USSR ACADEMY OF SCIENCES
STEKLOV MATHEMATICAL INSTITUTE
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L O M I   P R E P R I N T S

E - 17 - 87

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QUANTIZED UNIVERSAL ENVELOPINT ALGEBRAS, THE YANG-BAXTER EQUATION AND INVARIANTS OF LINKS. II

Recommended for publication
by the Soientific Council of
Steklov Mathematical Institute,
Leningrad Department (LOMI)

6, June, 1987

Leningrad
1988

This preprint is a continuation of LOMI-preprint E-4-87:

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Quantized Universal Enveloping Algebras, the Yang-Baxter Equation and Invariants of Links. I.

Bibliographical reference:
N.Yu.Reshetikhin, LOMI, preprint E-17-87, Leningrad 1988

8. The solutions of the Yang-Baxter equation, representations of a braid group and invariants of links

Now, we will show how to construct representations of the braid group and link invariants which corresponds to the R-matrices described above.

A detailed definition of the braid group is given in [1]. Some elementary facts are given in Appendix B.

There is a natural homomorphism of the braid group B_N onto the symmetric group S_N: $s_i \longrightarrow \sigma_i$, where s_i are the generators of B_N and σ_i are the generators (elementary transpositions) of S_N. Let us fix the Schur subgroup $S_N^{k_1 \ldots k_\ell} = S_{k_1} \times \ldots S_{k_\ell}$ in S_N $(k_1 + \ldots + k_\ell = N)$. A subgroup $B_N^{k_1 \ldots k_\ell} \subset B_N$ corresponding to $S_N^{k_1 \ldots k_\ell} \subset S_N$ is called a partial coloured braid group.

Let $(a_1, \ldots, a_N)$ be an ordered set of symbols $a_1 = \ldots = a_{k_1} = \lambda_1$, $a_{k_1+1} = \ldots = a_{k_2} = \lambda_2, \ldots$. The set $(a_1, \ldots, a_N)$ is invariant under the action of $S_N^{k_1 \ldots k_\ell}$.

Let us consider the spaces $\mathcal{H}^{(b_1 \ldots b_N)} = V^{(b_1)} \otimes \ldots \otimes V^{(b_N)}$, where $(b_1, \ldots, b_N)$ is some permutation of the set $(a_1, \ldots, a_N)$ and $V^{(b)} = V^{\lambda_m}$ if $b = a_i$ and $k_m \leq i < k_{m+1}$.

Let us assume that the matrices $M^{(b)}: V^{(b)} \longrightarrow V^{(b)}$ and $R^{(b,c)}: V^{(b)} \otimes V^{(c)} \longrightarrow V^{(c)} \otimes V^{(b)}$ for any $b, c \in (a_1, \ldots, a_N)$ satisfy the relations:

$$(R^{(a,b)} \otimes 1)(1 \otimes R^{(cb)})(R^{(ca)} \otimes 1) = (1 \otimes R^{(ca)})(R^{(cb)} \otimes 1)(1 \otimes R^{(ab)}) \quad (8.1)$$

$$(M^{(b)} \otimes M^{(a)}) R^{(a,b)} = R^{(a,b)} (M^{(a)} \otimes M^{(b)}), \quad (8.2)$$

$$tr_2((1 \otimes M^{(a)})(R^{(a,a)})^{\pm 1}) = z_a^{\pm 1} \cdot I \quad (8.3)$$

Here the trace in (8.3) is taken over the second multiplier in $V^{(a)} \otimes V^{(a)}$.

Let us define the matrices $R_i(b) : \mathcal{H}^{(\ldots b_i b_{i+1} \ldots)} \longrightarrow \mathcal{H}^{(\ldots, b_{i+1}, b_i, \ldots)}$.

$$R_i(b) = (I \otimes \ldots \otimes R^{(b_i b_{i-1})} \otimes \ldots \otimes I) \tag{8.4}$$

where $b = (b_1, \ldots, b_N)$.

THEOREM 8.1. If $\alpha \in B_N^{K_1 \ldots K_\ell}$ and $\alpha = S_{i_K}^{\varepsilon_K} \ldots S_{i_1}^{\varepsilon_1}$ is a decomposition of α in a product of generators of B_N then the map π

$$\pi(\alpha) = R_{i_K}({}^{a}\sigma_{i_{K-1}} \ldots \sigma_{i_1})^{\varepsilon_K} \ldots R_{i_2}({}^{a}\sigma_{i_1})^{\varepsilon_2} R_{i_1}(a)^{\varepsilon_1} \tag{8.5}$$

where $a_\sigma = (a_{\sigma_1}, \ldots, a_{\sigma_N})$, $\sigma \in S_N$ define a representation of $B^{K_1 \ldots K_\ell}$ in $Mat(V^{(a_1)} \otimes \ldots \otimes V^{(a_N)})$.

To prove this theorem it is sufficient to use the relation (8.2).

Let us define character of this representation by setting:

$$\chi_{\lambda_1 \ldots \lambda_\ell}^{K_1 \ldots K_\ell}(\alpha) = \prod_{j=1}^{\ell} z_j^{w_j(\alpha)} tr((M^{(a_1)} \otimes \ldots \otimes M^{(a_N)})\, \pi(\alpha)) \tag{8.6}$$

where $w_j(\alpha) = N_j^+(\alpha) - N_j^-(\alpha)$ and $N_j^\pm(\alpha)$ is the number of the generators $S_i^{\pm 1} \in B_{K_j}$ in the representation of the element α through the generators of B_N.

Using the relation (8.1)-(8.3) the following theorem is easily proved.

THEOREM 8.2. The character (8.6) satisfies the relations:

$$\chi_{\lambda_1 \lambda_2 \ldots \lambda_\ell}^{K_1 K_2 \ldots K_\ell + 1}(\alpha) = \chi_{\lambda_\ell}\, \chi_{\lambda_1 \ldots \lambda_\ell}^{K_1 \ldots K_\ell}(\alpha), \quad \alpha \in B_N^{K_1 \ldots K_\ell} \tag{8.7}$$

$$X^{K_1\ldots K_\ell}_{\lambda_1\ldots\lambda_\ell}(\beta\alpha\beta^{-1}) = X^{K_1\ldots K_\ell}_{\lambda_1\ldots\lambda_\ell}(\alpha), \quad \alpha, \beta \in B_N^{K_1\ldots K_\ell}, \tag{8.8}$$

$$X^{K_1,\ldots,K_{\ell+1}}_{\lambda_1,\ldots,\lambda_\ell}(\alpha S_N^{\pm 1}) = X^{K_1\ldots K_\ell}_{\lambda_1,\ldots,\lambda_\ell}(\alpha), \quad \alpha \in B_N^{K_1\ldots K_\ell} \tag{8.9}$$

$$X^{K_1\ldots K_\ell}_{\lambda_1\ldots\lambda_\ell}(\beta\alpha\beta^{-1}) = \tilde{X}(\alpha), \quad \beta\alpha\beta^{-1} \in B_N^{K_1\ldots K_\ell}, \beta \in B_N, \tag{8.10}$$

where $X_\lambda = tr_{V^\lambda}(M^\lambda)$ and $\tilde{X}$ is the character (8.6) with $a \mapsto \sigma a$, where σ is the element of S_N corresponding to $\beta \in B_N$.

COROLLARY 8.2.1. The character $X^{K_1\ldots K_\ell}_{\lambda_1\ldots\lambda_\ell}(\alpha)$ is a invariant of the link $L = \hat{\alpha}$, where $\hat{\alpha}$ is the closure of the braid $\alpha \in B_N^{K_1\ldots K_\ell}$.

Let us specify the general construction described above for the R-matrices connected with the algebras $U_q(\mathfrak{g})$. The symbols λ_i are now the highest weights of the finite-dimensional irreducible representations of $U_q(\mathfrak{g})$.

From the results of sections 1-3 we obtain:

PROPOSITION 8.1. Let $V^{\lambda_1}, \ldots, V^{\lambda_\ell}$ be the IFR of $U_q(\mathfrak{g})$ and $R^{\lambda_i\lambda_j}$ be the corresponding R-matrices. Then the matrix M^λ satisfying (8.2), (8.3) exists and is given by

$$M^\lambda = q^{-\rho}\Big|_{V^\lambda}, \quad Z_\lambda = q^{\frac{c(\lambda)}{2}} \tag{8.11}$$

where $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+} H_\alpha$, and the character (8.6) define the invariant of links.

REMARK 8.1. Let us consider the centralizer of $U_q(\mathfrak{g})$ in the tensor product of N vectorial representations of $U_q(\mathfrak{g})$ and define the functional

$$Tr(x) = \left(\frac{1}{\chi_{\omega_1}}\right)^N tr((M\otimes\ldots\otimes M)x), \quad x\in C_N^{\omega_1}(g).$$

This functional can be extended on $C_\infty^{\omega_1}(g)=\coprod_{n\geq 1} C_n^{\omega_1}(g)$. It has the following properties

$$Tr(1) = 1$$

$$Tr(ab) = Tr(ba)$$

$$Tr(a g_N^{\pm 1}) = z^{\pm 1} Tr(a), \quad a\in C_N^{\omega_1}(g)\subset C_\infty^{\omega_1}(g)$$

where

$$z = \frac{q^{+\frac{c(\omega_1)}{2}}}{\chi_{\omega_1}}$$

and g_i are the generator of $C_N^{\omega_1}(g)$ (3.2).

So, we see that $Tr(a)$ is the Ocneanu trace on Hecke algebra for $g=sl(n)$. This functional can be decomposed into the sum of characters of irreducible representations of $C_N^{\omega_1}(g)$. The latter are:

$$\chi^{(\lambda)}(x) = \sum_a \pi_{aa}^\lambda(x) = \frac{(\chi_{\omega_1}(q))^{|\lambda|}}{\chi_\lambda(q)} \sum_a tr(\mathcal{P}_\lambda(a)x)$$

where $\pi_{ab}^\lambda(x)$ are the matrix elements of the representation (3.7), (3.10) acting in W_λ and $\mathcal{P}_\lambda(a)$ are the projectors (3.19). Using the decomposition (3.3) we obtain the following expression for $Tr(x)$:

$$Tr(x) = \sum_{\lambda} W_{\lambda}(q;n)\, \chi^{(\lambda)}(x)$$

$$W_{\lambda}(q,n) = \frac{\chi_{\lambda}(q)}{(\chi_{\omega_1}(q))^{|\lambda|}} .$$

For $\mathfrak{g} = sl(n)$ this is known "Fourier transform" formula for Ocneanu trace [2]:

$$W_{\lambda}(q,n) = \left(\frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}\right)^{|\lambda|} \prod_{(i,j)\in\lambda} \frac{(q^{\frac{j-i+n}{2}} - q^{-\frac{j-i+n}{2}})}{(q^{\frac{h_{ij}}{2}} - q^{-\frac{h_{ij}}{2}})} .$$

Here $|\lambda|$ is the number of cells in Young's diagram λ, the ind ces i and j numerate rows and colomns (from top to bottom and from left to right) respectively, h_{ij} is the lenghth of the hook with coordinates (i,j) in λ.

REMARK 2. Let us note that the links $L = \hat{\alpha}$, $\alpha \in B_N^{k_1 \dots k_\ell}$ have ℓ or more components and if $\lambda_1 \neq \dots \neq \lambda_\ell$ the functional (8.6) gives the invariant of links depending on ℓ integer and one continuous parameters.

To study the invariants (8.6) it is useful to present them as functionals on the diagrams of links. To do so let us give some definitions.

DEFINITION 8.1. The elements

of the diagram $\mathcal{D}_L$ are called elementary fragments".

DEFINITION 8.2. Let us define the state functional on diagrams of oriented links by the rules:

i) let the diagram be in general position

ii) to each component of L we associate the higherst weight of algebra $U_q(\mathfrak{g})$; this h.w. is called <u>the colour of the component</u>; the diagram with the coloured components will be call <u>coloured diagram</u>.

iii) divide the diagram into elementary fragments

iv) to each edge connecting elementary fragments we associate the states i , $i \in V^{\lambda^*}$ if the edge is oriented upwards and $i \in V^{\lambda}$ if the edge is oriented downwards; here λ is the colour of the component

v) to each elementary fragment we associate a matrix using the rules (2.4)-(2.6) of the previous section.

vi) fix the states on the edges of the diagram $\mathcal{D}_L$; multiply the matrix elements between these states corresponding to elementary fragments; taking the sum of resulting product over all states on $\mathcal{D}_L$ we obtain the state functional on the diagram of coloured, oriented link; this sum is denoted by $Z_{\lambda_1 \dots \lambda_\ell}(\mathcal{D}_L)$ and is called the state functional.

PROPOSITION 8.3. Let $(\sigma_1, \dots, \sigma_\ell)$ denote the orientation of the link L with ℓ components, then

$$Z_{\lambda_1 \dots \lambda_\gamma \dots \lambda_\ell}(\mathcal{D}_{L(\sigma_1 \dots \sigma_\gamma \dots \sigma_\ell)}) = Z_{\lambda_1 \dots \lambda^*_\gamma \dots \lambda_\ell}(\mathcal{D}_{L(\sigma_1, \dots, -\sigma_\gamma, \dots \sigma_\ell)}).$$

COROLLARY. The state functional does not depend on the orientation of the component γ if $\lambda^*_\gamma = \lambda_\gamma$.

Let $N^{\pm}_\gamma(\mathcal{D}_L)$ are the numbers of positive and negative vertices where two strings of the component cross:

and

$$w_\gamma(\mathcal{D}_L) = N^+_\gamma(\mathcal{D}_L) - N^-_\gamma(\mathcal{D}_L).$$

THEOREM 8.3. The functional

$$\varphi_{\lambda_1\ldots\lambda_\ell}(L) = \prod_{\gamma} q^{w_\gamma(\mathcal{D}_L)\left(\frac{c(\lambda_\gamma)}{2}+i\pi\log(q)\right)} Z_{\lambda_1\ldots\lambda_K}(\mathcal{D}_L) \tag{8.12}$$

is the invariant of oriented links.

PROOF. From the definition of $Z_{\{\lambda\}}(\mathcal{D}_L)$ it follows that it is invariant under the regular isotopies. Using the relations (1.49) we derive that smoothing of a look results in the following transformation of

$Z($  $^{\lambda_\gamma}) = Z($  $^{\lambda_\gamma})\, q^{\frac{c(\lambda_\gamma)}{2}}(-1)^{[\lambda]}$.

The multiplier in (8.14) compensates this variation of $Z_{\lambda_1\ldots\lambda_K}(D_L)$. Hence the functional φ is the invariate of links.

THEOREM 8.4. The invariants φ and χ are equal:

$$\chi^{K_1\ldots K_\ell}_{\lambda_1\ldots\lambda_\ell}(\alpha) = \varphi_{\lambda_1\ldots\lambda_\ell}(\hat{\alpha}) \tag{8.13}$$

where we choose the standard orientation of $\hat{\alpha}$.

To prove this theorem it is sufficient to use the equality

$$(C^{-1})^{T} C = (-1)^{[\lambda]} q^{-\rho}$$

and the definitions of φ and χ .

Let us describe the combinatorial procedure, which reduces the calculation of the state functionals $Z_{\lambda_1\ldots\lambda_\ell}$ to the calculation of the basic state functionals $Z_{\omega_1\ldots\omega_1}$. For this purpose we need the following definition:

DEFINITION 8.3. The link $L_{N_1\ldots N_\ell}(\alpha_1\ldots\alpha_\ell)$ built on the link L in accordance to rules (a), (b) is called a <u>composite</u>

link if

(a) each component of the link L is divided in N_i parallel strands ($i=1,\dots,\ell$, ℓ is the number of components of L).

(b) each set of N_i parallel srands is braided; the resulting closed braid is $\hat{\alpha}_i \in B_{N_i}$.

THEOREM 8.5. If we fix a imbedding a_i ; $V^{\lambda_i} \subset (V^{\omega_1})^{\otimes N_i}$ (the parametrization and explicit description of such an imbedding are given in section 3 (3.11), (3.16), (3.19)), then

$$\text{a)}\quad Z_{\lambda_1\dots\lambda_\ell}(\mathcal{D}_L) = \sum_{\alpha(m_1)\dots\alpha(m_\ell)} Q(a_1,\alpha(m_1))\dots$$

$$\dots Q(a_\ell,\alpha(m_\ell))\, Z_{\omega_1\dots\omega_1}(\mathcal{D}_{L_{N_1\dots N_\ell}}(\alpha(m_1),\dots,\alpha(m_\ell)) \tag{8.14}$$

where the coefficients $Q(a,\alpha(m))$ are determined by (3.19) for $\mathcal{P}(a)$

$$\mathcal{P}(a) = \sum_{\alpha(m)} Q(a,\alpha(m))\, R_{\alpha(m)}$$

b) the result of the calculation according to (8.15) does not depend upon the choise of the imbeddings α_i .

PROOF. Using the orthogonality and crossing-symmetry of , we have:

$$Z_{\dots\lambda_i\dots}(\uparrow\lambda_i) = Z_{\dots\lambda_i\dots}\Big($$

$$\Big) =$$

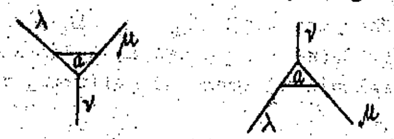
$$= Z_{\dots\underbrace{\omega_1\dots\omega_1}_{N_i}\dots}\Big($$

$$\Big) = \sum_{\alpha(m_i)} Q(a_i,\alpha(m_i))\, Z_{\dots\underbrace{\omega_1\dots\omega_1}_{N_i}\dots}\Big($$

$$\Big).$$

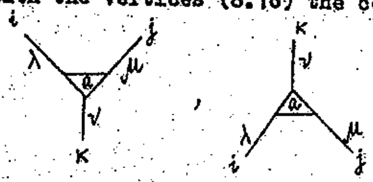
Applying such transformations to each component of L we obtain the formula (8.16). It is obvious that the answer does not depend on the choise of the imbeddings.

REMARK 8.3. The state functional $Z_{\lambda_1 \dots \lambda_k}$ can be defined also for diagrams $\mathcal{D}_\Gamma$ of knoting graphs with triple vertices. For this purpose the following elements should be added into the list of elementary fragments in Definition 8.1:



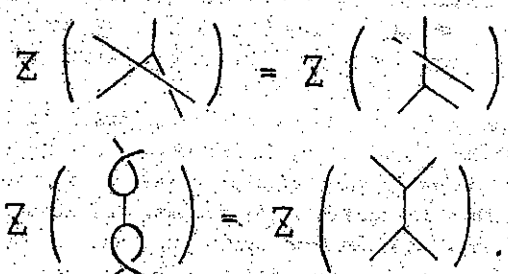
(8.16)

In definition 8.5 we now colour the dges the vertices of the graph Γ. According to rules of section 2 we assotiate with the vertices (8.16) the corresponding CGC matrices



$\longleftrightarrow (K_\nu^{\lambda\mu}(a))_k^{ij}$.

The following equations for state functionals $Z_{\lambda_1 \dots \lambda_\ell}(\mathcal{D}_\Gamma)$ are the result of the relations between the R-matrices and CGC, described in section 2:



(8.18)

9. The invariants of links connected with the algebras $U_q(sl(n)), U_q(so(2n+1)), U_q(so(2n)), U_q(sp(2n))$.

The basic invariant in the case $\mathcal{G}=sl(n)$ was considered by V. Jones [3]. He showed that this invariant is equal to FYLMOH invariant [4] at special values of parameters. To make the picture complete we add here a recursive prosedure for calculating this invariant

PROPOSITION 9.1. The functional $Z_{\omega_1\ldots\omega_1}(\mathcal{D}_L) \equiv \equiv Z_J(\mathcal{D}_L, n)$ for $\mathcal{G}=sl(n)$ gives an invariant for oriented links and is calculated from the following recursive relations

$$Z_J(\times) - Z_J(\times) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) Z_J()() \tag{9.1}$$

$$Z_J(\wp) = q^{-\frac{n}{2}} Z_J(\,)\,) , \tag{9.2}$$

$$Z_J(\wp) = q^{\frac{n}{2}} Z_J(\,)\,) , \tag{9.3}$$

$$Z_J(\bigcirc) = \chi_{\omega_1} = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} . \tag{9.4}$$

PROOF. The functional Z_J depends on the orientation of L because $\omega_1^* \neq \omega_1$ for $U_q(sl(n))$. The relations (9.1)-(9.4) follow from the properties of the matrix $R^{\omega_1\omega_1}$ given in section 4.

The link invariants corresponding to vectorial representations of $U_q(so(2n)), U_q(so(2n+1)), U_q(sp(2n))$ are equal to the Kauffmann invariant [5] for special values of parameters in the last one. This is Turaev's [6] result. The construction given in [6] slightly differs from our and that is why we give here the most important properties of corresponding state functional.

PROPOSITION 9.2. The functionals $Z_{\omega_1 \dots \omega_1}(\mathcal{D}_L) \equiv$ $\equiv Z_K(\mathcal{D}_L, n)$ for $\mathfrak{g} = so(2n+1), so(2n), sp(2n)$ do not depends on orientation of L and satisfy the following relations

$$Z_K(\times) - Z_K(\times) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(Z_K()() - Z_K(\asymp)) \tag{9.5}$$

$$Z_K(\rangle\!\!\bigcirc) = (-1)^{[\omega_1]} q^{-\frac{N-1}{2}} Z_K()) \tag{9.6}$$

$$Z_K(\rangle\!\!\bigcirc) = (-1)^{[\omega_1]} q^{\frac{N-1}{2}} Z_K()) \tag{9.7}$$

$$Z_K(\bigcirc) = X_{\omega_1} . \tag{9.8}$$

Here $N = 2n$ for $\mathfrak{g} = so(2n), sp(2n)$ and $N = 2n+1$ for $\mathfrak{g} =$ $= so(2n+1)$; $[\omega_1] = 0$ for $so(2n), so(2n+1)$, $[\omega_1] = 1$ for $sp(2n)$;

$X_{\omega_1} = (q^{\frac{N-2}{4}} + q^{-\frac{N-2}{4}})(q^{\frac{N}{4}} - q^{-\frac{N}{4}})/(q^{\frac{1}{2}} - q^{-\frac{1}{2}})$ for $\mathfrak{g} = so(2n+1)$, $so(2n)$, $X_{\omega_1} = (q^{\frac{N+2}{4}} + q^{-\frac{N+2}{4}})(q^{\frac{N}{4}} - q^{-\frac{N}{4}})/(q^{\frac{1}{2}} - q^{-\frac{1}{2}})$ for $\mathfrak{g} = sp(2n)$.

PROOF. The functional Z_K does not depend on orientation of L because $\omega_1^* = \omega_1$ for $\mathfrak{g} = so(2n+1), so(2n), sp(2n)$. The relations (9.5)-(9.8) follows from the properties of matrices $R^{\omega_1 \omega_1}$ described in sections 5 and 6.

The relations (9.5)-(9.8) can be considered as a recursive procedure for calculating the values of state functional Z_K [5]. Another simple combinatorial procedure for calculating Z_K use the fact that $U_q(sl(n)) \subset U_q(\mathfrak{g})$ for $\mathfrak{g} = so(2n+1)$, $so(2n)$, $sp(2n)$ as a Hopf subalgebra, or in another words the block structure (5.16) - (5.19), (6.9), (6.10) of the matrices $R^{\omega_1 \omega_1}$, $(R^{\omega_1 \omega_1})^{-1}$.

THEOREM 9.1. The calculation of the values of the functional $Z_K(\mathcal{D}_L)$ corresponding to $\mathfrak{g} = sp(2n)$, $so(2n)$ is reduced to the calculation of the functional Z_J according to the following combinatorial prosedure:

i) define some orientation on each edge of the diagram

ii) for each vertex with admissible orientation compare the weights:

$$W(\times) = W(\times) = W(\times) = W(\times) =$$

$$= W(\times) = W(\times) = W(\times) = W(\times) = 1$$

$$W(\cup) = \varepsilon q^{\frac{n-\varepsilon}{4}}, \quad W(\cup) = q^{\frac{n-\varepsilon}{4}}, \quad W(\cap) = \varepsilon q^{\frac{n-\varepsilon}{4}}, \quad W(\cap) = q^{\frac{n-\varepsilon}{4}}$$

$$\varepsilon = (-1)^{[\omega_1]}.$$

iii) the vertices with non admissible orientation we reconstruct according to the following rules:

$$\times = W()()\,)(+ W(\asymp)\asymp ,$$

$$\times = 0, \quad \cup = 0, \quad \cup = 0$$

$$\times = W(\asymp)\asymp + W()()\,)(,$$

$$\times = 0, \quad \cap = 0, \quad \cap = 0$$

where $W(\,)(\,) = -(q^{1/2} - q^{-1/2})$, $W()() = (q^{1/2} - q^{-1/2})$, $W(\asymp) = + (q^{1/2} - q^{-1/2})\, \varepsilon q^{\frac{n-\varepsilon}{2}}$, $W(\asymp) = -(q^{1/2} - q^{-1/2})\, \varepsilon q^{\frac{n-\varepsilon}{2}}$.

iv) with each reconstructed diagram $\hat{\mathcal{D}}_L$ we associate the weight

$$W(\mathcal{D}_L, \hat{\mathcal{D}}_L) = \prod_{\text{all verteces}} W\ (\text{vertex})$$

v) calculating the functionals $Z_J(\hat{\mathcal{D}}_L, n)$ and taking the sum over all orientations of edges we obtain the following representation for $Z_K(\mathcal{D}_L)$:

$$Z_K(\mathcal{D}_L) = \sum_{\hat{\mathcal{D}}_L} W(\mathcal{D}_L, \hat{\mathcal{D}}_L) Z_J(\hat{\mathcal{D}}_L) . \qquad (9.9)$$

PROOF. The representation (9.9) we obtain substituting the block structures (5.18), (5.19), (6.9), (6.10) of in the definition of $Z_{\omega_1 \dots \omega_1}(\mathcal{D}_L)$.

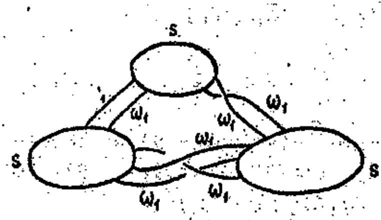
A similar combinatorial procedure for calculating of Z_K for $\mathcal{G}=SO(2N+1)$ we obtain from the block structure (5.16), (5.17) of corresponding matrix $R^{\omega_1\omega_1}$.

The calculation of link invariants connected with higher tensor representations of classical algebras is reduced to the calculation of the basic invariants according to the theorem 8.5.

The calculation of links invariants corresponding to the two-valued representations of $U_q(SO(2N))$ and $U_q(SO(2N+1))$ can also be reduced to calculating the invariants corresponding to vector representation. Let us describe the procedure, reducing the calculation of the spinor invariant of links to calculating the vector invariant for some "reconstructed" links.

PROPOSITION 9.3. The calculation of the state functional corresponding to spinor representation of $U_q(SO(2N))$ and $U_q(SO(2N+1))$ can be reduced to calculating the vector state functional by the following two-step prosedure:

i) using the formulae (5.11), (5.31)-(5.39) we obtain the representation of the state functional $Z_{S\dots S}(\mathcal{D}_L)$ as a sum of state functionals on the diagrams of knoting graphs with nonlinked and nonknoting loops colored in spinor representations jointed by the braids with strings colored in vector representation. An example of such a graph is given below:



ii) using the relation (5.35) the "spinorial loops is calculated" and after this procedure we obtain the representation of the state functional $Z_{s \dots s}$ as a sum of state functionals Z_K on some diagrams colored by ω_1 with some additional weights (lake $W(\mathcal{D}_L, \hat{\mathcal{D}}_L)$ in (9.9)) which can be found from (5.1), (5.31)-(5.35).

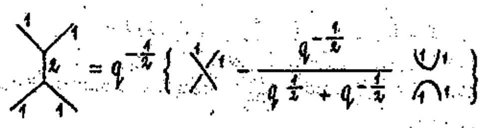
10. The invariant of links corresponding to the algebra G_2

Substituting an explicite expression for basic $U_q(G_2)$-R-matrix in the definition of the state functional $Z_{\omega_1 \dots \omega_1}$ we obtain the basic $U_q(G_2)$ invariant of links. Most nontrivial problem is the calculation of the values of this functional on concrete links. Let us show how the calculation of the $U_q(G_2)$ - invariant is reduced to the calculation of Jones invariant on "reconstructed" diagrams.

One can easy prove the following

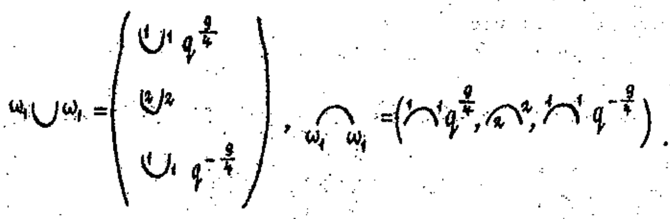
PROPOSITION 10.1. There are the following identities connected CGC and R-matrices corresponding to the algebra $U_q(sl(2))$:

$$= \sqrt{\frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{q + q^{-1}}} \qquad (10.1)$$



(10.2)

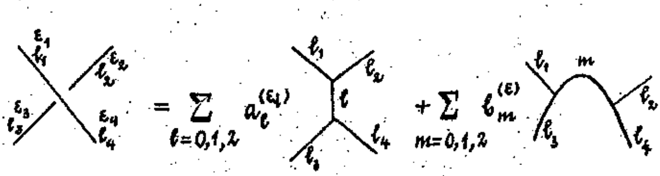
Moreover one can easy check that



(10.3)

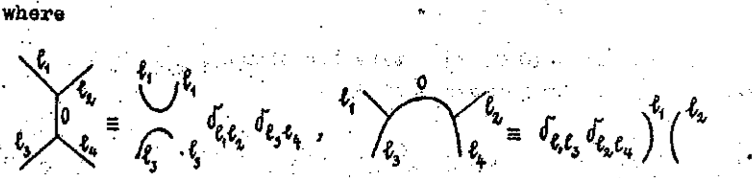
Here the l.h.s.'s are the elements of the graphical technique for $U_q(G_2)$.

Let us rewrite the equality (7.10) in the following form:



(10.4)

where



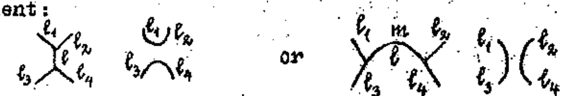
The numbers $\ell^{\varepsilon} = (1^+, 2, 1^-)$ numerate the irreducible $U_q(s\ell(2))$ components of vector representation of $U_q(G_2)$.

$|\ell_1-\ell_2|\leqslant \ell \leqslant \ell_1+\ell_2,\ \ |\ell_3-\ell_4|\leqslant \ell \leqslant \ell_3+\ell_4,\ \ |\ell_1-m|\leqslant$
$\leqslant \ell_3 \leqslant \ell_1+m,\ \ |m-\ell_2|\leqslant \ell_4 \leqslant m+\ell_2,\ \ \ell,\ m=0,1,2$
and the coefficients $a_\ell^{(\varepsilon)}$ and $b_m^{(\varepsilon)}$ can be founed from (7.10), (7.15), (7.17).

DEFINITION 10.1. A plane graph Γ with edges numerated by the numbers $\ell=1,2$ is called a reconstruction of the diagram $\mathcal{D}_L$ if it can be obtained from $\mathcal{D}_L$ by the following combinatorial procedure:

i) the edges of $\mathcal{D}_L$ are numerated by the numbers $\ell=1,2$.

ii) each vertex is replaced on one of the following element:

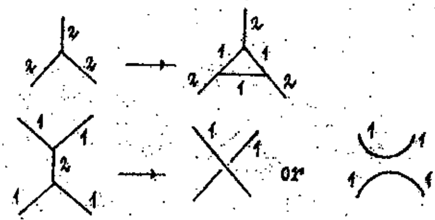


with some numbers $\ell, m = 1,2$.

DEFINITION 10.2. The link $\hat{L}$ we call a tangled reconstruction of $\mathcal{D}_L$ if the diagram $\hat{\mathcal{D}}_L$ of the link $\hat{L}$ can be obtained from Γ by the following procedure:

(a)

(b)



PROPOSITION 10.2. The state functional $Z_{\omega_1\omega_1\ldots\omega_1}(\mathcal{D}_L)$ has the following representation:

$$Z_{\omega_1\ldots\omega_1}(\mathcal{D}_L) = \sum_{\substack{\text{all tangled}\\ \text{reconstructions } \hat{\mathcal{D}}_L\\ \text{of } \mathcal{D}_L}} W(\mathcal{D}_L, \hat{\mathcal{D}}_L)\, Z_T(\hat{\mathcal{D}}_L, 2) \tag{10.5}$$

where $Z_J(\mathcal{D}_L, v)$ is the Jones state functional and the weights $W(\mathcal{D}_L, \hat{\mathcal{D}}_L)$ are determined by the formulae (10.1)-(10.4).

Conclusion

I do not discuss here deep connections of all these invariants with the structure of algebras $U_q(\mathfrak{g})$. It appears that for each algebra $U_q(\mathfrak{g})$ there exists a universal invariant of links. It takes values in $Z_q(\mathfrak{g})^{\otimes K}$ where K is the number of components of links and $Z_q(\mathfrak{g})$ is the center of the algebra $U_q(\mathfrak{g})$. This invariant will be described in a separate publication.

The work focuses on the invariants of links. However, a analogous objects can be easily constructed for the knotted graphs with vertices (2.6) (see the end of section 8). The study of these invariants will be given separately.

The action (3.7-9) of the elements in the representations W_λ of the algebra $C_N^\omega(q)$ give the key to understanding the sense of the so-called restricted solid on solid (RSOS) models of statistical mechanics [7] in the context of the representation theory. When $q = exp\left(\frac{2\pi i}{n}\right)$ the algebras $C_N^\omega(q)$ are no langer the simple. However, even in this case they have $*$- subfactors (see [8, 9] for Hecke algebra). The RSOS models are described in terms of irreducible representations of these $*$-subfactors.

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Appendix

Let us recall the definition of the braid group

DEFINITION 1. The braid is a set of smooth lines $\gamma_1 \cdots \gamma_N$ in $\mathbb{R}^2 \times I$, $I = [0,1]$ nonselfintersecting, not intersecting in pairs, and having nonzero tangent vectors $\dot{\gamma}_j \neq 0$, transversal to layers $\mathbb{R}^2 \times t$, fot all $0 \leq t \leq 1$ with bases in points $(P_1, 0), \ldots, (P_N, 0)$ and ending in points $(P_{\sigma_1}, 1), \ldots, (P_{\sigma_N}, 1)$ where σ is a certain permutation of numbers $(1, \ldots, N)$.

An example of a braid is given on Figure a.

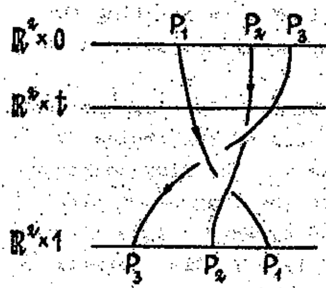

Fig. a

The isotopy classes of braids form the braid group B_N. The unit element of B_N is given on Figure b. The braid $\alpha\beta$ is the joint of the braids α and β where the beginning of the braid β is the end of the braid α (see Figure c)

Figure b

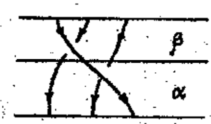

Figure c

PROPOSITION 1. The braid group B_N is formed by the generators s_i, $i=1,\dots,N-1$ corresponding to the simple transpositions $\sigma_i = \begin{pmatrix} \dots & i & i+1 & \dots \\ \dots & i+1 & i & \dots \end{pmatrix}$. The generators s_i satisfy the relations

$$s_i\, s_{i+1}\, s_i = s_{i+1}\, s_i\, s_{i+1}$$

$$s_i\, s_j = s_j\, s_i \quad , \quad |i-j|>1 .$$

The braids corresponding to the elements s_i and s_i^{-1} are given on the Figure d and e respectively.

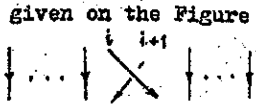

Figure d

Figure e

DEFINITION 2. The closed braid $\hat{\alpha}$ is the diagram of oriented links obtained from the braid α by jointing the end with the begining f this braid by nonlinked and nonknoting strings (see Figure f).

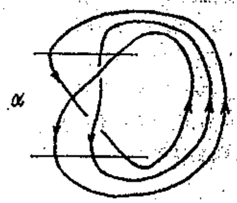

Figure f

PROPOSITION 2. The transformations of the braids α nonchanging the link corresponding to the diagram $\hat{\alpha}$ are generated by Markov type I and Markov type II transformations:

$$\text{I}: \quad \hat{\alpha} = \widehat{\beta \alpha \beta^{-1}} \quad , \qquad \forall \alpha, \beta \in B_N \, ,$$

$$\text{II}: \quad \hat{\alpha} = \widehat{\alpha s_N^{\pm 1}} \quad , \qquad \forall \alpha \in B_N \, , \; (\alpha s_N^{\pm 1} \in B_{N+1}) \, .$$

PROPOSITION 3. Any link L can be transformed by regular isotopies to some closed braid.

DEFINITION 3. The invariant of links is the functional $\varphi(L)$ (which does not change under isotopy transformations of L .

If $\mathcal{D}_L = \hat{\alpha}$; $\alpha \in B_N$ the isotopy invariance of the functional defined on $\hat{\alpha}$ means that this functional considered as a functional on braids satisfy the following relations

$$\varphi_N(\alpha) = \varphi_N(\beta \alpha \beta^{-1}), \quad \forall \alpha, \beta \in B_N$$

$$\varphi_{N+1}(\alpha s_N^{\pm 1}) = \varphi_N(\alpha), \quad \forall \alpha \in B_N \, .$$

Here $\varphi_N(\alpha) \equiv \varphi(\hat{\alpha})$ and index N means that $\alpha \in B_N$.

РТП ЛИЯФ,зак.61,тир.220,уч.-изд.л.1,0; 15/1-1988 г.
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