{c(\lambda)}{2}} \tag{8.11}
\]

where \( \rho = \frac{1}{2} \sum_{\alpha \in \Delta^{+}} \mathcal{H} \alpha \), and the character (8.6) define the invariant of links.

**Remark 8.1.** Let us consider the centralizer of \( U_q (g) \) in the tensor product of \( N \) vectorial representations of \( U_q (g) \) and define the functional
\[ \mathrm{Tr}(\lambda) = \left( \frac{1}{X_{\lambda}} \right)^N \mathrm{Tr}((M \otimes \cdots \otimes M)x), \quad x \in C_N^\omega(q). \]

This can be extended on \( C_N^\omega(q) = \bigcup_{n \geq 1} C_n^\omega(q) \). It has the following properties:

1. \( \mathrm{Tr}(1) = 1 \)
2. \( \mathrm{Tr}(ab) = \mathrm{Tr}(ba) \)
3. \( \mathrm{Tr}(a g^{\pm \frac{1}{2}}) = \mathbb{Z}^{\pm \frac{1}{2}} \mathrm{Tr}(a), \quad a \in C_N^\omega(q) \subseteq C_\infty^\omega(q) \)

where
\[ \mathbb{Z} = \frac{q^{1/2}}{X_{\lambda}}. \]

and \( q^\frac{1}{2} \) are the generator of \( \mathbb{C}_N^\omega(q) \) (3.2).

So, we see that \( \mathrm{Tr}(\lambda) \) is the Connes trace on Hecke algebra for \( q^\frac{1}{2} = s(\lambda) \). This functional can be decomposed into the sum of characters of irreducible representations of \( \mathbb{C}_N^\omega(q) \). The latter are:

\[ \chi^\lambda(x) = \sum_a \pi^\lambda_{a, \lambda}(x) = \frac{\chi^\lambda(q)}{X_{\lambda}(q)} \sum_a \mathrm{Tr}(\mathcal{P}_{\lambda}(a)x), \]

where \( \pi^\lambda_{a, \lambda}(x) \) are the matrix elements of the representation (3.7), (3.10) acting in \( W_\lambda \) and \( \mathcal{P}_\lambda^\lambda(a) \) are the projectors (3.19). Using the decomposition (3.3) we obtain the following expression for \( \mathrm{Tr}(\lambda) \):
\[ T_\lambda(x) = \sum_\lambda \mathcal{W}_\lambda(q_1, n) X^{(\lambda)}(x) \]

\[ \mathcal{W}_\lambda(q_1, n) = \frac{X_\lambda(q_1)}{(X_\lambda(q_1))^{1/\lambda}} \]

For \( q_1 = \mathfrak{sl}(n) \), this is known "Fourier transform" formula for Ocneanu trace [2]:

\[ \mathcal{W}_\lambda(q_1, n) = \left( \frac{q^{\frac{1}{n}} - q^{-\frac{1}{n}}}{q^{\frac{1}{n}} - q^{-\frac{1}{n}}} \right)^{|\lambda|} \prod_{(i,j) \in \lambda} \frac{q^{\frac{j-i+n}{n}} - q^{-\frac{j-i+n}{n}}}{q^{\frac{j+i+n}{n}} - q^{-\frac{j+i+n}{n}}} \]

Here \(|\lambda|\) is the number of cells in Young's diagram \( \lambda \), the index \( i \) and \( j \) numerate rows and columns (from top to bottom and from left to right) respectively, \( h_{ij} \) is the length of the hook with coordinates \((i,j)\) in \( \lambda \).

**Remark 2.** Let us note that the links \( \lambda = \hat{\alpha}_1 \), \( \kappa = \mathcal{B}^0 \), \( \cdots \), \( \mathcal{B}^N \) have \( \ell \) or more components and if \( \lambda_1 \neq \cdots \neq \lambda_\ell \), the functional (8.6) gives the invariant of links depending on \( \ell \) integer and one continuous parameter.

To study the invariants (8.6) it is useful to present them as functionals on the diagrams of links. To do so let us give some definitions.

**Definition 8.1.** The elements

\[ \begin{array}{c}
\bowtie \\
\cup \\
\times \\
\times \\
\end{array} \]

of the diagram \( \mathcal{D}_\ell \) are called elementary fragments".

**Definition 8.2.** Let us define the state functional on diagrams of oriented links by the rules:

1) let the diagram be in general position

2) to each component of \( \lambda \) we associate the
weight of algebra $U_q(g)$; this h.w. is called the colour of the component; the diagram with the coloured components will be called coloured diagram.

iii) divide the diagram into elementary fragments

iv) to each edge connecting elementary fragments we associate the states $i$, $i \in \Lambda^\vee$ if the edge is oriented upwards and $i \in \Lambda$ if the edge is oriented downwards; here $\Lambda$ is the colour of the component

v) to each elementary fragment we associate a matrix using the rules (2.4)-(2.6) of the previous section.

vi) fix the states on the edges of the diagram $\mathcal{G}_L$; multiply the matrix elements between these states corresponding to elementary fragments; taking the sum of resulting product over all states on $\mathcal{G}_L$ we obtain the state functional on the diagram of coloured, oriented link; this sum is denoted by $Z_{\lambda_1, \ldots, \lambda_t}(\mathcal{G}_L)$ and is called the state functional.

PROPOSITION 6.3. Let $(S'_1, \ldots, S'_t)$ denote the orientation of the link $L$ with $t$ components, then

$$Z_{\lambda_1, \ldots, \lambda_t}(\mathcal{G}_L)=Z_{\lambda_1^*, \ldots, \lambda_t^*}(\mathcal{G}_L(S'_1, \ldots, S'_t)).$$

COROLLARY. The state functional does not depend on the orientation of the component $f$ if $\lambda_t^*=\lambda_t$.

Let $N^+_f(\mathcal{G}_L)$ are the numbers of positive and negative vertices where two strings of the component cross:

$$\begin{array}{c}
\times \rightarrow (-) \\
\times \rightarrow (+)
\end{array}$$

and

$$\omega_f(\mathcal{G}_L) = N^+_f(\mathcal{G}_L) - N^-_f(\mathcal{G}_L).$$
THEOREM 8.3. The functional

$$\varphi_{\lambda_1, \ldots, \lambda_\ell}(L) = \prod_l |Q_l^{\omega_l}(\theta_l)| \prod |\lambda_l| \varphi_{\lambda_1, \ldots, \lambda_\ell}(\theta_l) \quad (8.12)$$

is the invariant of oriented links.

PROOF. From the definition of $Z_{\{\lambda\}}(\theta_L)$ it follows that it is invariant under the regular isotopies. Using the relations (1.49) we derive that smoothing of a loop results in the following transformation of

$$Z(\lambda_1^\prime \cdots \lambda_\ell^\prime) = Z(\lambda_1 \cdots \lambda_\ell) \frac{\varphi_{\lambda_1^\prime, \ldots, \lambda_\ell^\prime}}{\varphi_{\lambda_1, \ldots, \lambda_\ell}} \cdot (-1)^{\frac{\varphi_{\lambda_1^\prime, \ldots, \lambda_\ell^\prime}}{\varphi_{\lambda_1, \ldots, \lambda_\ell}}} [\lambda] \ .$$

The multiplier in (8.14) compensates this variation of $Z_{\lambda_1, \ldots, \lambda_\ell}(\theta_L)$. Hence the functional $\varphi$ is the invariant of links.

THEOREM 8.4. The invariants $\varphi$ and $\chi$ are equal:

$$\chi_{\lambda_1^\prime \cdots \lambda_\ell^\prime}(\alpha) = \varphi_{\lambda_1 \cdots \lambda_\ell}(-\alpha) \quad (8.13)$$

where we choose the standard orientation of $\alpha$.

To prove this theorem it is sufficient to use the equality

$$\left( c^{-1} \right)^T = (-1)\left[ \lambda \right] c^{-1} \varphi$$

and the definitions of $\varphi$ and $\chi$.

Let us describe the combinatorial procedure, which reduces the calculation of the state functionals $Z_{\lambda_1 \cdots \lambda_\ell}$ to the calculation of the basic state functionals $Z_{\omega_1 \cdots \omega_\ell}$. For this purpose we need the following definition:

DEFINITION 8.3. The link $L_{\alpha_1 \cdots \alpha_\ell}(\alpha_1 \cdots \alpha_\ell)$, built on the link $L$ in accordance to rules (a), (b) is called composite...
(a) each component of the link \( L \) is divided in \( N_L \) parallel strands (\( i=1, \ldots, \ell, \ell \) is the number of components of \( L \)).

(b) each set of \( N_i \) parallel strands is braided; the resulting closed braid is \( \hat{\omega}_i \in \mathbb{B}_{N_i} \).

THEOREM 6.5. If we fix a imbedding \( \alpha_i : Y^{\lambda_i} \subset (V \omega_i) \otimes N_i \) (the parametrization and explicit description of such an imbedding are given in section 3 \((3.11), (3.16), (3.19)\))

\[ Z_{\lambda_i} \ldots \lambda_1 (\hat{\Omega}_L) = \sum_{\alpha(m_i)} Q(a_i, \alpha(m_i)) \ldots Q(a_{\ell}, \alpha(m_{\ell})) Z_{\omega_{\ell} \ldots \omega_i} (\hat{\Omega}_{N_{\ell} \ldots N_i}, \alpha(m_{\ell}), \ldots, \alpha(m_i)) \] \( (8.14) \)

where the coefficients \( Q(a, \alpha(m)) \) are determined by \((3.19)\) for \( P(a) \)

\[ P(a) = \sum_{\alpha(m)} Q(a, \alpha(m)) R_{\alpha(m)} \]

b) the result of the calculation according to \((8.15)\) does not depend upon the choice of the imbeddings \( \alpha_i \).

PROOF. Using the orthogonality and crossing-symmetry of we have:

\[ Z_{\ldots \lambda_i \ldots (\lambda_i \lambda_i)^{\ell}} = Z_{\ldots \lambda_i \ldots} (\omega_i \omega_i) = \]

\[ = Z_{\omega_{\ell} \ldots \omega_i} = \sum_{\alpha(m_i)} Q(a_i, \alpha(m_i)) Z_{\omega_{\ell} \ldots \omega_i} (\alpha(m_i)) \]

\[ \]
Applying such transformations to each component of $\mathcal{L}$ we obtain the formula (8.16). It is obvious that the answer does not depend on the choice of the imbeddings.

Remark 8.3. The state functional $Z_{\lambda_1, \ldots, \lambda_K}$ can be defined also for diagrams $\mathcal{G}_{\Gamma}$ of knotting graphs with triple vertices. For this purpose the following elements should be added into the list of elementary fragments in Definition 8.1:

\[
\begin{align*}
\begin{array}{c}
\lambda \\
\gamma \\
\mu \\
\end{array} & \quad \begin{array}{c}
\lambda \\
\gamma \\
\mu \\
\end{array} \\
\begin{array}{c}
\lambda \\
\gamma \\
\mu \\
\end{array} & \quad \begin{array}{c}
\lambda \\
\gamma \\
\mu \\
\end{array}
\end{align*}
\]
(8.16)

In definition 8.5 we now colour the edges the vertices of the graph $\Gamma$. According to rules of section 2 we associate with the vertices (8.16) the corresponding CGO matrices

\[
\begin{align*}
(K_{ij}^{K}(a_{kl}))_{ij} \\
(K_{ij}^{K}(a_{kl}))_{ij}
\end{align*}
\]

The following equations for state functionals $Z_{\lambda_1, \ldots, \lambda_K}(\mathcal{G}_{\Gamma})$ are the result of the relations between the R-matrices and CGO, described in section 2:

\[
\begin{align*}
Z\left(\begin{array}{c}
\lambda \\
\gamma \\
\mu \\
\end{array}\right) &= Z\left(\begin{array}{c}
\lambda \\
\gamma \\
\mu \\
\end{array}\right) \\
Z\left(\begin{array}{c}
\lambda \\
\gamma \\
\mu \\
\end{array}\right) &= Z\left(\begin{array}{c}
\lambda \\
\gamma \\
\mu \\
\end{array}\right)
\end{align*}
\]
(8.16)
9. The invariants of links connected with
the algebras $U_q(sl(n))$, $U_q(so(2n+1))$, $U_q(so(2n))$, $U_q(sp(2n))$.

The basic invariant in the case $q = sl(n)$ was considered
by V. Jones [3]. He showed that this invariant is equal to
FXYLMOH invariant [4] at special values of parameters. To make
the picture complete we add, here a recursive procedure for
calculating this invariant.

PROPOSITION 9.1. The functional $Z_{\omega_1\ldots\omega_t} (Q_L) =
\equiv Z_J (Q_L, \omega)$ for $q = sl(n)$ gives an invariant for
oriented links and is calculated from the following recursive
relations

\[ Z_J (X) = Z_J (X') = (q^{\frac{k}{2}} - q^{-\frac{k}{2}}) Z_J (X') \]  \hspace{1cm} (9.1)

\[ Z_J (\Xi) = q^{-\frac{k}{2}} Z_J (\gamma) \]  \hspace{1cm} (9.2)

\[ Z_J (\chi) = q^{\frac{k}{2}} Z_J (\gamma) \]  \hspace{1cm} (9.3)

\[ Z_J (\emptyset) = X_{\omega} = \frac{q^{\frac{k}{2}} - q^{-\frac{k}{2}}}{q^{\frac{k}{2}} - q^{-\frac{k}{2}}} \]  \hspace{1cm} (9.4)

PROOF. The functional $Z_J$ depends on the orientation of $L$
because $\omega_i \not= \omega_i$ for $U_q (sl(n))$. The relations
(9.1)-(9.4) follow from the properties of the matrix $P_{\omega_i \omega_j}$
given in section 4.

The link invariants corresponding to vectorial represen-
tations of $U_q (so(2n))$, $U_q (so(2n+1))$, $U_q (sp(2n))$
are equal to the Kauffman invariant [5] for special values of parameters
in the last one. This is Turaev's [6] result. The construction
given in [6] slightly differs from our and that is why we
give here the most important properties of corresponding state
functional.

**PROPOSITION 9.2.** The functionals \( Z_{\omega_i} \ldots \omega_i (\mathcal{A}_L) = Z_{\mathcal{K}}(\mathcal{A}_L, \omega_i) \) for \( \mathcal{Q} = \mathfrak{so}(\mathfrak{L}+1), \mathfrak{so}(2\mathfrak{L}), \mathfrak{sp}(2\mathfrak{L}) \) do not depend on orientation of \( L \) and satisfy the following relations

\[
Z_{\mathcal{K}}(\mathcal{X}) - Z_{\mathcal{K}}(\mathcal{Y}) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(Z_{\mathcal{K}}(\mathcal{X}) - Z_{\mathcal{K}}(\mathcal{Y})) \quad (9.5)
\]

\[
Z_{\mathcal{K}}(\mathcal{Q}) = (-1)^{[\omega_i]} q^{-\frac{N-1}{2}} Z_{\mathcal{K}}(\mathcal{Q}) \quad (9.6)
\]

\[
Z_{\mathcal{K}}(\mathcal{O}_i) = (-1)^{[\omega_i]} q^{-\frac{N-1}{2}} Z_{\mathcal{K}}(\mathcal{O}) \quad (9.7)
\]

\[
Z_{\mathcal{K}}(\mathcal{O}) = X_{\omega_i} \quad (9.8)
\]

Here \( N = 2\mathfrak{L} \) for \( \mathcal{Q} = \mathfrak{so}(2\mathfrak{L}), \mathfrak{sp}(2\mathfrak{L}) \) and \( N = 2\mathfrak{L} + 1 \) for \( \mathcal{Q} = \mathfrak{so}(2\mathfrak{L}+1), \mathfrak{so}(2\mathfrak{L}), \mathfrak{sp}(2\mathfrak{L}) \);

\[
X_{\omega_i} = (q^{\frac{1}{2}} + q^{-\frac{1}{2}})(q^{\frac{N-1}{4}} - q^{-\frac{N-1}{4}})/(q^{\frac{1}{2}} - q^{-\frac{1}{2}})
\]

for \( \mathcal{Q} = \mathfrak{so}(2\mathfrak{L}+1), \mathfrak{so}(2\mathfrak{L}), \mathfrak{sp}(2\mathfrak{L}) \), \( X_{\omega_i} = (q^{\frac{1}{2}} + q^{-\frac{1}{2}})(q^{\frac{N-1}{4}} - q^{-\frac{N-1}{4}})/(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \)

for \( \mathcal{Q} = \mathfrak{sp}(2\mathfrak{L}) \).

**PROOF.** The functional \( Z_{\mathcal{K}} \) does not depend on orientation of \( L \) because \( \omega_i^\mathcal{K} = \omega_i \) for \( \mathcal{Q} = \mathfrak{so}(2\mathfrak{L}+1), \mathfrak{so}(2\mathfrak{L}), \mathfrak{sp}(2\mathfrak{L}) \). The relations (9.5)-(9.8) follows from the properties of matrices \( R_{\omega_i, \omega_i} \) described in sections 5 and 6.

The relations (9.5)-(9.8) can be considered as a recursive procedure for calculating the values of state functional \( Z_{\mathcal{K}} \) [5]. Another simple combinatorial procedure for calculating \( Z_{\mathcal{K}} \) via the fact that \( U_{\mathcal{K}}(\mathfrak{so}(\mathfrak{L})) \subset U_{\mathcal{K}}(\mathcal{Q}) \) for \( \mathcal{Q} = \mathfrak{so}(2\mathfrak{L}+1), \mathfrak{so}(2\mathfrak{L}), \mathfrak{sp}(2\mathfrak{L}) \) as a Hopf subalgebra, or in another words the block structure (5.16)-(5.19), (6.9)-(6.10) of the matrices \( R_{\omega_i, \omega_i} \), \( (R_{\omega_i, \omega_i})^{-1} \).

**THEOREM 9.1.** The calculation of the values of the functional \( Z_{\mathcal{K}}(\mathcal{A}_L) \) corresponding to \( \mathcal{Q} = \mathfrak{sp}(2\mathfrak{L}), \mathfrak{so}(2\mathfrak{L}) \) is
reduced to the calculation of the functional $Z_J$ according to the following combinatorial procedure:

i) define some orientation on each edge of the diagram

ii) for each vertex with admissible orientation compare the weights:

$$W(\mathcal{X}) = W(\mathcal{X}) = W(\mathcal{X}) = W(\mathcal{X}) =$$

$$w(\mathcal{X}) = w(\mathcal{X}) = w(\mathcal{X}) = w(\mathcal{X}) = 1$$

$$w(\mathcal{U}) = \varepsilon q^{\frac{M}{2} \varepsilon}, \quad w(\mathcal{V}) = q^{\frac{1}{2} \varepsilon}, \quad w(\mathcal{Z}) = \varepsilon q^{\frac{1}{2} \varepsilon}, \quad w(\mathcal{W}) = q^{\frac{1}{2} \varepsilon},$$

$$\varepsilon = (-1)^{[\omega, \varepsilon]}.$$

iii) the vertices with non admissible orientation we reconstruct according to the following rules:

$$\mathcal{X} = W(\mathcal{X}) \mathcal{X} + W(\mathcal{X}) \mathcal{X},$$

$$\mathcal{X} = 0, \quad \mathcal{U} = 0, \quad \mathcal{V} = 0,$$

$$\mathcal{X} = W(\mathcal{X}) \mathcal{X} + W(\mathcal{X}) \mathcal{X},$$

$$\mathcal{X} = 0, \quad \mathcal{Z} = 0, \quad \mathcal{Z} = 0$$

where

$$W(\mathcal{X}) = -(q^{1/2} - q^{-1/2}), \quad W(\mathcal{X}) = (q^{1/2} - q^{-1/2}), \quad W(\mathcal{X}) =$$

$$= + (q^{1/2} - q^{-1/2}) \varepsilon q^{\frac{1}{2} \varepsilon}, \quad W(\mathcal{X}) = -(q^{1/2} - q^{-1/2}) \varepsilon q^{\frac{1}{2} \varepsilon}.$$

iv) with each reconstructed diagram $\mathcal{D}_\mu$ we associate the weight

$$W(\mathcal{D}_\mu, \mathcal{D}_\mu) = \prod_{\text{vertex}} W(\text{vertex}),$$

all vertices

v) calculating the functionals $Z_J(\mathcal{D}_\mu, \mu)$ and taking the sum over all orientations of edges we obtain the follo-
Wing representation for $Z_k(\mathcal{A}_L)$:

$$Z_k(\mathcal{A}_L) = \sum_{\mathcal{A}_{L'}} w(\mathcal{A}_L, \mathcal{A}_{L'}) Z_f(\mathcal{A}_{L'}). \tag{9.9}$$

**Proof.** The representation (9.9) we obtain by substituting the block structures (5.19), (5.19), (6.9), (6.10) of $Z_{\omega_1...\omega_k}(\mathcal{A}_L)$ in the definition of $Z_{\omega_1...\omega_k}(\mathcal{A}_L)$.

A similar combinatorial procedure for calculating of $Z_k$ for $g=so(2n+1)$ we obtain from the block structure (5.16), (5.17) of corresponding matrix $B_{\omega_1\omega_2}$.

The calculation of link invariants connected with higher tensor representations of classical algebras is reduced to the calculation of the basic invariants according to the theorem 8.5.

The calculation of links invariants corresponding to the two-valued representations of $U_q(so(4n))$ and $U_q(so(2n+1))$ can also be reduced to calculating the invariants corresponding to vector representation. Let us describe the procedure, reducing the calculation of the spinor invariant of links to calculating the vector invariant for some "reconstructed" links.

**Proposition 9.3.** The calculation of the state functional corresponding to spinor representation of $U_q(so(4n))$ and $U_q(so(2n+1))$ can be reduced to calculating the vector state functional by the following two-step procedure:

1) using the formulae (5.11), (5.31)-(5.39) we obtain the representation of the state functional $Z_{\omega_1...\omega_k}(\mathcal{A}_L)$ as a sum of state functionals on the diagrams of knotting graphs with nonlinked and nonknotting loops colored in spinor representations jointed by the braids with strings colored in vector representation. An example of such a graph is given below:
ii) using the relation (5.35) the "spinorial loops is calculated" and after this procedure we obtain the representation of the state functional $Z_\omega$, as a sum of state functionals $Z_{\omega_i}$ on some diagrams colored by $\omega_i$ with some additional weights (take $W(\mathcal{A}_u, \mathcal{A}_v)$ in (9.9)) which can be found from (5.1), (5.31)-(5.35).

10. The invariant of links corresponding to the algebra $G_2$.

Substituting an explicit expression for basic $U_q(G_2)$-R-matrix in the definition of the state functional $Z_{\omega_1 \cdots \omega_t}$, we obtain the basic $U_q(G_2)$-invariant of links. Most non-trivial problem is the calculation of the values of this functional on concrete links. Let us show how the calculation of the $U_q(G_2)$-invariant is reduced to the calculation of Jones invariant on "reconstructed" diagrams.

One can easily prove the following

PROPOSITION 10.1. There are the following identities connected CGC and R-matrices corresponding to the algebra $U_q(\mathfrak{sl}(2))$:

$$2 \quad 2 = \sqrt{\frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{q + q^{-1}}}$$

(10.1)
Moreover one can easily check that

\[
\omega_i \cup \omega_i = \begin{pmatrix}
\cup_i q^{\frac{g}{4}} \\
\cap_i q^{\frac{g}{4}} \\
\cup_i q^{-\frac{g}{4}}
\end{pmatrix}, \quad \omega_i \cap \omega_i = \begin{pmatrix}
\cap_i q^{\frac{g}{4}} \\
\cup_i q^{\frac{g}{4}} \\
\cap_i q^{-\frac{g}{4}}
\end{pmatrix},
\]

(10.3)

Here the l.h.s.'s are the elements of the graphical technique for \( U_q(G_2) \).

Let us rewrite the equality (7.10) in the following form:

\[
\begin{align*}
\delta_i^1 &+ \delta_i^2 + \delta_i^3 = \sum_{\ell = 0,1,2} a^{(\ell)}_i e_t \quad + \sum_{m = 0,1,2} b^{(m)}_i e_t
\end{align*}
\]

(10.4)

where

\[
\begin{align*}
\delta_i^0 &\equiv \delta_{i_1} \delta_{i_2} \delta_{i_3} \delta_{i_4}, \\
\delta_i^1 &\equiv \delta_{i_1} (e_{i_2} e_{i_1}) (e_{i_2}), \\
\delta_i^2 &\equiv \delta_{i_2} (e_{i_2} e_{i_3}) (e_{i_2}), \\
\delta_i^3 &\equiv \delta_{i_3} (e_{i_2} e_{i_3}) (e_{i_2}).
\end{align*}
\]

The numbers \( \ell^\varepsilon = (1^+, 1^-, 1^-) \) numerate the irreducible \( U_q(sl(2)) \) components of vector representation of \( U_q(G_2) \).
\[ |l_1 - l_2| \leq l \leq l_1 + l_2, \quad |l_3 - l_4| \leq l \leq l_3 + l_4, \quad |l_5 - l_6| \leq l_5 + m, \quad |m - l_6| \leq m, \quad l_5, m = 0, 1, 2 \]

and the coefficients \( a_{l}^{(1)} \) and \( a_{m}^{(2)} \) can be found from (7.10), (7.15), (7.17).

**Definition 10.1.** A plane graph \( \Gamma \) with edges numerated by the numbers \( l = 1, 2 \) is called a reconstruction of the diagram \( \mathcal{D}_L \) if it can be obtained from \( \mathcal{D}_L \) by the following combinatorial procedure:

1) the edges of \( \mathcal{D}_L \) are numerated by the numbers \( l = 1, 2 \);

2) each vertex is replaced on one of the following elements:

\[ \begin{array}{c}
\begin{array}{c}
\ell_1 \ell_2 \\
\ell_3 \ell_4 \\
\ell_5 \\
\ell_6
\end{array}
\end{array} \quad \text{or} \quad \begin{array}{c}
\begin{array}{c}
\ell_1 \\
\ell_2 \\
\ell_3 \\
\ell_4 \\
\ell_5
\end{array}
\end{array} \]

with some numbers \( l, m = 1, 2 \).

**Definition 10.2.** The link \( L \) we call a tangled reconstruction of \( \mathcal{D}_L \) if the diagram \( \mathcal{D}_L \) of the link \( L \) can be obtained from \( \Gamma \) by the following procedure:

(a) \[ \begin{array}{c}
\begin{array}{c}
\ell_1 \\
\ell_2
\end{array}
\end{array} \quad \text{or} \quad \begin{array}{c}
\begin{array}{c}
\ell_1 \\
\ell_2
\end{array}
\end{array} \]

(b) \[ \begin{array}{c}
\begin{array}{c}
\ell_1 \\
\ell_2
\end{array}
\end{array} \quad \text{or} \quad \begin{array}{c}
\begin{array}{c}
\ell_1 \\
\ell_2
\end{array}
\end{array} \]

**Proposition 10.2.** The state functional \( \mathcal{Z}_{\omega_1, \omega_2, \ldots, \omega_t}(\mathcal{D}_L) \) has the following representation:

\[ \mathcal{Z}_{\omega_1, \omega_2, \ldots, \omega_t}(\mathcal{D}_L) = \sum \mathcal{W}(\mathcal{D}_L, \mathcal{D}'_L) \mathcal{Z}_T(\mathcal{D}'_L, L) \quad (10.5) \]

all tangled reconstructions \( \mathcal{D}'_L \) of \( \mathcal{D}_L \).
where $\mathcal{Z}_j(\mathcal{H}_L, \nu)$ is the Jones state functional and the weights $\mathcal{W}(\mathcal{H}_L, \hat{\mathcal{H}}_L)$ are determined by the formulae (10.1)-(10.4).

**Conclusion**

I do not discuss here deep connections of all these invariants with the structure of algebras $\mathcal{U}_q(\mathfrak{g})$. It appears that for each algebra $\mathcal{U}_q(\mathfrak{g})$ there exists a universal invariant of links. It takes values in $\mathbb{Z}_q(\mathfrak{g})^\mathbf{k}$ where $\mathbf{k}$ is the number of components of links and $\mathbb{Z}_q(\mathfrak{g})$ is the center of the algebra $\mathcal{U}_q(\mathfrak{g})$. This invariant will be described in a separate publication.

The work focuses on the invariants of links. However, analogous objects can be easily constructed for the knotted graphs with vertices (2.6) (see the end of section 8). The study of these invariants will be given separately.

The action (3.7-9) of the elements in the representations $\mathcal{W}_\lambda$ of the algebra $\mathcal{C}_N^\omega(\mathfrak{g})$ give the key to understanding the sense of the so-called restricted solid on solid (RSOS) models of statistical mechanics [7] in the context of the representation theory. When $\mathfrak{g} = \exp\left(\frac{2\pi i}{n}\right)$ the algebras $\mathcal{C}_N^\omega(\mathfrak{g})$ are no longer the simple. However, even in this case they have $\ast$-subfactors (see [8, 9] for Hecke algebras). The RSOS models are described in terms of irreducible representations of these $\ast$-subfactors.

**References**


Appendix

Let us recall the definition of the braid group.

DEFINITION 1. The braid is a set of smooth lines $y_1, \ldots, y_n$ in $\mathbb{R}^2 \times I$, $I = [0,1]$ non-selfintersecting, not intersecting in pairs, and having nonzero tangent vectors $\dot{y}_j \neq 0$, transversal to layers $\mathbb{R}^2 \times t$, for all $0 < t < 1$, with bases in points $(P_1, 0), \ldots, (P_n, 0)$ and ending in points $(P_{\sigma_1}, 1), \ldots, (P_{\sigma_n}, 1)$ where $\sigma$ is a certain permutation of numbers $\{1, \ldots, N\}$.

An example of a braid is given on Figure a.
The isotopy classes of braids form the braid group $B_N$.

The unit element of $B_N$ is given on Figure b. The braid $\alpha \beta$ is the joint of the braids $\alpha$ and $\beta$, where the beginning of the braid $\beta$ is the end of the braid $\alpha$ (see Figure c).

**Figure a**

**Figure b**

**Figure c**

**PROPOSITION 1.** The braid group $B_N$ is formed by the generators $S_i$, $i=1,\ldots,N-1$, corresponding to the simple transpositions $S_i = (\cdots i i+1 \cdots)$. The generators $S_i$ satisfy the relations

$$S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1},$$

$$S_i S_j = S_j S_i, \quad |i-j| > 1.$$

The braids corresponding to the elements $S_i$ and $S_i^{-1}$ are given on the Figure d and e, respectively.

**Figure d**

**Figure e**

**DEFINITION 2.** The closed braid $\hat{\omega}$ is the diagram of
oriented links obtained from the braid \( \hat{\alpha} \) by jointing the end with the beginning \( f \) this braid by nonlinked and nonknotting strings (see Figure 1).

\begin{figure}[H]
\centering
\includegraphics[width=0.3\textwidth]{figure1}
\caption{Figure 1}
\end{figure}

**PROPOSITION 2.** The transformations of the braids \( \hat{\alpha} \) nonchanging the link corresponding to the diagram \( \hat{\alpha} \) are generated by Markov type I and Markov type II transformations:

\[ \begin{align*}
I : & \hat{\alpha} = \hat{\beta} \alpha \beta^{-1}, & \forall \alpha, \beta \in B_N,
\Pi : & \hat{\alpha} = \alpha \hat{s}_N^{\pm 1}, & \forall \alpha \in B_N, (\alpha \hat{s}_N^{\pm 1} \in B_{N+1}).
\end{align*} \]

**PROPOSITION 3.** Any link \( L \) can be transformed by regular isotopies to some closed braid.

**DEFINITION 3.** The invariant of links is the functional \( \Phi(L) \) (which does not change under isotopy transformations of \( L \)).

If \( \Phi(L) = \hat{\alpha} ; \alpha \in B_N \), the isotopy invariance of the functional defined on \( \hat{\alpha} \) means that this functional considered as a functional on braids satisfy the following relations

\[ \begin{align*}
\Phi_N(\alpha) &= \Phi_N(\beta \alpha \beta^{-1}), & \forall \alpha, \beta \in B_N,
\Phi_{N+1}(\alpha \hat{s}_N^{\pm 1}) &= \Phi_N(\alpha), & \forall \alpha \in B_N.
\end{align*} \]

Here \( \Phi_N(\alpha) = \Phi(\hat{\alpha}) \) and index \( N \) means that \( \alpha \in B_N \).