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Contents

Introduction

The interest in the quantum inverse scattering method (QISM) and in the solutions of Yang-Baxter equations 1 has significantly increased recently, for it happened that interesting mathematical constructions lie at the basis of QISM - these are noncommutative and noncocommutative Hopf algebras. The first example of such an algebra was the q-deformation of a universal enveloping algebra $U_q(sl(2))$ [4, 5]. The algebras $U_q(\hat{\mathfrak{g}})$ where $\hat{\mathfrak{g}}$ is a simple or a Kac-Moody algebra, have been constructed (in works) by Drinfeld and Jimbo [4, 5]. All these Hopf algebras are connected with an entire class of solutions of Yang-Baxter equation.

V.Jones [6] has recently shown that under certain additional conditions solutions of the Yang-Baxter equations can be used for constructing invariants of links. He also showed that two-parameters invariant of links constructed by Freyd-Yetter, Lickorish-Millet, Ocneanu and Hoste (FILMOH) [7] is connected with the solution of the Yang-Baxter equation found by Jimbo and Cherednik [3, 8]. Following this idea Turaev [9] showed that the R-matrices found by Jimbo [10] are connected with the Kauffman invariant [1].

This paper argues that to each algebra $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a simple Lie algebra, we can associate a countable set of invariants of links. If a link contains K components then an invariant depending on one continuous parameter and K discrete parameters can be defined for it. These discrete parameters correspond to a highest weight of the algebra $U_q(\mathfrak{g})$. Solutions of the Yang-Baxter equations (they are what the link invariants are built from) will be called R-matrices following the terminology of QISM.

In the beginning of section 1 some information on the algebras $U_q(\mathfrak{g})$ and on universal R-matrices connected with them is given. Next the following important theorem is proved.

THEOREM (1.5). If V^ν, V^λ and V^μ are irreducible representations of $U_q(\mathfrak{g})$ with highest weights ν, λ, μ, $V^\nu \subset V^\lambda \otimes V^\mu$ with multiplicity equal to one, $R^{\lambda\mu}$ is R-matrix, $R^{\lambda\mu}: V^\lambda \otimes V^\mu \to V^\mu \otimes V^\lambda$ and $K_\nu^{\lambda\mu}: V^\lambda \otimes V^\mu \to V^\nu$ is the matrix of Clebsh-Gordan q-coefficients (CGC), then

$$K_\nu^{\mu\lambda} R^{\lambda\mu} = (-1)^{\bar\nu} q^{\frac{1}{4}(c(\nu)-c(\lambda)-c(\mu))} K_\nu^{\lambda\mu}$$

where $\bar\nu = 0,1$, $c(\nu) = \langle \nu,\nu\rangle + 2\langle \rho,\nu\rangle$, $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+} \alpha$,

Δ_+ is a set of positive roots of the algebra $\mathfrak{g}$.

A fusion prosedure [12] for R-matrices connected with the algebras $U_q(\mathfrak{g})$ is given by theorem 1.6.

THEOREM (1.6). Let $K_\nu^{\lambda\mu}(a)$ be a CGC matrix, $K_\nu^{\lambda\mu}(a)$: $V^\lambda \otimes V^\mu \to V^\nu \subset V^\lambda \otimes V^\mu$ (the index a numbers V^ν if the multiplicity of V^ν in $V^\lambda \otimes V^\mu$ is greates than one) then the matrices $R^{\lambda\tau}$, $R^{\mu\tau}$ and $R^{\nu\tau}$ are connected by the following relation

$$R^{\nu\tau}(K_\nu^{\lambda\mu}(a))_{12} = (K_\nu^{\lambda\mu}(a))_{23} R_{12}^{\lambda\tau} R_{23}^{\mu\tau} .$$

Their exact meaning is given in section 2.

Section 2 introduces a graphical interpretation of R-matrices and of CGC in order to clarify visually their relations. All of these relations are expressed graphically. The last theorem 1.8 of section 1 is proved with the help of this graphical interpretation.

This graphical interpretation is actively exploited in subsequent sections.

Section 3 explores the structure of the centralizer of the algebra $U_q(\mathfrak{g})$ in the space $(V^\omega)^{\otimes N}$, where ω is the basic representation of $U_q(\mathfrak{g})$ (see section 1). These results are applied for calculating the matrices $R^{\lambda\mu}$ through the matrix $R^{\omega\omega}$. It is shown that the matrices

$$g_i = 1 \otimes \ldots \otimes R^{\omega\omega} \otimes \ldots \otimes 1$$

form the centralizer $C_N^\omega(\mathfrak{g}) = C(U_q(\mathfrak{g}), (V^\omega)^{\otimes N})$. From the complete irreducibility of finite dimensional representations of $U_q(\mathfrak{g})$ follows the analog of Weyl's duality:

$$(V^\omega)^{\otimes N} = \sum_\lambda V^\lambda \otimes W_\lambda$$

where V^λ is an irreducible finite dimensional $U_q(\mathfrak{g})$-module and W_λ is an irreducible $C_N^\omega(\mathfrak{g})$-module. In the spaces W_λ an orthogonal basis analogous to the Young basis for S_N is built. It is shown how to calculate the matrix elements of g_i in this basis, when CGC $K^{\lambda\omega}_{\mu}$ are known. A q-analog of the Young basis is built out of g_i. (A q-analog of the Young projectors). Using the expression for the Young projectors through the matrices $R^{\omega\omega}$ it is shown how the matrices $R^{\lambda\mu}$ are expressed through $R^{\omega\omega}$.

The results of sections 1-3 are applied to the R-matrices connected with the algebra $U_q(gl(n))$ in section 4. This section has a methodological character. Most probably an explicit description of the ideal by which the Hecke algebra should be factorized in order to obtain $C^{\omega}_{N}(gl(n))$ is new. Other results (obtained in [5, 6]) are given merely for the purpose of drawing a fuller picture.

Section 5 focuses on R-matrices connected with the algebras $U_q(so(2n+1))$ and $U_q(so(2n))$. R-matrices corresponding to tensorial representation of these algebras are analysed. The matrices $R^{\omega_1\omega_1}$, found earlier by Jimbo [10], are constructed using the results of sections 1-3. It is shown that $C^{\omega_1}_{N}(so(2n+1))$ and $C^{\omega_1}_{N}(so(2n))$ are both equal to a factor of Birman-Wenzl algebra [13] over an ideal given explicitly. A block structure of the matrices $R^{\omega_1\omega_1}$ corresponding to the embeddings $so(2n+1)\supset gl(n)$, $so(2n)\supset gl(n)$ is given. The second half of section 5 deals with spinorial representations of $U_q(so(2n+1))$ and $U_q(so(2n))$. It is shown that the matrices $K^{\omega_1 s}_{s}$, where s are spinorial representations, form a q-analog of the Clifford algebra. Finally, it is shown how spinorial R-matrices are expressed through the matrices $K^{\omega_1 s}_{s}$ and $R^{\omega_1\omega_1}$.

Section 6 is concerned with the solutions of Yang-Baxter equation connected with $U_q(sp(2n))$. All of the results are quite similar to tensorial representations of the algebras $U_q(so(2n+1))$ and $U_q(so(2n))$.

A matrix $R^{\omega_1\omega_1}$ for the algebra $U_q(G_2)$ is found

in section 7, based on the results of sections 1-3. A block structure of $R^{\omega_1\omega_1}$ corresponding to the embedding $G_2 \supset sl(2)$ is obtained.

Section 8 applies the result obtained in the previous sections to knot theory. It is shown how to build with the help of the solutions of Yang-Baxter equation satisfying certain auxiliary condition representations of the group of partially coloured braids and invariants of links as characters of these representations. This result is a natural generalization of Jones construction [6]. Next, it is shown that the matrices $R^{\lambda\mu}$ connected with irreducible representations of the algebras $U_q(\mathfrak{g})$ satisfy the necessary conditions and generate a class of invariants of links. Two ways of description are offered for these invariants. In the first way the invariants are interpreted as characters of the braid group representations. In the second way the invariants are looked upon as statistical sums (state functionals) on the diagram of the link. At the end of the section a combinatorial rule, wich helps to reduce the calculation of invariants built on the matrices $R^{\lambda\mu}$ to the calculation of the basic invariant (built on the matrix $R^{\omega_1\omega_1}$).

Section 9 discusses the invariants connected with the q-deformations of classical Lie algebras. The basic invariants connected with these algebras give invariant and the Kauffman invariant. Theorem 9.1 is a new result which allows us to reduce the calculation of the Kauffman invariant to the calculation of the FILMOH invariant for some reconstructed links. At the end of the section it is shown how the calculation of invariants built on spinorial R-matrices is reduced to the calculation of Kauffman invariants for reconstructed links.

In Section 10 a rule by which a basic invariant connected with $U_q(G_2)$ can be calculated by means of Jones's invariant [14] is given.

Finally, in conclussion some hypotheses are formulated.

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1. The solutions of Yang-Baxter equation connected with the q-deformations of universal enveloping algebras of simple Lie algebras

The q-deformation of the universal enveloping algebras (or quantum universal enveloping algebras QUEA) $U_q(\mathfrak{g})$ arose when studing some special class of integrable systems. The definition of $U_q(\mathfrak{g})$ for any simple Lie algebra $\mathfrak{g}$ was given in [4, 5]. One can define these algebras by generators and relations.

DEFINITION 1.1 [4, 5]. QUEA $U_q(\mathfrak{g})$ is the algebra with generators $X_i^\pm$, H_i, $i=1,\dots,r=\mathrm{rank}\,\mathfrak{g}$ and with the relations:

$$[H_i, H_j] = 0, \quad [H_j, X_i^\pm] = \pm(\alpha_j \alpha_i) X_i^\pm \tag{1.1}$$

$$[X_i^+, X_j^-] = \delta_{ij} \frac{sh(\frac{h}{2} H_i)}{sh(\frac{h}{2})}, \quad q = e^h \tag{1.2}$$

$$i \neq j, \ \sum_{k=0}^{n} (-1)^k \binom{n}{k}_{q_i} q_i^{-\frac{k(n-k)}{2}} (X_i^\pm)^k X_j^\pm (X_i^\pm)^{n-k} = 0, \quad q_i = q^{\frac{(\alpha_i \alpha_i)}{2}} \tag{1.3}$$

where A_{ij} is the Cartan matrix of $\mathfrak{g}$, (α, β) is the scalar product of the roots $(A_{ij} = (\alpha_i \alpha_j)(\alpha_j \alpha_j))$, $n = 1 - A_{ij}$, $i \neq j$.

The generators $X_i^\pm$, H_i play the role of Chevalley basis in $U(\mathcal{G})$. The elements H_i form the commutative subalgebra $U(\mathfrak{h}) \subset U_q(\mathcal{G})$. This subalgebra is the analog of Cartan subalgebra in $U(\mathcal{G})$. The elements H_i correspond to the simple roots of $\mathcal{G}$. Let $\alpha = \sum_{i=1}^{r} n_i \alpha_i$ be the root of $\mathcal{G}$, a corresponding element of $\mathfrak{h}$ we denote as $H_\alpha = \sum_{i=1}^{r} n_i H_i$.

THEOREM 1.1. [4, 5]. The algebra $U_q(\mathcal{G})$ is the Hopf algebra [14] (see also Appendix A) with comultiplication and with the antipode γ defined on the generators by the following formulae:

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i , \tag{1.4}$$

$$\Delta(X_i^{\pm}) = X_i^{\pm} \otimes e^{\frac{h}{4}H_i} + e^{-\frac{h}{4}H_i} \otimes X_i^{\pm} , \tag{1.5}$$

$$\gamma(H_i) = -H_i , \quad \gamma(X_i^{\pm}) = -e^{\frac{h}{2}\rho} X_i^{\pm} e^{-\frac{h}{2}\rho} , \tag{1.6}$$

where $\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} H_\alpha$ is the element of $\mathfrak{h}$ and Δ_+ is the positive roots of $\mathcal{G}$.

The parameter q is the deformation parameter. At $q=1$ the algebra $U_q(\mathcal{G})$ is the universal enveloping algebra of $\mathcal{G}$. For simple Lie algebras of the algebras $U_q(\mathcal{G})$ are simple for general q. The special values of q are situated in the rational points on the unit cyrcle.

Let $\mathcal{b}_\pm \subset \mathcal{G}$ be the Borel subalgebras in $\mathcal{G}$. The algebras $U_q(\mathcal{b}_+)$ and $U_q(\mathcal{b}_-)$ with generators (H_i, X_i^+) and (H_i, X_i^-) are the Hopf subalgebras in $U_q(\mathcal{G})$. There is a duality between $U_q(\mathcal{b}_+)$ and $U_q(\mathcal{b}_-)$. Let $\{e_s\}$ is the basis in $U_q(\mathfrak{b}_-)$ and $\{e^s\}$ be the dual basis in $U_q(\mathfrak{b}_+)$.

It is not difficult to prove that the map $\sigma\circ\Delta$: $U_q(\mathfrak{g}) \longrightarrow U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$, where σ is the permutation in $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ also defines the structure of Hopf algebra on $U_q(\mathfrak{g})$ with antipode $\gamma' = \gamma^{-1}$.

THEOREM 1.2 [4]. The comultiplication Δ and $\sigma\circ\Delta$ are connected by the following relation

$$\sigma\circ\Delta(a) = \mathcal{R}\,\Delta(a)\,\mathcal{R}^{-1} \quad \forall a \in U_q(\mathfrak{g}) \tag{1.7}$$

where σ is the permutation operator in $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ and $\mathcal{R} = \sum\limits_s e_s \otimes e^s \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$.

The proof of this theorem the following one is based on the construction of d Hopf algebra [5].

THEOREM 1.3. The element $\mathcal{R}$ satisfies the relations

$$(\Delta \otimes id)\,\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} \tag{1.8a}$$

$$(id \otimes \Delta)\,\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \tag{1.8b}$$

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \tag{1.8c}$$

$$(id \otimes \gamma)\,\mathcal{R} = \mathcal{R}^{-1} \tag{1.8d}$$

Here $\mathcal{R}_{12}, \mathcal{R}_{13}, \mathcal{R}_{23} \in U_q(\mathfrak{g})^{\otimes 3}$, $\mathcal{R}_{12} = e_s \otimes e^s \otimes 1$, $\mathcal{R}_{13} = e_s \otimes 1 \otimes e^s$, $\mathcal{R}_{23} = 1 \otimes e_s \otimes e^s$.

PROOF. Let m, μ and γ be the matrices of multiplication, comultiplication and antipode in algebra $U_q(\mathfrak{b}_-)$:

$$e_s e_t = m^r_{st} e_r, \quad \Delta(e_s) = \mu^{tr}_s e_t \otimes e_r, \quad \gamma(e_s) = \gamma^t_s e_t.$$

The matrices μ, m' and γ^{-1} are the matrices of multiplication, comultiplication and antipode in $U_q(\mathfrak{b}_+)$:

$$e^s e^t = \mu^{st}_r e^r, \quad \Delta(e^s) = m^s_{tr} e^r \otimes e^t, \quad \gamma(e^s) = (\gamma^{-1})^s_t e^t,$$

From (1.7) we have the commutation relations between the elements e^s and e_t in $U_q(\mathfrak{g})$:

$$\mu_s^{pq} m_{tp}^{r} e^t e_q = \mu_s^{pq} m_{qt}^{r} e_p e^t .$$

From this commutation relations and from the definition of $\mathcal{R}$ follows (1.8c). To prove (1.8d) we use the definition of antipode (A.):

$$\mathcal{R}(id \otimes \gamma)\mathcal{R} = e_s e^t \otimes e^s \gamma(e^t) = m_{st}^{r} e_r \otimes \gamma_p^t \mu_q^{sp} e^q =$$

$$= m_{st}^{r} \gamma_p^t \mu_q^{sp} e_r \otimes e^q = e \otimes e .$$

Similarly

$$(id \otimes \gamma)(\mathcal{R}) \cdot \mathcal{R} = m_{st}^{r} \gamma_p^s \mu_q^{pt} e_r \otimes e^q = e \otimes e .$$

The following formulae prove the relations (1.8a) and (1.8b)

$$(\Delta \otimes id)\mathcal{R} = \Delta(e_s) \otimes e^s = \mu_s^{rt} e_r \otimes e_t \otimes e^s =$$

$$= e_r \otimes e_t \otimes e^r e^t = \mathcal{R}_{13} \mathcal{R}_{23} ,$$

$$(id \otimes \Delta)\mathcal{R} = e_s \otimes \Delta(e^s) = e_s \otimes m_{rt}^{s} e^t \otimes e^r =$$

$$= e_r e_t \otimes e^t \otimes e^r = \mathcal{R}_{13} \mathcal{R}_{12} .$$

The theorem is proved.

The algebras $U_q(\mathfrak{g})$ and $\mathfrak{g}$ have a common Cartan subalgebra $\mathfrak{h}$ generated by the elements H_i. This means that the theories of the finite dimensional representations of $U_q(\mathfrak{g})$ and $\mathfrak{g}$ are quite parallel. The following two statements will play the main role in the text.

PROPOSITION 1.1. The finite dimensional representations of $U_q(\mathfrak{g})$ are completely irreducible.

PROPOSITION 1.2. (a) The irreducible finite-dimensional representations (IFR) of $U_q(\mathfrak{g})$ are parametrized by the highest weights λ of the algebra $\mathfrak{g}$. (b) Any irreducible finite dimensional modul V^λ is decomposed into the sum of the weight spaces $V^\lambda = \bigoplus_\mu V^\lambda(\mu)$ and the dimensions of $V^\lambda(\mu)$ are the same as for IFR of $\mathfrak{g}$.

From the complete irreducibility of finite dimensional representations of $U_q(\mathfrak{g})$ it follows that any IFR can be considered as some irreducible component of corresponding tensorial power of basic representations. The list of basic representation is given in table 1.

category \ $\mathfrak{g}$	A_n	B_n	C_n	$\mathcal{D}_n$	G_2	E_6	E_7	E_8	F_4
finite dimensional	ω_1	ω_1	ω_1	ω_1	ω_1	ω_1	ω_7	ω_8	ω_4
tensorial represent	-	ω_n	-	ω_1, ω_n	-	-	-	-	-

Table 1

For algebras B_n and $\mathcal{D}_n$ it is natural to separate from the finite dimentional representations the category of tensorial representations (with integers highest weights). In this category the basic representation is vectorial represen ation with h.w. ω_1 in both cases.

It is very important for our purposes that all finite dimension l representations of $U_q(\mathfrak{g})$ are contained in tensorial powers of the basic representations.

Below we study the restrictions of universal R-matrices on the IFR's. If ρ_λ and ρ_μ are IFR of $U_q(\mathfrak{g})$ and $P^{\lambda\mu}$ is the permutation operator on $V^\lambda \otimes V^\mu$ ($P^{\lambda\mu}$: $V^\lambda \otimes V^\mu \to V^\mu \otimes V^\lambda$, $P^{\lambda\mu}(f \otimes g) = g \otimes f$) then we denote the matrix $P^{\lambda\mu}(\rho_\lambda \otimes \rho_\mu)(\mathcal{R})$ as $R^{\lambda\mu}$ and we call it the R-matrix in λ, μ representation (or λ, μ R-matrix). This matrix map $V^\lambda \otimes V^\mu$ on $V^\mu \otimes V^\lambda$ and satisfies the following relations:

$$(R^{\lambda\mu}\otimes I)(I\otimes R^{\nu\mu})(R^{\nu\lambda}\otimes I)=(I\otimes R^{\nu\lambda})(R^{\nu\mu}\otimes I)(I\otimes R^{\lambda\mu}). \quad (1.9)$$

The left and the right sides of these relations act from $V^{\nu}\otimes V^{\lambda}\otimes V^{\mu}$ to $V^{\mu}\otimes V^{\lambda}\otimes V^{\nu}$.

To find the matrices $R^{\lambda\mu}$ one can use two ways. The first one is th explicit projection of the universal R-matrix by factorising both copies of $U_q(g)$ over corresponding ideals. The second way uses the complete irreducibility of IFR of $U_q(g)$. Because all of IFR are contained in tensorial powers of basic representations of $U_q(g)$ one can construct the matrices $R^{\lambda\mu}$ from a tensorial product of the R-matrics acting in basic representations. The second way gives also the important relations between the matrices $R^{\lambda\mu}$ and we will use this way below.

Consider a Cartan antiinvolution in $U_q(g)$

$$\theta(X_i^{\pm}) = X_i^{\mp}, \quad \theta(H_i) = H_i$$

and authomorphism σ corresponding to an authomorphism of Dynkin diagram:

$$\sigma(X_i^{\pm}) = X_{i*}^{\pm}, \quad \sigma(H_i) = H_{i*}.$$

We define a q-analog C of the element with the maximal length of the Weyl group by the formula

$$\gamma = \sigma \circ Ad_C \circ \theta .$$

It is not difficult to check that in basic representations the element C has the form

$$C = S q^{\rho/2}$$

where S is the corresponding element of the Weyl group for $U(g)$.

(b) The element $\mathcal{R}$ satisfies the relation

$$(id \otimes \sigma)(\mathcal{R}^{-1}) = (1 \otimes C)(id \otimes \theta)(\mathcal{R})(1 \otimes C^{-1}).$$

DEFINITION 1.2. If $V^{\nu} \subset V^{\lambda} \otimes V^{\mu}$, the projection matrix $K_{\nu}^{\lambda\mu}(q, a): V^{\lambda} \otimes V^{\mu} \to V^{\nu}$ is called the matrix of Klebsch-Gordan coefficient (KGC) (here index a numers the components V^{ν} in $V^{\lambda} \otimes V^{\mu}$ if the multiplicity of V^{ν} is more than one).

The matrices $K_{\nu}^{\lambda\mu}$ and $K_{\nu'}^{\mu\lambda}$ are ortogonal if $\nu \neq \nu'$. They are normalized when $mult(V^{\nu}) = mult(V^{\nu'}) = 1$

$$K_{\nu}^{\lambda\mu}(K_{\nu'}^{\mu\lambda})^{T} = \delta_{\nu\nu'} I_{\nu} \tag{1.14}$$

THEOREM 1.4. The matrices $R^{\lambda\mu}$ and $K_{\nu}^{\lambda\mu}$ for $mult(V^{\nu}) = 1$ satisfy the following relations

$$P^{\mu\lambda} R_{(q)}^{\lambda\mu} P^{\mu\lambda} = S \otimes S R_{(q)}^{\mu\lambda} S^{-1} \otimes S^{-1} = R^{\lambda\mu}(q^{-1})^{-1}, \tag{1.15}$$

$$C^{-1} K_{\nu}^{\lambda\mu}(q)(C \otimes C) = (-1)^{\bar{\nu}} K_{\nu^*}^{\lambda^*\mu^*}(q^{-1}), \tag{1.16}$$

$$K_{\nu}^{\lambda\mu}(q) P^{\mu\lambda} = (-1)^{\bar{\nu}} K_{\nu}^{\mu\lambda}(q^{-1}), \tag{1.17}$$

$$R^{\lambda\mu}(q)^{T} = R^{\lambda\mu}(q). \tag{1.18}$$

Here $\bar{\nu} = 0,1$ is the parity of V^{ν} in $V^{\lambda} \otimes V^{\mu}$ and T is the transposition.

Following theorems are the keys ones to study the matrices $R^{\lambda\mu}$.

THEOREM 1.5. Let $V^{\nu} \subset V^{\lambda} \otimes V^{\mu}$ and $mult(V^{\nu}) = 1$ then

$$R^{\lambda\mu}(q)(K_{\nu}^{\lambda\mu}(q))^{T} = (-1)^{\bar{\nu}} q^{\frac{c(\nu)-c(\lambda)-c(\mu)}{4}} K_{\nu}^{\mu\lambda}(q)^{T} \tag{1.19}$$

$$K_\nu^{\mu\lambda}(q)\,R^{\lambda\mu}(q) = (-1)^{\bar\nu}\, q^{\frac{c(\nu)-c(\lambda)-c(\mu)}{4}}\, K_\nu^{\lambda\mu}(q) \tag{1.20}$$

where $\bar\nu$ is the parity of V^ν in $V^\lambda \otimes V^\mu$ and $c(\nu)$ is the value of Casimir operator of $\mathcal{G}$ on the irreducible representation with hi-ghest weight ν :

$$c = \sum_{i=1}^{r} H^{i^2} + \sum_{\alpha\in\Delta} X_\alpha X_{-\alpha}\,, \quad H^i = (H, \omega^i) \tag{1.21}$$

$$c(\nu) = \nu^2 + 2\rho\nu\,. \tag{1.22}$$

PROOF. From the definition of universal R-matrix follows the equality

$$R^{\lambda\mu}\Delta^{\lambda\mu}(a) = \Delta^{\mu\lambda}(a)\,R^{\lambda\mu} \tag{1.23}$$

where $\Delta^{\lambda\mu}(a) \equiv (\rho^\lambda \otimes \rho^\mu)\Delta(a)$. The representation $\Delta^{\lambda\mu}$ is reducible and $K_\nu^{\lambda\mu}$ are the projectors on the irreducible components:

$$\Delta^{\lambda\mu}(a)\,K_\nu^{\lambda\mu^T} = K_\nu^{\lambda\mu^T}\rho^\nu(a) \tag{1.24}$$

$$\Delta^{\mu\lambda}(a)\,K_\nu^{\mu\lambda^T} = K_\nu^{\mu\lambda^T}\rho^\nu(a)\,. \tag{1.25}$$

From (1.23) and (1.24) we have

$$R^{\lambda\mu}K_\nu^{\lambda\mu^T}\rho_\nu(a) = \Delta^{\mu\lambda}(a)\,R^{\lambda\mu}K_\nu^{\lambda\mu^T}. \tag{1.26}$$

Comparing with (1.25) we obtain

$$R^{\lambda\mu}K_\nu^{\lambda\mu^T} = R_\nu(q)\,K_\nu^{\mu\lambda^T} \tag{1.27}$$

where $R_\nu(q)$ is some function of q, λ, μ, ν . To find $R_\nu(q)$ we consider the limit $q \to 1$. Let $\varphi_\nu^{\lambda\mu}(q)$ be the highest weight vector in $V^\nu \subset V^\lambda \otimes V^\mu$. From (1.27) follows the equality

$$R^{\lambda\mu}(q)\,\varphi_\nu^{\lambda\mu}(q) = R_\nu(q)\,\varphi_\nu^{\mu\lambda}(q) \tag{1.28}$$

At $q \to 1$ we have

$$R^{\lambda\mu} = P^{\lambda\mu}(1+(q-1)\,r + O((q-1)^2)), \tag{1.29}$$

$$r = \frac{1}{2}\sum_i H^i \otimes H^i + \sum_{\alpha\in\Delta_+} X_\alpha \otimes X_{-\alpha} \tag{1.30}$$

$$\varphi_\nu^{\lambda\mu}(q) = \varphi^{\lambda\mu} + (q-1)\,\psi_\nu^{\lambda\mu} + O((q-1)^2), \tag{1.31}$$

$$R_\nu(q) = (-1)^{\bar\nu}(1+(q-1)\,s_\nu + O((q-1)^2)). \tag{1.32}$$

Here $r \in g \otimes g$ $(g \subset U(g))$ is the so-called classical r-matrix [16,c], $\varphi_\nu^{\lambda\mu}$ is the $h.w.v.$ of g, and $\psi_\nu^{\lambda\mu}$ and s_ν some unknown vectors and constants.

Theorem 1.4 implies symmetry relations for $\varphi_\nu^{\lambda\mu}$ and $\psi_\nu^{\lambda\mu}$:

$$P^{\lambda\mu}\varphi_\nu^{\lambda\mu} = (-1)^{\bar\nu}\varphi_\nu^{\mu\lambda}, \quad P^{\lambda\mu}\psi_\nu^{\lambda\mu} = (-1)^{\bar\nu+1}\psi_\nu^{\mu\lambda}. \tag{1.33}$$

Comparing the coefficients at $(q-1)$ in (1.28) we obtain the equation for s_ν:

$$P^{\lambda\mu} r\,\varphi_\nu^{\lambda\mu} + P^{\lambda\mu}\psi_\nu^{\lambda\mu} = (-1)^{\bar\nu}\psi_\nu^{\mu\lambda} + (-1)^{\bar\nu} s_\nu\,\varphi_\nu^{\mu\lambda}$$

Multiplying this equality by $(\varphi_\nu^{\mu\lambda})^T$ from the left and using the symmetry relations (1.33) we obtain the following expression for s_ν

$$s_\nu = (\varphi_\nu^{\lambda\mu})^T\, r\, \varphi_\nu^{\lambda\mu}.$$

Using the symmetry relations (1.33) and the property of highest weight vectors the value of S_ν one can express through the Casimir operators

$$S_\nu = \frac{1}{4}\left(c(\nu) - c(\lambda) - c(\mu)\right). \tag{1.34}$$

From the symmetry relations (1.15) and from the definition of $R^{\lambda\mu}$ we have

$$R_\nu(q)\, R_\nu(q^{-1}) = 1 \tag{1.35}$$

$$R_\nu(q) \simeq (-1)^{\bar{\nu}} q^{\alpha}, \quad q \to \infty \tag{1.36}$$

Moreover, the function $R_\nu(q)$ has no poles and zeros for finite values of q. From Liouville theorem it follows that there is only one function $R_\nu(q)$ satisfying the relations (1.29), (1.34)-(1.36)

$$R_\nu(q) = (-1)^{\bar{\nu}} q^{\frac{c(\nu)-c(\lambda)-c(\mu)}{4}}.$$

The equality (1.20) is the transposition of (1.19). The Theorem is proved.

CONSEQUENCE. When all irreducible components in $V^\lambda \otimes V^\mu$ have the multiplicity equal to one the matrices $R^{\lambda\mu}$ and $(R^{\lambda\mu})^{-1}$ have the following decompositions:

$$R^{\lambda\mu}(q) = \sum_{\nu\in\lambda\otimes\mu} (-1)^{\bar{\nu}} q^{\frac{c(\nu)-c(\lambda)-c(\mu)}{4}} \mathcal{P}_\nu^{\lambda\mu}(q) \tag{1.37}$$

$$(R^{\lambda\mu}(q))^{-1} = \sum_{\nu\in\lambda\otimes\mu} (-1)^{\bar{\nu}} q^{-\frac{c(\nu)-c(\lambda)-c(\mu)}{4}} \mathcal{P}_\nu^{\mu\lambda}(q) \tag{1.38}$$

where

$$\mathcal{P}_\nu^{\lambda\mu}(q) = K_\nu^{\mu\lambda}(q)^T K_\nu^{\lambda\mu}(q) \tag{1.39}$$

These R-matrices will be called R-matrices with simple spectrum.

Let us consider the space $V^\tau \otimes V^\lambda \otimes V^\mu$ and the matrices $R_{12}^{\tau\lambda} = R^{\tau\lambda} \otimes 1 : V^\tau \otimes V^\lambda \otimes V^\mu \to V^\lambda \otimes V^\tau \otimes V^\mu$, $R_{23}^{\tau\mu} = 1 \otimes R^{\tau\mu} : V^\lambda \otimes V^\tau \otimes V^\mu \to V^\lambda \otimes V^\mu \otimes V^\tau$, $(K_\nu^{\lambda\mu})_{23} = 1 \otimes K_\nu^{\lambda\mu} : V^\tau \otimes V^\lambda \otimes V^\mu \to V^\tau \otimes V^\nu$, $(K_\nu^{\lambda\mu})_{12} = K_\nu^{\lambda\mu} \otimes 1 : V^\lambda \otimes V^\mu \otimes V^\tau \to V^\nu \otimes V^\tau$, $R^{\tau\nu} : V^\tau \otimes V^\nu \to V^\nu \otimes V^\tau$ and similar matrices $R_{23}^{\mu\tau}$, $R_{12}^{\lambda\tau}$, $(K_\nu^{\lambda\mu})_{23}$, $(K_\nu^{\lambda\mu})_{23}$, $(K_\nu^{\lambda\mu})_{12}$, $R^{\nu\tau}$.

THEOREM 1.6. The matrices introduced above satisfy the following relations

$$R^{\tau\nu}(K_\nu^{\lambda\mu})_{23} = (K_\nu^{\lambda\mu})_{12} R_{23}^{\tau\mu} R_{12}^{\tau\lambda} \tag{1.40}$$

$$R^{\nu\tau}(K_\nu^{\lambda\mu})_{12} = (K_\nu^{\lambda\mu})_{23} R_{12}^{\lambda\tau} R_{23}^{\mu\tau} \tag{1.41}$$

PROOF. The restriction of the relation (1.8) on the irreducible representation gives:

$$\sum_s (\rho_\nu(e_s) \otimes \rho_\tau(e^s))(K_\nu^{\lambda\mu} \otimes I) = (K_\nu^{\lambda\mu} \otimes I) \cdot \tag{1.42}$$

$$\cdot \sum_{s,t} \rho_\lambda(e_t) \otimes \rho_\mu(e_s) \otimes \rho_\tau(e^t e^s).$$

Multiplying this formula by the permutation operator $P^{\nu\tau}$: $V^\nu \otimes V^\tau \to V^\tau \otimes V^\nu$ from the left and using the equality

$$P^{\nu\tau}(K_\nu^{\lambda\mu} \otimes I) P_{13} P_{23} = I_\tau \otimes K_\nu^{\lambda\mu}$$

we obtain (1.41). The relation (1.40) is proved similarly.

Crossing-symmetry of the universal R-matrices (1.8d) implies crossing-symmetry of the matrices $R^{\lambda\mu}$ and $K_\nu^{\lambda\mu}$:

THEOREM 1.7. The matrices $R^{\lambda\mu}$ and $K_\nu^{\lambda\mu}$ satisfy the following crossing-symmetry relations:

$$(P^{\mu\lambda}R^{\lambda\mu})^{-1} = (C\otimes 1)(P^{\mu\lambda}R^{\lambda\mu})^{t_1}(C^{-1}\otimes I) \qquad (1.43)$$

$$\sum_j (K_\nu^{\lambda\mu})_k^{ij}(C)_j^{l} = \sqrt{\frac{\chi_\lambda}{\chi_\nu}}\,(K_\lambda^{\nu\mu^*})_i^{kl} \qquad (1.44)$$

$$\sum_i (K_\nu^{\lambda\mu})_k^{ij}(C)_i^{l} = \sqrt{\frac{\chi_\mu}{\chi_\nu}}\,(K_\mu^{\lambda^*\nu})_j^{lk} \qquad (1.45)$$

where t_1 is trasposition over the first space, λ^* is the highest weight (h.w.) vector conjugated with λ by Cartan automorphism,

$$C = Sq^{-\frac{\rho}{2}}\,, \quad \chi_\lambda = \mathrm{tr}_{V^\lambda}(q^{-\rho}). \qquad (1.46)$$

Using the explicit formula for coproduct in it is not difficult to prove the first statement of the following theorem.

THEOREM 1.8. (a) The matrix

$$(K^{\lambda\lambda^*})^{ij} = \frac{1}{\sqrt{\chi_\lambda}}\, C^{ji} \qquad (1.47)$$

is the projector on to the one-dimensional subrepresentation in $V^\lambda\otimes V^{\lambda^*}$

(b) The matrices $R^{\lambda\lambda^*}$ and $R^{\lambda\lambda}$ satisfy the relations

$$R^{\lambda\lambda^*}(K^{\lambda\lambda^*})^T = q^{-\frac{c(\lambda)}{2}}(-1)^{[\lambda]}(K^{\lambda^*\lambda})^T \qquad (1.48)$$

$$tr_2((I\otimes q^{-\rho})(R^{\lambda\lambda})^{\pm 1}) = q^{\pm\frac{c(\lambda)}{2}}\cdot I . \qquad (1.49)$$

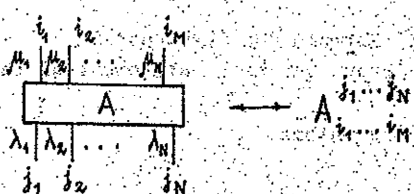
Here $[\lambda] = 0,1$, tr_2 is the matrix trace over the second space in $V^\lambda \otimes V^\lambda$.

The proof of the second statement of this theorem is given in the next section, where we introduce graphical representation for the R-matrices and for KGC.

2. Graphical representation of the R-matrices

The graphical representation given below is very useful for showing the relations between the R-matrices and KGC. This representation shows the combersome formulae in terms of simple figures and makes these relations intuitively understandable.

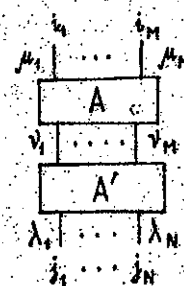
i) The matrix A mapping the space $V^{\lambda_1}\otimes\ldots\otimes V^{\lambda_N}$ in $V^{\mu_1}\otimes\ldots\otimes V^{\mu_M}$ is represented by $(N+M)$-edged diagram:



The indices $\{i\}$, $\{j\}$ and $\{\lambda\}$, $\{\mu\}$ are called states on the edges and the colours of the edges correspondingly. The states numerate the bases in the spaces V^λ ($i_\alpha = 1,\ldots, dim\, V^{\lambda_\alpha}$, $j_\alpha = 1,\ldots, dim\, V^{\lambda_\alpha}$).

ii) The product of the matrices A and A' is represented by an ordered from top to bottom combination of the diagrams A and A'. If $A': V^{\lambda_1}\otimes\ldots\otimes V^{\lambda_N} \rightarrow$

$V^{\nu_1}\otimes\ldots\otimes V^{\nu_L}$ and $A': V^{\nu_1}\otimes\ldots\otimes V^{\nu_L}\longrightarrow$
$\longrightarrow V^{\mu_1}\otimes\ldots\otimes V^{\mu_M}$ the diagram AA' is:



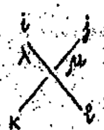
$$\longleftrightarrow (AA')^{j_1\ldots j_N}_{i_1\ldots i_M} = \sum_{k_1\ldots k_L} A^{k_1\ldots k_L}_{i_1\ldots i_M} A'^{j_1\ldots j_N}_{k_1\ldots k_L} \quad . \tag{2.2}$$

The junction of the edges corresponds to summation over the states on these edges.

iii) The unit matrix acting in V^{λ} is represented by vertical line:

$$\longleftrightarrow \delta_{ij}\ , \quad i,j=1,\ldots,\dim V^{\lambda}\ . \tag{2.3}$$

Let us represent the matrices $R^{\lambda\mu}$, $(R^{\lambda\mu})^{-1}$, $K^{\lambda\mu}$, C and C^{-1} by the following diagrams:

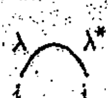


$$\longleftrightarrow (R^{\lambda\mu})^{k\ell}_{ij} = (e^t_i\otimes e^t_j)\,R^{\lambda\mu}(e_k\otimes e_\ell) \tag{2.4}$$

$$\longleftrightarrow ((R^{\mu\lambda})^{-1})^{k\ell}_{ij} = (e^t_i\otimes e^t_j)(R^{\mu\lambda})^{-1}(e_k\otimes e_\ell) \tag{2.5}$$

 $\longleftrightarrow (K^{\lambda\mu}_{\nu}(q,a))^{ij}_{k}$ $\longleftrightarrow$  (2.6)

 $\longleftrightarrow (C^{-1})^{ij}$,  $\longleftrightarrow (C^{-1})^{ji}$ (2.7)

 $\longleftrightarrow C_{ij}$,  $\longleftrightarrow C_{ji}$ (2.8)

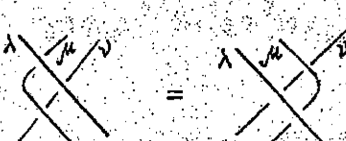
Further on we assume that each line acquires the colour λ which defines the states on this line.

Using the rules (2.1)-(2.8) we obtain the graphical representation of the formulae from section 1:

i) the relation $R^{\lambda\mu}(R^{\mu\lambda})^{-1} = (R^{\mu\lambda})^{-1} R^{\lambda\mu}$ is represented by the following graphical equality

 (2.9)

ii) the relation (1.12) has the following representation

 (2.10)

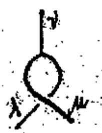
iii) the representation of $\mathcal{P}^{\lambda\mu}_{\nu}$ from (1.39) is


$$\longleftrightarrow \quad \mathcal{T}_{\nu}^{\lambda\mu} \qquad (2.11)$$

iv) the theorem 1.5 is represented by the equalities


$$=$$

$$(-1)^{\nu} q^{\frac{c(\nu)-c(\lambda)-c(\mu)}{4}} , \qquad (2.12)$$


$$=$$

$$(-1)^{\nu} q^{\frac{c(\nu)-c(\lambda)-c(\mu)}{4}} \qquad (2.13)$$

v) the representations of the decompositions (1.37) and (1.38) are:

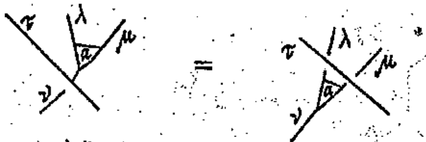

$$= \sum_{\nu}$$

$$(-1)^{\bar{\nu}} q^{\frac{c(\nu)-c(\lambda)-c(\mu)}{4}} , \qquad (2.14)$$


$$= \sum_{\nu}$$

$$(-1)^{\bar{\nu}} q^{-\frac{c(\nu)-c(\lambda)-c(\mu)}{4}} \qquad (2.15)$$

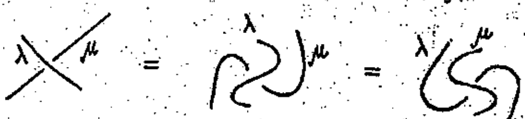
vi) the theorem 1.6 is represented by the following equalities



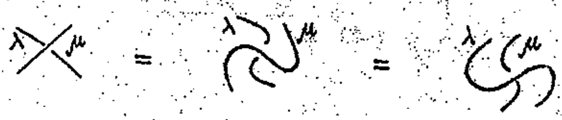
(2.16)

(2.17)

vii) crossing-symmetry means that the diagrams representing $R^{\lambda\mu}$ and $(R^{\lambda\mu})^{-1}$ can be rotated by 90°:

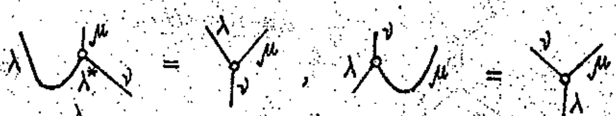


(2.18)

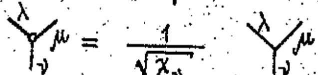


(2.19)

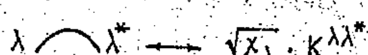
viii) the representation of the crossing symmetries of $K_{\nu}^{\lambda\mu}$ is



(2.20)

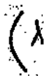
where 

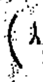
ix) the theorem 1.8 is represented by the following graphical equalities:



(2.21)

(2.22)

 $=$  $\cdot\, q^{-\frac{c(\lambda)}{2}}$ (2.23)

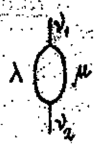
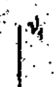
 $=$  $\cdot\, q^{\frac{c(\lambda)}{2}}$ (2.24)

x) Taking the crossing conjugation of the decompositions (1.37) and (1.38) we obtain the equalities

 $= \sum_{\nu\in\mu^*\otimes\lambda} (-1)^{\bar{\nu}} q^{-\frac{c(\nu)-c(\lambda)-c(\mu)}{4}} \cdot$  (2.25)

 $= \sum_{\nu\in\mu^*\otimes\lambda} (-1)^{\bar{\nu}} q^{\frac{c(\nu)-c(\lambda)-c(\mu)}{4}} \cdot$  (2.26)

xi) the orthogonality relations (1.14) have the following representation

 $=$  $\delta_{\nu_1, \nu_2}\,.$ (2.27)

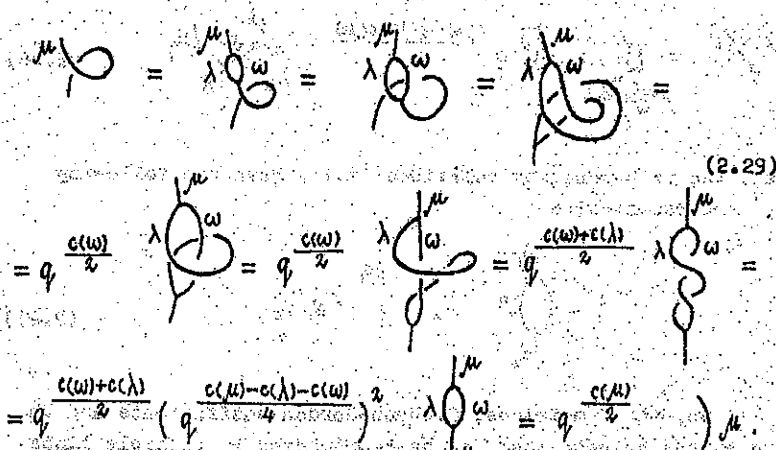
So, the R-matrices, Clebsch-Gordan coefficients and the relations between them can be represented by flexible lines on the surface. Moreover, these lines can be glued into a graph with three edged vertices. These graphs can be deformed according to the rules i)-xi).

This representation of the matrices is very convenient for manipulations with the matrices $R^{\lambda\mu}$, $K_{\nu}^{\lambda\mu}$ and we will use it many times.

THE PROOF OF THE THEOREM 1.8. Using the crossing-symmetry of R-matrices (1.43) and the relation

$$(C^{-1})^{t}\, C = (-1)^{[\lambda]}\, q^{-\rho} \tag{2.28}$$

we obtain that the relations (1.48) and (1.49) are equivalent. For the matrices $R^{\lambda\lambda^*}$ with simple spectrum the relation (1.48) follows from the theorem 1.5. In the general case we use an induction procedure. The relation (1.49) takes place if λ is the basic representation. Let us prove that it holds for any $\mu \subset \lambda \otimes \omega$ if it holds for λ. To prove this statement it is sufficient to consider a set of equalities following from (2.4)-(2.7)



The equality (1.49) is proved. Using the crossing-symmetry of $R^{\lambda\mu}$ we obtain (1.48) from (1.49).

3. The q-analog of Brauer-Weyl duality

In this section we study the structure of centralizater of $U_q(\mathfrak{g})$ in $(V^{\omega})^{\otimes N}$, where V^{ω} is the basic representation.

Let $V^{\{\lambda\}} = V^{\lambda_1} \otimes \ldots \otimes V^{\lambda_N}$ and $V^{\{\lambda\sigma\}} = V^{\lambda_{\sigma_1}} \otimes \ldots \otimes V^{\lambda_{\sigma_N}}$, where $\sigma \in S_N$ is the permutation of the set $(1, \ldots, N)$.

DEFINITION 3.1. An algebra of the maps $\psi : V^{\{\lambda\sigma\}} \to V^{\{\lambda\sigma'\}}$ commuting with the action of $V_q(\mathfrak{g})$ in these spaces is called the centralizer of $U_q(\mathfrak{g})$ in $V^{\lambda_1} \otimes \ldots \otimes V^{\lambda_N}$ and denoted by $C^{\lambda_1 \ldots \lambda_N}(\mathfrak{g})$.

Here we will consider only the algebras $C^{\omega \ldots \omega}(\mathfrak{g}, q) \equiv C^{\omega}_N(\mathfrak{g}, q)$ *) for the representations ω, when multiplicity of the irreducible components in the decomposition

$$V^{\lambda} \otimes V^{\omega} = \sum_{\mu} \otimes V^{\mu} \tag{3.1}$$

is equal to one for any IRR V^{λ}.

PROPOSITION 3.1. The algebra $C^{\omega}_N(\mathfrak{g}, q)$ is simple for general q and is generated by the elements

$$g_i = 1 \otimes \ldots \otimes R^{\omega\omega}_{(i,i+1)} \otimes \ldots \otimes 1 \tag{3.2}$$

---

*) We will write $C^{\omega}_N(\mathfrak{g})$ when it will not be ambiguous.

if the spectrum of the Casimir operator is simple in the decomposition (3.1)

The proof of this proposition is based on the spectral decomposition of the matrix $R^{\omega\omega}$.

The representations of $C_N^{\omega}(q)$ are given by the decomposition of $(V^{\omega})^{\otimes N}$ on the $U_q(g)$ -irreducible components:

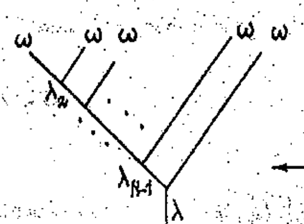
$$(V^{\omega})^{\otimes N} = \sum_{\lambda} W_{\lambda} \otimes V^{\lambda} . \qquad (3.3)$$

The space W_λ describes the multiplicity of the h.w. λ in (3.3): $\dim W_\lambda = mult\,(V^{\lambda})$.

PROPOSITION 3.2. The representations W_λ in (3.3) form the list of all irreducible representations of $C_N^{\omega}(q)$.

Let us use now the graphical representation of the matrices $R^{\lambda\mu}$ and $K_{\nu}^{\lambda\mu}$ developed in section 2 to describe the basis in W_λ .

PROPOSITION 3.3. The elements



$\longleftrightarrow K_\lambda(q,a)$, (3.4)

where $a = (\omega, \lambda_2, \dots, \lambda_{N-1}, \lambda)$ and $V^{\lambda_i} \subset V^{\omega} \otimes V^{\lambda_{i-1}}$ form the orthogonal basis in W_λ . More precisely

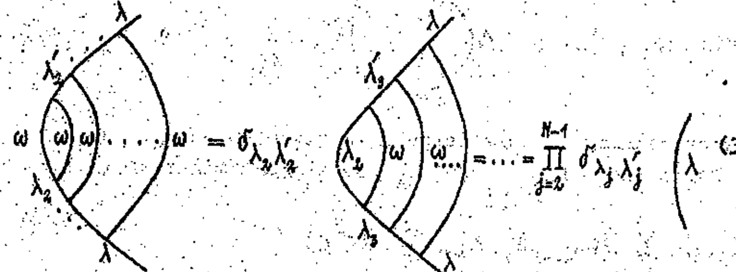
$$K_\lambda(q,a) = E_\lambda(a) \otimes I_\lambda$$

where $\{E_\lambda(a)\}$ is the basis in W_λ and I_λ is the unit matrix in V^{λ}.

The orthogonality of the elements $K_\lambda(q,a)$ means that

$$K_\lambda(q,b)K_\lambda(q,a)^t = \delta_{a,b}\cdot I \quad , \tag{3.5}$$

where $a=(\omega,\lambda_2,\dots,\lambda_{N-1},\lambda)$, $b=(\omega,\lambda_2',\dots,\lambda_{N-1}',\lambda)$. K_λ^t is the transposition of the matrix K_λ. The relation (3.5) is proved by the following equalities founded on the orthogonality of CGC:



(3.6)

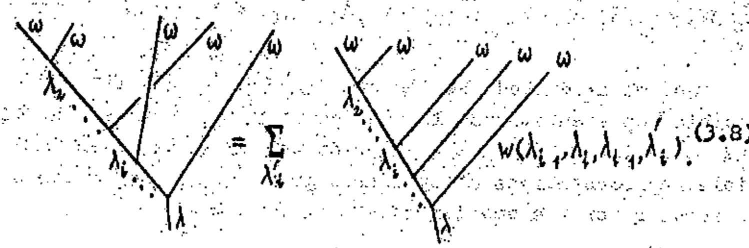
The completeness of this basis in W_λ is evident.

PROPOSITION 3.4. The action of the elements g_i in basis have the following form:

$$g_i K_\lambda(q,a) = \sum_{\lambda_i'} W(\lambda_{i-1},\lambda_i,\lambda_{i+1},\lambda_i') K_\lambda(q,a') \tag{3.7}$$

where $a=(\omega,\lambda_2,\dots,\lambda_{N-1},\lambda)$, $a'=(\omega,\lambda_2,\dots,\lambda_i',\dots,\lambda_{N-1},\lambda)$ and $W(\lambda_{i-1},\lambda_i,\lambda_{i+1},\lambda_i')$ are some constants calculated below.

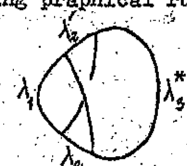
The relations (3.7) have a simple graphical representation:



(3.8)

Using the orthogonality of the basis $K_\lambda(q,a)$ we obtain a formula for the coefficients W :

PROPOSITION 3.5. The coefficients W in (3.8) are calculated by the following graphical rule:

$W(\lambda_1,\lambda_2,\lambda_3;\lambda_2') =$ 

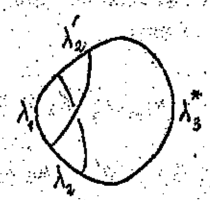
$\frac{1}{\chi_{\lambda_3}}$. (3.9)

An exact meaning of this formula is given in section 8.

The action of g_i^{-1} in K_λ basis is similar to (3.7):

$$g_i^{-1} K_\lambda(a) = \sum_{\lambda_i'} \overline{W}(\lambda_{i-1},\lambda_i,\lambda_{i+1},\lambda_i')\, K_\lambda(a') \tag{3.10}$$

where the coefficient $\overline{W}$ has the following form:

$\overline{W}(\lambda_1,\lambda_2,\lambda_3,\lambda_2') =$ 

$\frac{1}{\chi_{\lambda_3}}$ (3.11)

From $g_i g_i^{-1} = 1$ follows the relation

$$\sum_{\lambda_i'} W(\lambda_{i-1},\lambda_i,\lambda_{i+1},\lambda_i')\, \overline{W}(\lambda_{i-1}\,\lambda_i',\lambda_{i+1}\,\lambda_i'') = \delta_{\lambda_i \lambda_i''} . \tag{3.12}$$

So, we described the algebra $C_N^\omega(q)$ and its irreducible representations. It is necessary to note that the basis (3.4) is exactly a q-analog of the Young basis in irreducible representations of symmetric group S_N . This basis is regular for the embeddings

$$C_1^\omega(q) \subset C_2^\omega(q) \subset \dots \subset C_N^\omega(q) \tag{3.13}$$

where $C_N^{\omega}(q)$ is formed by the elements $1,\dots,g_{N-1}$ and $C_{N-1}^{\omega}(q)$ is formed by the elements $1,\dots,g_{N-2}$. Indeed, if we consider the restriction

$$W_\lambda(C_N^{\omega})\Big|_{C^{\omega}_{N-1}} = \sum_{\lambda'} \oplus W_\lambda(C^{\omega}_{N-1}) \tag{3.14}$$

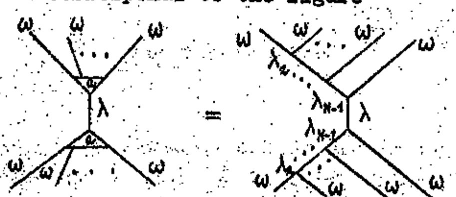
the basis in $W_{\lambda'}(C^{\omega}_{N-1})$ will be formed by the elements $K_\lambda(a_{\lambda'})$, where $a_{\lambda'} = (\omega,\dots,\lambda_{N-2},\lambda',\lambda)$.

Let us also note that the decompositions (3.14) determine the graph of the algebra $C_N^{\omega}(q)$ [15]. It is evident that really this graph is fixed by decomposition (3.3).

Let us consider now the q-analog of Young symmetrizers. For this purpose we introduce the matrices $\mathcal{P}_\lambda(q,a)$: $(V^{\omega})^{\otimes N} \longrightarrow (V^{\omega})^{\otimes N}$, $\mathrm{Im}(\mathcal{P}_\lambda(q,a)) \simeq V^\lambda$:

$$\mathcal{P}_\lambda(q,a) = K^{\dagger}_\lambda(q,a) K_\lambda(q,a) = (E_\lambda(a)\otimes E^{\dagger}_\lambda(a))\otimes I_\lambda . \tag{3.15}$$

This matrix corresponds to the figure

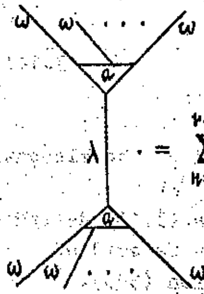


(3.16)

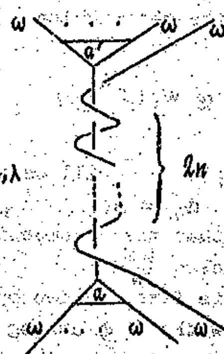
Proposition 3.3 follows that the set $\mathcal{P}_\lambda(q,a)$ forms an orthogonal set of projectors in $(V^{\omega})^{\otimes N}$:

$$\mathcal{P}_\lambda(q,a)\,\mathcal{P}_\mu(q,b) = \delta_{\lambda\mu}\,\delta_{a,b} . \tag{3.17}$$

THEOREM 3.1. The elements $\mathcal{P}_\lambda$ satisfy the following relations:



$$= \sum_{n=0}^{\nu-1} q_{n,\lambda}$$



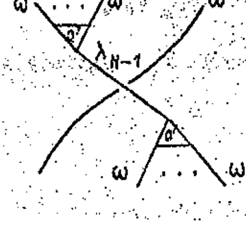
$2n$ crosses (3.18)

where $a = (\omega, \lambda_2, \ldots, \lambda_{N-1}, \lambda)$, $a' = (\omega, \lambda_2, \ldots, \lambda_{N-1})$, $q_{n,\lambda}$ are the matrix elements of the matrix :

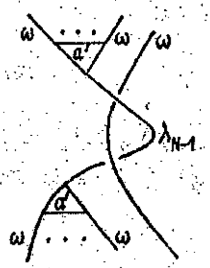
$$A = \left(q^{n\frac{c(\lambda)-c(\lambda_{N-1})-c(\omega)}{2}} \right)_{\lambda\in\lambda_{N-1}\otimes\omega,\ n=0,\ldots,\nu-1} \qquad (3.19)$$

ν is the number of irreducible components in $\lambda_{N-1}\otimes\omega$, the index n numbers the rows of A and λ numbers the colomns of A.

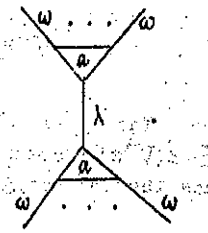
PROOF. Let us consider the matrix $R^{\lambda_{N-1}\omega}(q, a')$ corresponding to the figure



From the theorem 1.5 we have the decomposition of matrix $R^{\lambda_{N-1}\omega}(q,a')R^{\omega\lambda_{N-1}}(q,a')$



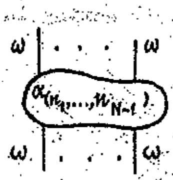
$$= \sum_{\lambda\in\lambda_{N-1}\otimes\omega} q^{\frac{c(\lambda)-c(\lambda_{N-1})-c(\omega)}{2}}$$



Taking powers of degrees $n = 0,1,\dots,r-1$ of this equality we obtain a system for the projectors $\mathcal{P}_\lambda(q,a)$. Solving this system we obtain the relation (3.18).

Considering the relation (3.18) as an inductive process for calculating $\mathcal{P}_\lambda(q,a)$ we obtain the explicit expression for $\mathcal{P}_\lambda(q,a)$:

$$\mathcal{P}_\lambda(q,a)=\sum_{n_1=0}^{r_1-1}\cdots\sum_{n_{N-1}=0}^{r_{N-1}-1} q_{n_1\lambda}\,q_{n_2\lambda_{N-1}}\cdots q_{n_{N-1},\omega}\;\times$$

$\times$  (3.19)

Here $\alpha_{(n_1,\dots,n_{N-1})}$ is a braid (see section 8) where the string with number i turns round the strings with numbers $1,\dots,i-1$ (from left to right) $2n_i$ times moving counter-clockwise and form the top to the bottom.

Now we can describe the structure of the matrices $R^{\lambda\mu}$.

Consider a decomposition of $V^\lambda \otimes V^\mu$ into the sum of $U_q(\mathfrak{g})$ irreducible components:

$$V^\lambda \otimes V^\mu = \sum_\nu W_\nu^{\lambda\mu} \otimes V^\nu \tag{3.20}$$

The matrix $R^{\lambda\mu}$ is an element of $End(V^\lambda \otimes V^\mu) \to End(V^\lambda \otimes V^\mu)$ commuting with the action of $U_q(\mathfrak{g})$. Hence it has the following form:

$$R^{\lambda\mu} = \sum_\nu R_\nu^{\lambda\mu} \otimes I_\nu \tag{3.21}$$

where $R_\nu^{\lambda\mu} : W_\nu^{\lambda\mu} \to W_\nu^{\lambda\mu}$.

Let W_ν be the C^ω_{N+M} -irreducible module. The decomposition of this module into the sum of $C^\omega_N \times C^\omega_M$ -irreducible components is:

$$W_\nu \Big|_{C^\omega_N \times C^\omega_M} = \sum_{\lambda\mu} W_\nu^{\lambda\mu} \otimes W_\lambda \otimes W_\mu \tag{3.22}$$

where the spaces $W_\nu^{\lambda\mu}$ are the same as in (3.20).

Let $R^{(N,M)}$ be the matrix acting in $(V^\omega)^{\otimes(N+M)}$ of the form:

$R^{(N,M)} \longleftrightarrow$  (3.23)

In accordance with (3.3) this matrix has the following decomposition

$$R^{(N,M)} = \sum_\nu R_\nu^{(N,M)} \otimes I_\nu$$

$$R_{\nu}^{(N,M)} = \sum_{\lambda,\mu} R_{\nu}^{\lambda\mu} \otimes \sigma^{\lambda\mu} \qquad (3.25)$$

where $\sigma^{\lambda\mu}: W_{\lambda} \otimes W_{\mu} \to W_{\mu} \otimes W_{\lambda}$ is the permutation operator; $\sigma^{\lambda\mu}(f_{\lambda} \otimes g_{\mu}) = g_{\mu} \otimes f_{\lambda}$.

The decomposition (3.24) and (3.25) is the generalization of the formula (1.37) on matrices $R^{\lambda\mu}$ with arbitrary λ and μ. The formula (3.25) gives the method for calculating the matrices $R^{\lambda\mu}$.

4. $\mathfrak{g} = s\ell(n)$.

The basic representation of $V_q(s\ell(n))$ have the h.w. ω_1 and $V^{\omega_1} \simeq \mathbb{C}^n$. The matrix elements of the generators do not depend on q in this representation:

$$\pi(X_i^+) = E_{i\,i+1}, \quad \pi(X_i^-) = E_{i+1\,i}, \quad \pi(H_i) = E_{ii} - E_{i+1,i+1} \qquad (4.1)$$

where $(E_{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta}$ are the basic matrices in $\mathbb{C}^n$.

The Tensorial square of V^{ω_1} is decomposed into the sum of two irreducible components

$$(V^{\omega_1})^{\otimes 2} = V^{2\omega_1} \oplus V^{\omega_2}. \qquad (4.2)$$

The KGC of this decomposition are determined by the relations:

$$e_{ij}^{2\omega_1} = \frac{1}{\sqrt{1+q^{-1}}}(e_i \otimes e_j + q^{-\frac{1}{2}} e_j \otimes e_i), \; i<j, \; e_{ii}^{2\omega_1} = e_i \otimes e_i \qquad (4.3)$$

---

*) From this point we will use the normalization and the notation of books [16] for the highest weights and the roots of simple Lie algebras.

$$e^{\omega_2}_{ij} = \frac{1}{\sqrt{1+q^{-1}}} \left(e_i \otimes e_j - q^{-1/2} e_j \otimes e_i \right), \quad i<j \tag{4.4}$$

where $e^{2\omega_1}_{ij}$ and $e^{\omega_2}_{ij}$ are the basices in $V^{2\omega_1}$ and in V^{ω_2} respectively.

From the theorem 1.5 we find the basic R-matrix for $U_q(gl(n))$:

$$R^{\omega_1\omega_1} = \mathcal{P}^{\omega_1\omega_1}_{2\omega_1} q^{1/2} - \mathcal{P}^{\omega_1\omega_1}_{\omega_2} q^{-1/2} \tag{4.5}$$

or using (4.3), (4.4)

$$R^{\omega_1\omega_1} = q^{1/2} \sum_{i=1}^{n} E_{ii} \otimes E_{ii} + \sum_{i\neq j}^{n} E_{ij} \otimes E_{ji} -$$

$$- (q^{-1/2} - q^{1/2}) \sum_{j>i} E_{jj} \otimes E_{ii} . \tag{4.6}$$

In the form (4.6) the matrix $R^{\omega_1\omega_1}$ was found by Jimbo [5].

For $g = gl(n)$ the decomposition of $V^\lambda \otimes V^{\omega_1}$ is well known:

$$V^\lambda \otimes V^{\omega_1} = \sum_{K=1}^{n}{}' \oplus V^{\lambda^{(K)}} \tag{4.7}$$

where $\lambda^{(K)} = (\lambda_1, \ldots, \lambda_K + 1, \lambda_{K+1}, \ldots, \lambda_n)$ if $\lambda = (\lambda_1, \ldots, \lambda_n)$, $(\lambda_i \in \mathbb{Z}_+ , \lambda_i \geq \lambda_{i+1})$ and the sum is taken only over K satisfying the condition $\lambda_K > \lambda_{K+1}$. Using the rule (4.7) and the theorem 3.1 one can calculate the elements $\mathcal{P}_\lambda(q,a)$. The representations $\lambda = K\omega_1$ and $\lambda = \omega_K$ are contained in $(V^{\omega_1})^{\otimes K}$ with multiplicity equal to one. It is not difficult to calculate the corresponding projectors $\mathcal{P}_\lambda$:

$$\mathcal{P}_{\kappa\omega_1}=\frac{q^{\frac{\kappa(\kappa-1)}{4}}}{[\kappa]!}\sum_{w\in S_\kappa} q^{\frac{\ell(w)}{2}}\pi_w , \qquad (4.8)$$

$$\mathcal{P}_{\omega_\kappa}=\frac{q^{-\frac{\kappa(\kappa-1)}{4}}}{[\kappa]!}\sum_{w\in S_\kappa} (-q)^{-\frac{\ell(w)}{2}}\pi_w , \qquad (4.9)$$

where the sum taken is over the element of symmetric group S_κ , $\pi_{s_i} = g_i$ if s_i is the elementary transposition in S_κ , the elements g_i are defined by (3.2) and $\pi_w = \pi_{s_1} \dots \pi_{s_\kappa}$ if $w = s_1 \dots s_\kappa$ is the representation of w in the nonreducible product of the transpositions, $[\kappa]! = [\kappa] \dots [1]$, $[\kappa] = (q^{\kappa/2} - q^{-\kappa/2})/(q^{1/2} - q^{-1/2})$.

The formulae (4.8), (4.9) were obtained in [5] as the q-analoges of the Young sym (antisym) metrisators. The following teorem describes the connection between the algebra $C_N^{\omega_1}(gl(n))$ and Hecke algebra $\mathcal{H}_N(q)$. The last one is the assotiative algebra with units generated by the elements $1, \hat{g}_i$, $i=1,\dots,N-1$ satisfying the following relations:

$$\hat{g}_i \hat{g}_{i+1} \hat{g}_i = \hat{g}_{i+1} \hat{g}_i \hat{g}_{i+1}$$
$$\hat{g}_i \hat{g}_j = \hat{g}_j \hat{g}_i , \quad |i-j|>1 \qquad (4.10)$$
$$\hat{g}_i^2 = (q-1)\hat{g}_i + q .$$

THEOREM 4.1. The algebra $C_N^{\omega_1}(gl(n))$ is the factor of Hecke algebra $\mathcal{H}_N(q)$ over the ideal J_N formed by the elements

$$\sum_{w\in S_{n+1}} (-q)^{-\frac{\ell(w)}{2}} \pi_w = 0$$

where π_w are the same as in (4.8) and (4.9). If $n+1>N$ $C_N^{\omega_1}(g\ell(n)) \simeq \mathcal{H}_N(q)$. The generators g_i of $C_N^{\omega_1}(g\ell(n))$ correspond to the generators $\hat{g}_i$ of Hecke algebra $\hat{g}_i = q^{1/2} g_i$.

This theorem is equivalent to the fact that in the space $(V^{\omega_1})^{\otimes N}$ the representation of Hecke algebra $\alpha(\hat{g}_i)=q^{1/2}g_i$ (this fact was noted by Jimbo [5]) is defined. The theorem 4.1 means that the irreducible components of this representation of $\mathcal{H}_N(q)$ have the Young diagrams λ with no more than n rows.

An explicit expression for matrix elements of g_i in the basis (3.4) of the representation W_λ can be found from (3.9). Omitting the details, we give the answer:

$$W(\lambda,\lambda+e_k,\lambda+e_k+e_\ell,\lambda+e_\ell)=\frac{\sqrt{(q^{\frac{d+1}{2}}-q^{-\frac{d+1}{2}})(q^{\frac{d-1}{2}}-q^{-\frac{d-1}{2}})}}{(q^{\frac{1}{2}}+q^{-\frac{1}{2}})(q^{\frac{d}{2}}-q^{-\frac{d}{2}})}, \qquad (4.11)$$

$$W(\lambda,\lambda+e_k,\lambda+e_k+e_\ell,\lambda+e_k)=\begin{cases}\dfrac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{q^{d}-1}, & k\neq \ell\\ q^{-\frac{1}{2}}, & k=\ell\\ -q^{\frac{1}{2}}, & \ell=k+1,\ \lambda_k=\lambda_\ell\end{cases} \qquad (4.12)$$

where $d = \lambda_k - k - \lambda_\ell + \ell$

. This expression was given in [17] and [18] following a different argumentation.

5. $\mathfrak{g}=so(2n+1)$ and $\mathfrak{g}=so(2n)$.

The h.w. of these algebras can be integer or halfinteger vectors. For $\mathfrak{g}=so(2n+1)$, $\lambda=(\lambda_1,\dots,\lambda_n)$, $\lambda_1\geq\lambda_2\geq\dots\geq\lambda_n$, $\lambda_i\in\frac{1}{2}\mathbb{Z}_+$, on the h.w. vector e_λ : $H_i e_\lambda = (\lambda_i -$

$i=1,\dots,n-1$, $H_n e_\lambda = \lambda_n e_\lambda$. For $\mathcal{G} = so(2n)$, $\lambda = (\lambda_1,\dots,\lambda_{n-1},\lambda_n)$, $\lambda_1 \geq \dots \geq \lambda_{n-1} \geq \lambda_n$, $\lambda_i \in \frac{1}{2}\mathbb{Z}_+$, $i=1,\dots,n-1$, $\lambda_n \in \frac{1}{2}\mathbb{Z}$, $H_i e_\lambda = (\lambda_i - \lambda_{i+1})e_\lambda$, $i=1,\dots,n-1$, $H_n e_\lambda = (\lambda_{n-1}+\lambda_n)$.
The representation with integers h.w. λ form a cathegory of tensorial representations of $\mathcal{G}$. The basic representation in this category is the representation with h.w. $\omega_1=(1,0,\dots,0)$ In the category of finite dimensional representations basic representation of $so(2n+1)$ have h.w. $\omega_n = (\frac{1}{2},\dots,\frac{1}{2})$. For $so(2n)$ there are two basic representations in this category. These representations are ω_1 and ω_n or ω_1 and ω_{n-1} or ω_n and ω_{n-1} . All three possibilities are equivalent. Let us start to study the algebras $C_N^\omega(\mathcal{G})$ and R-matrices connected with algebras $U_q(so(2n))$ and $U_q(so(2n+1))$ from the category of a tensorial representation.

5.1. Tensorial representations of $U_q(so(2n))$ and $U_q(so(2n+1))$
The vectorial representation V^{ω_1} of $U_q(so(2n+1))$ and $U_q(so(2n))$ do not depend on q :

$U_q(so(2n+1))$

$$\pi(X_j^+) = E_{j\,j+1} - E_{N-j,N+1-j}\,, \quad j=1,\dots,$$

$$\pi(X_n^+) = \frac{1}{\sqrt{2}}(E_{n+1,n+2} + E_{n\,n+1}) \tag{5.1}$$

$$\pi(H_j) = E_{jj} - E_{j+1,j+1} - E_{N+1-j,N+1-j} + E_{N-j,N-j}$$

$$j=1,\dots,n-1$$

$$\pi(H_n) = E_{nn} - E_{n+2,n+2}$$

$$\pi(X_j^-) = \pi(X_j^+)^t$$

$$U_q(so(2n))$$
$$\pi(X_j^+) = E_{jj+1} - E_{N-j,N+1-j}, \quad j = 1,\dots,n-1$$
$$\pi(X_n^+) = E_{nn-1} - E_{nn+2},$$
$$\pi(H_j) = E_{jj} - E_{j+1\,j+1} - E_{N+1-j} + E_{N-j\,N-j},$$
$$j = 1,\dots,n-1 \tag{5.2}$$
$$\pi(H_n) = E_{nn} + E_{n-1,n-1} - E_{n+1n+1} - E_{n+2,n+2}$$
$$\pi(X_j^-) = \pi(X_j^+)^t .$$

Here E_{ij} are the basic matrices in $V^{\omega_1} \simeq \mathbb{C}^N$, $N = 2n+1$, for $U_q(so(2n$ and $N = 2n$ for $U_q(so(2n))$, t is the transposition. The vector representations of $U_q(so(2n))$ and $U_q(so(2n+1))$ are self-adjoint: $\omega_1^* = \omega_1$. The tensor product of two vectorial representations is decomposed into the sum of three irreducible components

$$(V^{\omega_1})^{\otimes 2} = V^{2\omega_1} \oplus V^{\omega_2} \oplus V^0 . \tag{5.3}$$

After some calculations one can find the matrices $K^{\omega_1\omega_1}_{2\omega_1}$, $K^{\omega_1\omega_1}_{\omega_2}$, $K^{\omega_1\omega_1}_{0}$. They are determined by the following formulae

$$i \neq j' \; i < j, \quad e^{2\omega_1}_{ij} = \frac{1}{\sqrt{1+q}}\left\{ e_i \otimes e_j + q^{1/2} e_j \otimes e_i \right\},$$
$$i \neq i' \quad e^{2\omega_1}_{ii} = e_i \otimes e_i ,$$
$$e^{2\omega_1}_{ii'} = n_i \Big\{ q^{1/2} e_i \otimes e_{i'} + q^{-1/2} e_{i'} \otimes e_i - \tag{5.4}$$
$$- \frac{(q^{-1/2} - q^{1/2})}{q^{-1/2} - q^{\frac{N-1}{2}}} \Big(q^{\frac{N-1}{2}} \sum_{K>i} q^{\frac{\bar{i}-\bar{K}}{2}} e_{K'} \otimes e_K + q^{-1/2} \sum_{K \leq i} q^{\frac{\bar{i}-\bar{K}}{2}} e_{K'} \otimes e_K \Big) \Big\}$$

$$e^{\omega_2}_{ij} = \frac{e_i \otimes e_j - q^{1/2} e_j \otimes e_i}{1+q}, \quad i < j$$

$$e^{\omega_2}_{ii'} = n_i \Big\{ e_i \otimes e_{i'} - e_{i'} \otimes e_i + \tag{5.5}$$

$$+ \frac{(q^{-\frac{1}{2}} - q^{\frac{1}{2}})}{q^{\frac{1}{2}} + q^{\frac{N-1}{2}}} \Big(q^{\frac{N}{2}} \sum_{k>i} q^{\frac{\bar{i}-\bar{k}}{2}} e_{k'} \otimes e_k + q \sum_{k \le i} q^{\frac{\bar{i}-\bar{k}}{2}} e_{k'} \otimes e_k \Big) \Big\}.$$

$$e^{0}_{ij} = \sqrt{\frac{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})}{(q^{-\frac{N}{4}} - q^{-\frac{N}{4}})(q^{\frac{N-2}{4}} + q^{-\frac{N-2}{4}})}}\ \delta_{ij'} \sum_{k} q^{\frac{\bar{i}-\bar{k}}{2}} e_{k'} \otimes e_k \tag{5.6}$$

where n_i are normalization constants, $j' = N+1-j$

$e^{2\omega_1}_{ij}$, $e^{\omega_2}_{ij}$, e^{0}_{ij} are the bases in $V^{2\omega_1}$, V^{ω_2}, V^{0} respectively,

$\bar{i} = \begin{cases} i+\frac{1}{2}, & i \le n \\ i, & i = n+1 \\ i-\frac{1}{2}, & i > n+1 \end{cases}$ for $U_q(SO(2n+1))$, $\bar{i} = \begin{cases} i+\frac{1}{2}, & i \le n \\ i-\frac{1}{2}, & i > n \end{cases}$ for $U_q(SO(2n))$

From the theorem 1.5 we obtain the matrix $R^{\omega_1 \omega_1}$:

$$R^{\omega_1\omega_1} = \mathcal{P}^{\omega_1\omega_1}_{2\omega_1} q^{-1/2} - \mathcal{P}^{\omega_1\omega_1}_{\omega_2} q^{1/2} + \mathcal{P}^{\omega_1\omega_1}_{0} q^{-\frac{N-1}{2}} . \tag{5.7}$$

Substituting the expressions (5.4)-(5.6) in this formula we obtain the representation of $R^{\omega_1\omega_1}$ as $N^2\times N^2$ matrix:

$$R^{\omega_1\omega_1} = q^{1/2} \sum_{i\neq\frac{N-1}{2}} E_{ii}\otimes E_{ii} + E_{\frac{N+1}{2},\frac{N+1}{2}} \otimes E_{\frac{N+1}{2},\frac{N+1}{2}} +$$

$$+\sum_{i\neq j,j'} E_{ji}\otimes E_{ij} + q^{-1/2} \sum_{i\neq i'} E_{ii'}\otimes E_{i'i} - \tag{5.8}$$

$$-(q^{-1/2}-q^{1/2}) \sum_{i>j} E_{jj}\otimes E_{ii} + (q^{-1/2}-q^{1/2}) \sum_{i>j} q^{\frac{\bar{j}-\bar{i}}{2}} E_{i'j}\otimes E_{ij'} ,$$

where E_{ij} are the basic matrices in $\mathbb{C}^N$ and the second term is present only for $U_q(so(2n+1))$.

The matrix (5.8) can be extracted from the work [10]: $R^{\omega_1\omega_1} = \lim_{x\to\infty} x^{-1/2}\check{R}(x)$, where $\check{R}(x)$ is the solution of Yang-Baxter equation corresponding to the series $B_n^{(1)}$ and $\mathcal{D}_n^{(1)}$.

The following decompositions are well known:

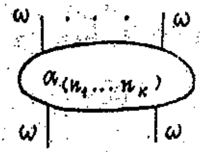
$$V^{\omega_1}\otimes V^{\lambda} = \sum_{k=1}^{2n+1} \oplus\, V^{\lambda^{(k)}} , \quad g = so(2n+1), \tag{5.9}$$

$$V^{\omega_1}\otimes V^{\lambda} = \sum_{k=1}^{2n}{}' \oplus\, V^{\lambda^{(k)}} , \quad g = so(2n) . \tag{5.10}$$

Here $\lambda^{(k)} = (\lambda_1, \ldots, \lambda_k + 1, \ldots, \lambda_n)$, $\lambda^{(n+1+k)} = (\lambda_1, \ldots \lambda_k - 1, \ldots, \lambda_n)$, $\lambda^{(n+1)} = \lambda$ in (5.9) and $\lambda^{(n+k)} = (\lambda_1, \ldots, \lambda_k - 1, \ldots \ldots, \lambda_n)$ in (5.10), $1 \leqslant k \leqslant n$.

Using the theorem 3.1 and the decompositions (5.9) (5.10) we obtain the formulae for projectors $\mathcal{P}_\lambda(q, a)$. For $\mathcal{P}_{\omega_k}(q)$ ($mult\,(\omega_k \subset (\omega_1)^{\otimes k}) = 1$) we have the following representation

$$\mathcal{P}_{\omega_k}(q) = \sum_{n_1, \ldots, n_k = 0}^{2} \prod_{i=1}^{k} q_{n_i, 1} \tag{5.11}$$



where

$$Q = (q_{i,j})_{\substack{i=0,1,2 \\ j=0,1,2}} = \begin{pmatrix} 1 & 1 & 1 \\ q & q^{-k} & q^{-N+k} \\ q^2 & q^{-2k} & q^{-2N+2k} \end{pmatrix}^{-1} \tag{5.12}$$

and $\alpha_{(n_1, \ldots, n_k)}$ is defined by (3.19).

The algebras $C_M^{\omega_1}(so(2n-1))$ and $C_M^{\omega_1}(so(2n+1))$ are connected with the Birman-Wenzl algebra. The last one is an associative algebra formed by the generators $1, G_i$, E_i, $i = 1, \ldots, M-1$ with the following relations

$$G_i G_j = G_j G_i, \quad |i-j| \geqslant 2$$

$$G_i G_{i+1} G_i = G_{i+1} G_i G_{i+1},$$

$$G_i + G_i^{-1} = m(1 + E_i),$$

$$
\begin{aligned}
&E_i E_{i+1} E_i = E_i = E_i E_{i-1} E_i \\
&G_{i+1} G_i E_{i+1} = E_i G_{i+1} G_i = E_i E_{i+1} \\
&G_{i-1} G_i E_{i-1} = E_i G_{i-1} G_i = E_i E_{i-1} \\
&G_{i+1} E_i G_{i+1} = G_i^{-1} E_{i+1} G_i^{-1} \\
&G_{i-1} E_i G_{i-1} = G_i^{-1} E_{i-1} G_i^{-1} \\
&G_{i+1} E_i E_{i+1} = G_i^{-1} E_{i+1} \\
&G_{i-1} E_i E_{i-1} = G_i^{-1} E_{i-1} \\
&E_{i+1} E_i G_{i+1} = E_{i+1} G_i^{-1} \\
&E_{i-1} E_i G_{i+1} = E_{i-1} G_i^{-1} \\
&G_i E_i = E_i G_i = \ell^{-1} E_i \\
&E_i G_{i+1} E_i = E_i G_{i-1} E_i = \ell E_i \, .
\end{aligned}
\tag{5.13}
$$

THEOREM 5.1. The algebras $C_M^{\omega_1}(SO(2n))$ and $C_M^{\omega_1}(SO(2n+1))$ are factors of Birmann-Wenzl algebra with $m = i(q^{1/2} - q^{-1/2})$, $\ell = -iq^{\frac{n-1}{2}}$ over the ideals formed by the elements

$$J = \mathcal{P}_{\omega_{n+1}}(q)^{1,\dots,n+1}$$

acting nontrivially only in the multipliers of $(V^{\omega_1})^{\otimes M}$ with numbers $i \leq n+1$. In other multipliers J acts as a unit matrix. Here $\mathcal{P}_{\omega_{n+1}}(q)$ is defined by (5.11), (5.12). The generators G_i of B.W. algebra are connected with the generators of $C_M^{\omega_1}(q)$ by the identification $G_j = ig_j$, the generators E_j are determined by (5.13). For $n+1 > M$, $C_M^{\omega_1} \simeq \simeq BW_M$.

CONSEQUENCE. The representation α of B.W. algebra with $m = i(q^{1/2} - q^{-1/2})$, $\ell = -iq^{\frac{N-1}{2}}$, $\alpha(G) = i(1 \otimes \ldots \otimes R^{\omega_1\omega_1} \otimes \ldots \otimes 1)$ defined in the space $(V^{\omega_1})^{\otimes M}$.

The structure of algebras $C_M^{\omega_1}(so(2n))$, $C_M^{\omega_1}(so(2n+1))$ and $C_M^{\omega_1}(s\ell(n))$ is in agreement with the natural embeddings $s\ell(n) \subset so(2n+1)$, $s\ell(n) \subset so(2n)$.

THEOREM 5.2. The algebras $U_q(so(2n))$ and $U_q(so(2n+1))$ contain $U_q(s\ell(n))$ as a Hopf subalgebra.

This theorem follows from the commutation relations (1.1)-(1.3) and from the formulae (1.4) (1.5) for coproduct in $U_q(g)$.

Let us consider the restriction of the vector representation of $U_q(so(2n))$ and $U_q(so(2n+1))$ on the subalgebra $U_q(s\ell(n))$. We have

$$V^{\omega_1} = \tilde V^{\omega_1} \oplus \tilde V^{\omega_1^*} \oplus V^0 \qquad \text{for } U_q(so(2n+1))$$
$$V^{\omega_1} = \tilde V^{\omega_1} \oplus \tilde V^{\omega_1^*} \qquad \text{for } U_q(so(2n)).$$

Here $\tilde V^{\omega_1}$ and $\tilde V^{\omega_1^*}$ are vector and conjugated to vector ($\tilde V^{\omega_1^*} = \tilde V^{\omega_{n-1}}$) representations of $U_q(s\ell(n))$. The $U_q(s\ell(n))$ R-matrices $R^{\omega_1\omega_1}$, $R^{\omega_1\omega_1^*}$, $R^{\omega_1^*\omega_1}$ and $R^{\omega_1^*\omega_1^*}$ in accordance with the notations of the section 2 we represent by the graphs:

From the symmetrics of R-matrices we have $R^{\omega_1\omega_1} = R^{\omega_1^*\omega_1^*}$, $R^{\omega_1^*\omega_1} = R^{\omega_1\omega_1^*\,T}$.

THEOREM 5.3. The basic R-matrices of $U_q(so(2n))$ and $U_q(so(2n+1))$ have the following block structures at the restriction of $U_q(so(2n))$ and $U_q(so(2n+1))$ on $U_q(s\ell(n))$

for $U_q(so(2n+1))$:

$\times =$

	$n\times n$	n	$n\times n$	n	1	n	$n\times n$	n	$n\times n$
$n\times n$	$\times$	0	0	0	0	0	0	0	0
n	0	$a\uparrow$	0	$\uparrow$	0	0	0	0	0
$n\times n$	0	0	$a(\,)(\,-q^{-\frac{n-3}{2}}$ [illegible]	0	$aq^{-\frac{n-3}{4}}\cup$	0	$\times$	0	0
$= n$	0	$\uparrow$	0	0	0	0	0	0	0
1	0	0	$aq^{-\frac{n-3}{4}}\cap$	0	1	0	0	0	0
n	0	0	0	0	0	$a\downarrow$	0	$\downarrow$	0
$n\times n$	0	0	$\times$	0	0	0	0	0	0
n	0	0	0	0	0	$\downarrow$	0	0	0
$n\times n$	0	0	0	0	0	0	0	0	$\times$

(5.16)

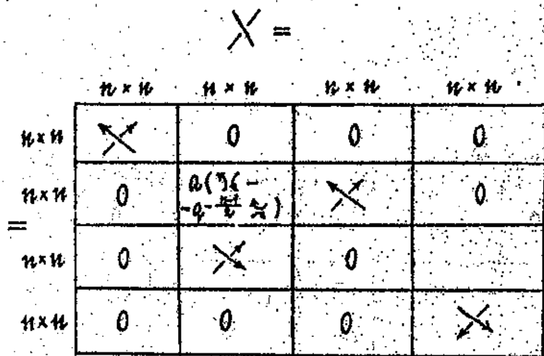
$$a = (q^{-\frac{1}{2}} - q^{\frac{1}{2}})$$

=

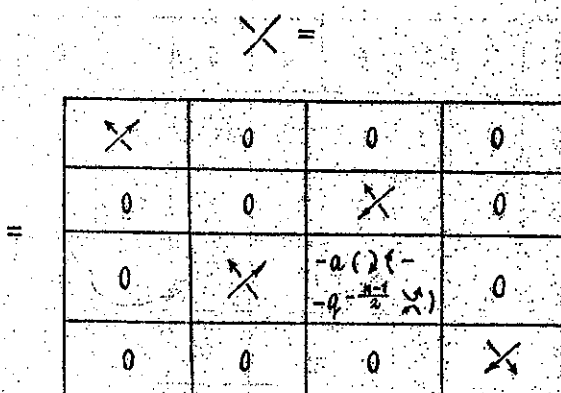
	$n\times n$	n	$n\times n$	n	1	n	$n\times n$	n	$n\times n$
$n\times n$		0	0	0	0	0	0	0	0
n	0	0	0	$\uparrow$	0	0	0	0	0
$n\times n$	0	0	0	0	0	0		0	0
n	0	$\uparrow$	0	$-a\uparrow$	0	0	0	0	0
1	0	0	0	0	1	0	$-aq^{\frac{n-3}{4}}\cap$	0	0
n	0	0	0	0	0	0	0	$\downarrow$	0
$n\times n$	0	0	0	0	$-aq^{\frac{n-3}{4}}\cup$	0	$-a()(1-$ $-q^{\frac{n-3}{2}}$)	0	0
n	0	0	0	0	0	$\downarrow$	0	$-a\downarrow$	0
$n\times n$	0	0	0	0	0	0	0	0	

(5.17)

for $U_q(so(2n))$:



(5.18)



(5.19)

This theorem follows from the formulae (5.8) and (4.6).

COROLLARY. There exist the following embeddings of the algebras $C_N^{\omega_1}$:

$$C_M^{\omega_1}(so(2n+1)) \supset C_M^{\omega_1}(s\ell(n)) \times C_M^{\omega_1}(s\ell(n)).$$

Using the theorem 5.3 and the action (4.11), (4.12) of $C_M^{\omega_1}(s\ell(n))$ in K_λ-basis in W_λ one can find the action of $C_M^{\omega_1}(so(2n+1))$ and $C_M^{\omega_1}(so(2n))$ in K_λ-basis in W_λ. Explicit formulae will be given in a separate publication.

5.2. The spinor representations. The spinor representations V^{ω_n} and $V^{\omega_{n-1}}$ of $U_q(so(2n))$ each have a dimension 2^{n-1}. For $U_q(so(2n+1))$ spinor representation V^{ω_n} has dimension 2^n. The bases in spinor representations are parametrized by the weights $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, $\varepsilon_i = \pm\frac{1}{2}$. For $U_q(so(2n))$ there are auxiliary retrictions on ε: $\prod_{i=1}^{n} \varepsilon_i = 2^{-n}$ for V^{ω_n} and $\prod_{i=1}^{n} \varepsilon_i = -2^{-n}$ for $V^{\omega_{n-1}}$. The spinor representations of $U_q(so(2n+1))$ and $U_q(so(2n))$ are defined by the action of the generators on the basic vectors e_ε:

$U_q(so(2n+1))$:

$$X_i^+ e_{\dots \varepsilon_i \varepsilon_{i+1} \dots} = -e_{\dots \varepsilon_{i+1}, \varepsilon_{i-1}, \dots},$$

$$X_i\, e_{\dots \varepsilon_n} = \frac{1}{\sqrt{2ch(\frac{h}{4})}}\, e_{\dots \varepsilon_n + 1}, \tag{5.20}$$

$$H_i e_\varepsilon = (\varepsilon_i - \varepsilon_{i+1}) e_\varepsilon, \quad i = 1, \dots, n-1$$

$$H_n e_\varepsilon = \varepsilon_n e_\varepsilon$$

$$X_i^- = (X_i^+)^t.$$

$U_q \mid so(2n)$:

$$X_i^+ e_{\ldots \varepsilon_i \varepsilon_{i+1} \ldots} = - e_{\ldots \varepsilon_i+1,\, \varepsilon_{i+1}-1, \ldots},$$

$$X_n^+ e_{\ldots \varepsilon_{n-1} \varepsilon_n} = - e_{\ldots \varepsilon_{n-1}+1,\, \varepsilon_n+1, \ldots}; \tag{5.21}$$

$$H_i e_\varepsilon = (\varepsilon_i - \varepsilon_{i+1}) e_\varepsilon ,$$

$$H_n e_\varepsilon = (\varepsilon_{n-1} + \varepsilon_n) e_\varepsilon ,$$

$$X_i^- = (X_i^+)^t$$

Here r.h.s. = 0 if the weight ε' of $e_{\varepsilon'}$ does not belong to the set of weights of the space V^{ω_n} (or $V^{\omega_{n-1}}$). The tensor product of spinor representations has the following spectral decomposition

$U_q (so(2n))$:

$$V^{\omega_{n-1}} \otimes V^{\omega_{n-1}} \simeq V^{\omega_n} \otimes V^{\omega_n} = \begin{cases} \left(\sum_{s=0}^{\frac{n}{2}-1} \oplus V^{\omega_{2s}} \right) \oplus V^{2\omega_n}, & n\text{-even} \\ \left(\sum_{s=1}^{\frac{n-1}{2}} \oplus V^{\omega_{2s-1}} \right) \oplus V^{2\omega_n} & \end{cases} \tag{5.22}$$

$$V^{\omega_n} \otimes V^{\omega_{n-1}} = \begin{cases} \left(\sum_{s=1}^{\frac{n}{2}-1} \oplus V^{\omega_{2s-1}} \right) \oplus V^{\omega_n + \omega_{n-1}}, & n\text{-even} \\ \left(\sum_{s=0}^{\frac{n-2}{2}} \oplus V^{\omega_{2s}} \right) \oplus V^{\omega_n + \omega_{n-1}}, & n\text{-odd} \end{cases} \tag{5.23}$$

$$V^{\omega_1} \otimes V^{\omega_n} \begin{cases} V^{\omega_{n-1}} \oplus V^{\omega_n+\omega_1}, & n\text{-}even \\ V^{\omega_n} \oplus V^{\omega_n+\omega}, & n \ odd \end{cases} \tag{5.24}$$

$$V^{\omega_1} \otimes V^{\omega_{n-1}} = \begin{cases} V^{\omega_n} \oplus V^{\omega_{n-1}+\omega_1}, & n\text{-}even \\ V^{\omega_{n-1}} \oplus V^{\omega_{n-1}+\omega_1}, & n \ odd \end{cases} \tag{5.25}$$

$U_q(so(2n))$:

$$\begin{aligned} & V^{\omega_n} \otimes V^{\omega_n} = \Big(\sum_{s=0}^{n-1} \oplus V^{\omega_s}\Big) \oplus V^{2\omega_n} \\ & V^{\omega_1} \otimes V^{\omega_n} = V^{\omega_n} \oplus V^{\omega_1+\omega_n} \ . \end{aligned} \tag{5.26}$$

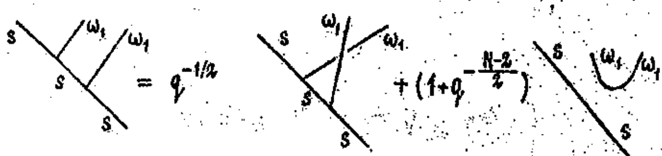
Let us describe the KGC of these decompositions. As in the previous part of this section we shall use the graphical representation of the matrices $R^{\lambda\mu}$ and $K^{\lambda\mu}_{\nu}$ given in section 2.

It is convenient to consider the space $V^s = V^{\omega_n} \oplus V^{\omega_{n-1}}$ for $U_q(so(2n))$ and the matrix

$$K_s^{s\omega_1} = \begin{pmatrix} K^{\omega_n\omega_1}_{\omega_n} & K^{\omega_n\omega_1}_{\omega_{n-1}} \\ K^{\omega_{n-1}\omega_1}_{\omega_n} & K^{\omega_{n-1}\omega_1}_{\omega_{n-1}} \end{pmatrix}$$

For $U_q(so(2n+1))$ we shall write $V^s = V^{\omega_n}$.

THEOREM 5.5. The matrices K^{ω_i} for $U_q(so(2n))$ and for $U_q(so(2n+1))$ satisfy the following relations:



(5.27)

The proof of this theorem follows from theorem 3.1.

It is interesting that the relations ( ) between the matrix elements of $K^{\omega_n \omega_1}_{\omega_n}$ and $K^{s\omega_1}_{s}$ permit us to introduce q-analog of Clifford algebra. If we write

$(\gamma_i)_{\alpha\beta} =$  (5.28)

the relations (5.27) give the q-anticommutation relations for the matrices γ_i :

$$\gamma_i \gamma_j = -q^{-1/2} \gamma_j \gamma_i \quad , \quad i<j \ , \ i \neq j'$$
$$\gamma_i^2 = 0 \tag{5.29}$$
$$\gamma_i \gamma_{i'} = -\gamma_{i'} \gamma_i - (q^{-1}-1) \sum_{k>i} q^{\frac{\bar{i}-\bar{k}}{2}} \gamma_{k'} \gamma_k + (1+q^{-\frac{N-2}{2}})\, q^{\frac{\bar{i}}{2}}$$

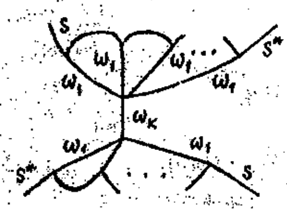
with normalization condition

$$\sum_i \gamma_i \gamma_{i'} = \chi_{\omega_1} .$$

Using explicit form of the spinorial and vector representations the matrices $K_s^{s\omega_1}$ can be calculated exactly. But we will not present here these calculations because they take up too much space and are not very instructive for our purposes.

The matrices $K_{\omega_1}^{ss}$ are the crossing transformations of the matrices $K_s^{s\omega_1}$ (see (2.20)). Using the formulae of the section 5.1 and (5.27) we obtain an expression for the projector $\mathcal{P}_{\omega_K}^{ss}$:

$= N_K^s$

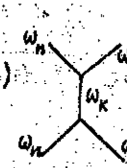


(5.30)

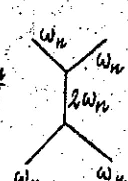
where N_K^s are some normalization constants and projector $\mathcal{P}: (V^{\omega_1})^{\otimes K} \longrightarrow V^{\omega_K}$ was given by (5.11).

So, we have the expression for spinor-spinorial R-matrices through the matrices $K_{\omega_1}^{ss}$ and $R^{\omega_1\omega_1}$:

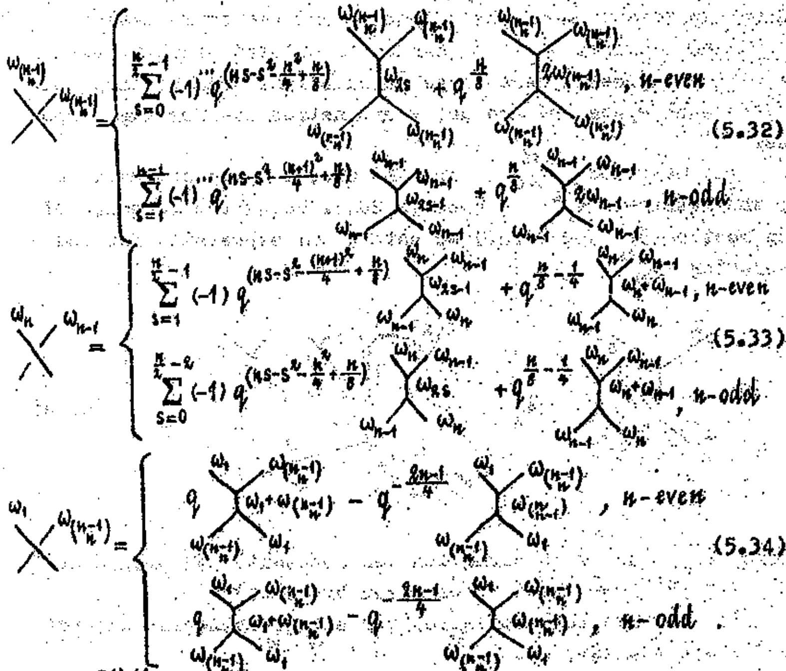
$= \sum_{s=0}^{n-1} q^{\frac{1}{4}((2n+1)K - K^2 - n^2 - \frac{n}{2})}$



$+ q^{\frac{n}{2}}$

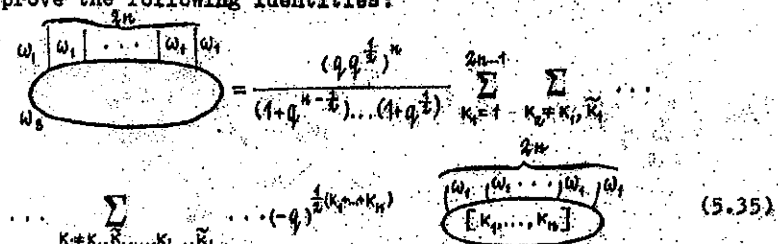


(5.31)



Here $\mathcal{G}^{\omega_i \omega_s}_{\omega_k}$ are given by (5.30).

Using the commutation relations (5.27) it is easy to prove the following identities:



$$\tilde{k}_i = \max_j \{ j \in (1, \ldots, 2n) \setminus (k_1, \tilde{k}_1, \ldots, k_{i-1}, \tilde{k}_{i-1}) \} .$$

Here $[k_1 \dots k_n]$ is the joint in which the line numbered (the numeration being from left to write) is connected with the $\tilde{k}_i$ - line in such a way that the k_i-line gous above k_j -line if $i<j$.

6. $\mathfrak{g} = sp(2n)$.

All formulae in this case are very similar to those in part one of the previous section.

The h.w. of finite dimensional representation of are parametrized by the numbers $\lambda = (\lambda_1, \dots, \lambda_n)$ $\lambda_1 \geqslant \dots \geqslant \lambda_n$, $\lambda_i \in \mathbb{Z}_+$. If e_λ is the h.w. vector, $H_i e_\lambda = (\lambda_i - \lambda_{i+1}) e_\lambda$, $i = 1, \dots, n-1$, $H_n e_\lambda = \lambda_n e_\lambda$, $X_i^+ e_\lambda = 0$.

The basic representation of $U_q(sp(2n))$ have the h.w. $\omega_1 = (1, 0, \dots, 0)$, $V^{\omega_1} \simeq \mathbb{C}^{2n}$

$$\begin{aligned}
&\pi(X_i^+) = E_{i i+1} - E_{N-i, N-i+1}, \quad i = 1, \dots, n-1 \\
&\pi(X_n^-) = E_{n n+1}, \quad \pi(X_i^-) = \pi(X_i^+)^t \\
&\pi(H_i) = E_{ii} - E_{i+1\, i+1} - E_{N-i+1, N-i+1} + E_{N-i, N-i} \\
&\pi(H_n) = 2E_{nn} - 2E_{n+1\, n+1}
\end{aligned} \tag{6.1}$$

The tensor square of the basic representation is the sum of three irreducible components:

$$V^{\omega_1} \otimes V^{\omega_1} = V^{2\omega_1} \oplus V^{\omega_2} \oplus V^{(0)} \tag{6.2}$$

and we have the following spectral decomposition of $R^{\omega_1 \omega_1}$:

$$R^{\omega_1 \omega_1} = q^{\frac{1}{2}} \mathcal{P}^{\omega_1 \omega_1}_{2\omega_1} - q^{-\frac{1}{2}} \mathcal{P}^{\omega_1 \omega_1}_{\omega_2} - q^{-\frac{2n+1}{2}} \mathcal{P}^{\omega_1 \omega_1}_{0} \tag{6.3}$$

As in the cases of $\mathfrak{g}=SO(2n+1)$ and $SO(2n)$ one can calculate the matrices $K^{\omega_1\omega_1}_{2\omega_1}$, $K^{\omega_1\omega_1}_{\omega_2}$, and $K^{\omega_1\omega_1}_{0}$. Substituting these matrices into (6.3) we obtain the matrix $R^{\omega_1\omega_1}$:

$$R^{\omega_1\omega_1}=q^{\frac{1}{2}}\sum_i E_{ii}\otimes E_{ii}+\sum_{i\neq j,j'} E_{ji}\otimes E_{ij}+$$

$$+q^{-\frac{1}{2}}\sum_{i\neq i'} E_{ii'}\otimes E_{i'i}+(q^{\frac{1}{2}}-q^{-\frac{1}{2}})\sum_{i>j} E_{ii}\otimes E_{jj}- \tag{6.4}$$

$$-(q^{\frac{1}{2}}-q^{-\frac{1}{2}})\sum_{i>j}\epsilon_i\epsilon_j\, q^{\frac{\bar{j}-\bar{i}}{2}} E_{i'j}\otimes E_{ij'}$$

where $i'=2n+1-i$; $\epsilon_i=1$, $i=1,\dots,n$, $\epsilon_i=-1$, $i=n+1,\dots,2n$; $\bar{i}=i-\frac{1}{2}$, $i\le n$, $\bar{i}=i+\frac{1}{2}$, $i\ge n+1$.
This matrix can also be extracted from [10].

The matrices $R^{\lambda\mu}$ and $\mathcal{P}_\lambda$ for any h.w. λ,μ can be found from the theorems 2. - 3. and from the ramification rule:

$$V^{\lambda}\otimes V^{\omega_1}=\sum_{k=1}^{2n}\oplus V^{\lambda^{(k)}} \tag{6.5}$$

where $\lambda^{(k)}=(\lambda_1,\dots,\lambda_k+1,\dots,\lambda_n)$, $\lambda^{(n+k)}=(\lambda_1,\dots,\lambda_k-1,\dots\lambda_n)$, $k=1,\dots,n$.

As in the case $\mathfrak{g}=SO(2n+1),SO(2n)$ we have the following propositions.

PROPOSITION 6.1. The algebra $C_N^{\omega_1}(Sp(2n))$ is the factor of B.-W. algebra with $m=(i(q^{1/2}-q^{-1/2})$, $\ell=iq^{-\frac{2n+1}{2}}$ over the ideal formed by the elements

$$J=\mathcal{P}_{\omega_{n+1}}(q)^{1,\dots,n+1} \tag{6.6}$$

acting nontrivially only in the multipliers of $(V^{\omega_1})^{\otimes N}$ with numbers $i \leq n+1$. The elements $\mathcal{P}_k(q)$ are defined by (5.11) with the matrix

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ q & q^{-k} & q^{k-2-2n} \\ q^{2} & q^{-2k} & q^{2(k-2-2n)} \end{pmatrix} \qquad (6.7)$$

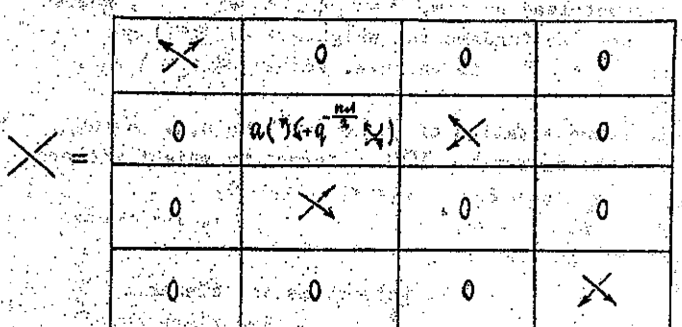
and $C^{\omega_1}_{M}(sp(2n)) \simeq BW_M$ for $n+1 > M$.

If we consider the block basis in V^{ω_1} connected with the embedding $U_q(sp(2n)) \supset U_q(sl(n))$

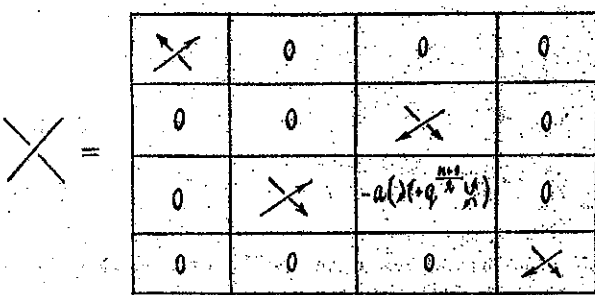
$$V^{\omega_1} = \tilde{V}^{\omega_1} \oplus \tilde{V}^{\omega_1^*}, \quad \dim \tilde{V}^{\omega_1} = \dim \tilde{V}^{\omega_1^*} = n \qquad (6.8)$$

we obtain the representation of $R^{\omega_1\omega_1}$ in the block form with the blocks constructed from $U_q(sl(n))$ R-matrices.

PROPOSITION 6.3. The matrices $(R^{\omega_1\omega_1})^{\pm 1}$ have the following block structure in the basis (6.9):



(6.9)



(6.10)

$a = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$

where we use the notations of sections 2 and 5.

7. $\mathcal{G} = G_2$.

The irreducible finite dimensional representations of $U_q(G_2)$ are parametrized by h.w. $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$, where ω_1 and ω_2 are the fundamental weights (see [16]) and $\lambda_i \in \mathbb{Z}_+$. If $e_\lambda \in V^\lambda$ is the h.w. vector, $H_1 e_\lambda = \lambda_1 e_\lambda$, $H_2 e_\lambda = 3\lambda_2 e_\lambda$.

The basic representation of $U_q(G_2)$ have h.w. $\lambda = \omega_1$. Let us consider the basics in V^{ω_1} formed by weight vectors, $e_1 = e_{\omega_1}$, $e_2 = e_{\alpha_1+\alpha_2}$, $e_3 = e_{\alpha_1}$, $e_4 = e_0$, $e_5 = e_1$ $e_6 = e - \alpha_1 - \alpha_2$, $e_7 = e - \omega_1$. The roots, are shown on the figure 1.

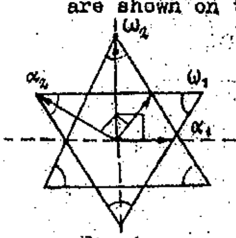


Fig.1

In this basis the elements H_i, $X_i^\pm$ are represented by the matrices:

$$\pi(X_1^+)=\left(\begin{array}{ccc|c|ccc} 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & c & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad \pi(X_2^+)=\left(\begin{array}{ccc|c|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad (7.1)$$

$$\pi(H_1)=\left(\begin{array}{ccc|c|ccc} 1 & & & & & & \\ & -1 & & & & 0 & \\ & & 2 & & & & \\ \hline & & & 0 & & & \\ \hline & & & & -2 & & \\ & 0 & & & & 1 & \\ & & & & & & -1 \end{array}\right), \quad \pi(H_2)=\left(\begin{array}{ccc|c|ccc} 0 & & & & & & \\ & 1 & & & & 0 & \\ & & -1 & & & & \\ \hline & & & 0 & & & \\ \hline & & & & 1 & & \\ & 0 & & & & -1 & \\ & & & & & & 0 \end{array}\right) \qquad (7.2)$$

$$\pi(X_i^-) = \pi(X_i^+)^t .$$

Here $b=\sqrt{\dfrac{sh(\frac{h}{2})}{sh(\frac{3h}{2})}}$, $c=\sqrt{\dfrac{sh(h)}{sh(\frac{3h}{2})}}$.

In this normalization the commutation relations between the generators H_i , $X_i^{\pm}$ have the following form:

$$[H_1, X_1^{\pm}] = \pm 2X_1^{\pm} , \quad [H_2, X_1^{\pm}] = \mp 3X_1^{\pm}$$

$$[H_1, X_2^{\pm}] = \mp 3X_2^{\pm} , \quad [H_2, X_2^{\pm}] = \pm 6X_2^{\pm}$$

$$[X_i^+, X_j^-] = \frac{sh(\frac{h}{2}H_i)}{sh(\frac{3h}{2})} , \quad q = e^h .$$

The representation V^{ω_1} is selfdual: $V^{\omega_1^*} = V^{\omega_1}$.

A tensorial product of the two basic representations of $U_q(G_2)$ has the following decomposition:

$$V^{\omega_1} \otimes V^{\omega_1} = V^{2\omega_1} \oplus V^{\omega_2} \oplus V^{\omega_1} \oplus V^0 . \qquad (7.3)$$

In this decomposition $\overline{2\omega_1}=0$, $\overline{\omega_2}=\overline{\omega_1}=1$, $\overline{0}=0$.
It is not difficult to calculate the matrix $K^{\omega_1\omega_1}_{\omega_1}$:

$$e_1 = g(cq^{\frac{3}{4}} e_2 \otimes e_3 - bq^{\frac{3}{2}} e_1 \otimes e_4 + bq^{-\frac{3}{2}} e_4 \otimes e_1 - cq^{-\frac{3}{4}} e_3 \otimes e_2)$$

$$e_2 = g(bq^{\frac{1}{2}} e_2 \otimes e_4 - cq^{\frac{5}{4}} e_1 \otimes e_5 + cq^{-\frac{5}{4}} e_5 \otimes e_1 - bq^{-\frac{1}{2}} e_4 \otimes e_2)$$

$$e_3 = g(bq^{\frac{1}{2}} e_3 \otimes e_4 - cq^{\frac{5}{4}} e_1 \otimes e_6 + cq^{-\frac{5}{4}} e_6 \otimes e_1 - bq^{-\frac{1}{2}} e_4 \otimes e_3)$$

$$e_4 = gb(-q e_1 \otimes e_7 - q^{\frac{3}{2}} e_2 \otimes e_6 + e_3 \otimes e_5 + (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \cdot$$

$$\cdot\, e_4 \otimes e_4 - e_5 \otimes e_3 + q^{-\frac{3}{2}} e_6 \otimes e_2 + q^{-1} e_7 \otimes e_1)$$

$$e_5 = g(bq^{\frac{1}{2}} e_4 \otimes e_5 - cq^{\frac{5}{4}} e_2 \otimes e_7 + cq^{-\frac{5}{4}} e_7 \otimes e_2 - \qquad (7.4)$$

$$-bq^{-\frac{3}{2}} e_5 \otimes e_4)$$

$$e_6 = g(bq^{\frac{1}{2}} e_4 \otimes e_6 - cq^{\frac{5}{4}} e_3 \otimes e_7 + cq^{-\frac{5}{4}} e_7 \otimes e_3 - bq^{-\frac{1}{2}} e_6 \otimes e_4);$$

$$e_7 = g(cq^{\frac{3}{4}} e_5 \otimes e_6 - bq^{\frac{3}{2}} e_4 \otimes e_7 + bq^{-\frac{3}{2}} e_7 \otimes e_4 - cq^{-\frac{3}{4}} e_6 \otimes e_5),$$

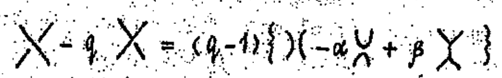
$$g = \frac{1}{\sqrt{q^2+q^{-2}}} .$$

So the matrix $(K^{\omega_1\omega_1}_{\omega_1})^{jk}_i$

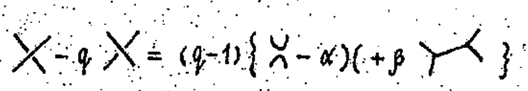
$$e_i = \sum_{j,k=1}^{7} (K^{\omega_1\omega_1}_{\omega_1})^{jk}_i \, e_j \otimes e_k \tag{7.5}$$

is given explicitly.

THEOREM 7.1. The matrix $R^{\omega_1\omega_1}$ satisfies the following relations:



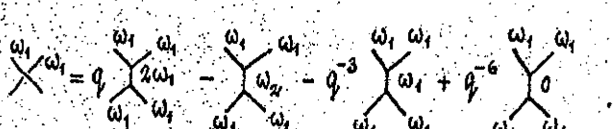
(7.6)



(7.7)

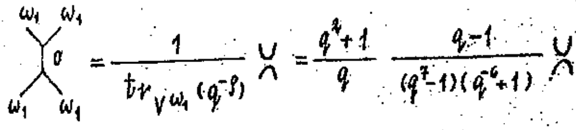
Here $\alpha = q + q^{-1}$, $\beta = \dfrac{q^{-3}(q^3-1)(q^4+1)}{q-1}$.

PROOF. From the theorem 1.5 we have the spectral decomposition of $R^{\omega_1\omega_1}$:



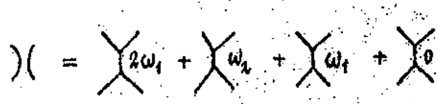
(7.8)

One-dimensional projector is defined in section 1 (1.47):



(7.9)

Substituting (7.8), (7.9) in the l.h.s. of 7.6 and using the decomposition of units in $V^{\omega_1\otimes 2}$:

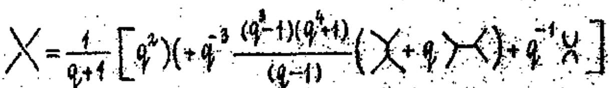


we obtain (7.6).

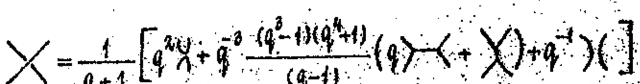
The equality (7.7) is obtained by rotating the pictures (7.6) by 90° (making crossing-transformation).

Considering the equalities (7.6) and (7.7) as a system of linear equations for $R^{\omega_1\omega_1}$ and $(R^{\omega_1\omega_1})^{-1}$ we obtain an expression for these matrices in terms of the matrix $K^{\omega_1\omega_1}_{\omega_1}$ (7.4):

PROPOSITION 7.1.



(7.10)



(7.11)

Let us consider now the restriction of G_2 to the $s\ell(2)$ triple formed by $X_1^{\pm}$, H_1. The representation V^{ω_1} is the sum of three $s\ell(2)$-irreducible components.

$$V^{\omega_1}\Big|_{sl(2)} = V^{\omega_1}\oplus V^{2}\oplus \tilde{V}^{1}. \qquad (7.12)$$

Here $dim\, V^1 = dim\, \tilde{V}^1 = 2,\quad dim\, V^2 = 3$. The spaces V^1 V^2 and $\tilde{V}^1$ are formed by the vectors (e_1, e_2) , (e_3, e_4, e_5) and (e_6, e_7) correspondingly.

For $sl(2)$ CGC we have:

$$V^2 \subset V^1 \otimes V^1$$

$$e^2 = e^1 \otimes e^1,$$

$$e^0 = \frac{1}{\sqrt{q^{1/2}+q^{-1/2}}}\left(q^{\frac{1}{4}} e^{-1}\otimes e^1 + q^{-\frac{1}{4}} e^1 \otimes e^{-1}\right), \qquad (7.12)$$

$$e^{-2} = e^{-1}\otimes e^{-1},$$

where $e^{\pm 1}$ are the bases in 2-dimensional representation of $U_q(sl(2))$ and e^2, e^0, e^{-2} are the bases of a 3-dimensional one.

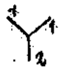
$$V^2 \subset V^2 \otimes V^2$$

$$e^2 = \frac{1}{\sqrt{q+q^{-1}}}\left(q^{\frac{1}{2}} e^2 \otimes e^0 - q^{-\frac{1}{2}} e^0 \otimes e^2\right),$$

$$e^0 = \frac{1}{\sqrt{q+q^{-1}}}\left(e^2 \otimes e^{-2} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\, e^0 \otimes e^0 - e^{-2}\otimes e^2\right), \qquad (7.13)$$

$$e^{-2} = \frac{1}{\sqrt{q+q^{-1}}}\left(q^{\frac{1}{2}} e^0 \otimes e^{-2} - q^{-\frac{1}{2}} e^{-2}\otimes e^0\right).$$

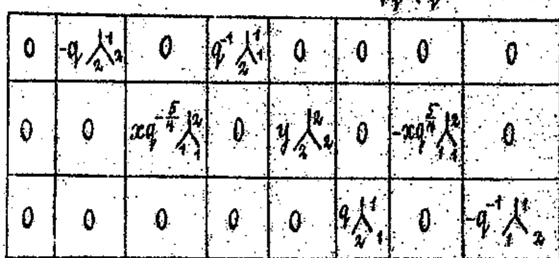
In accordance with the graphical representation of CGC given in section 2 we can associate the following pictures with the formulae (7.12) and (7.13)

 $\longleftrightarrow$ (7.12),  $\longleftrightarrow$ (7.13) (7.14)

An index 1 corresponds to a 2-dimensional representation. An index 2 corresponds to a 3-dimensional representation.

From (7.4) and (7.12)-(7.14) we obtain the $s\ell(2)$-block form of the matrix $K^{\omega_1\omega_1}_{\omega_1}$:

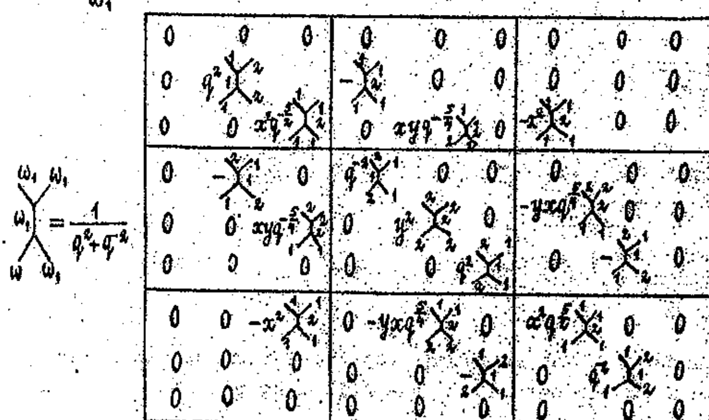
$$K^{\omega_1\omega_1}_{\omega_1} = $$  $$= \frac{1}{\sqrt{q^2+q^{-2}}}$$



(7.15)

where
$$x = \left(\frac{1-q^2}{1-q^3}\right)^{\frac{1}{2}} q^{\frac{1}{4}}, \quad y = \left(\frac{(1-q)(1+q^2)}{(1-q^3)}\right)^{1/2}.$$

From (7.15) we obtain the following expression for projector $\mathcal{P}^{\omega_1\omega_1}_{\omega_1}$:



(7.16)

and the similar expression for crossing-conjugate matrix. For one dimensional projector there is a similar representation:

$$\genfrac{}{}{0pt}{}{\omega_1 \smile \omega_1}{\omega_1 \frown \omega_1} =$$

$$= \left(\begin{array}{ccc|ccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q^{\frac{9}{2}} \genfrac{}{}{0pt}{}{\overset{1}{\cup}}{\underset{1}{\cap}} & 0 & q^{\frac{9}{4}} \genfrac{}{}{0pt}{}{\overset{1}{\cup}}{\underset{2}{\cap}} & 0 & \genfrac{}{}{0pt}{}{\overset{1}{\cup}}{\underset{1}{\cap}} & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q^{\frac{9}{4}} \genfrac{}{}{0pt}{}{\overset{2}{\cup}}{\underset{1}{\cap}} & 0 & \genfrac{}{}{0pt}{}{\overset{2}{\cup}}{\underset{2}{\cap}} & 0 & \genfrac{}{}{0pt}{}{\overset{2}{\cup}}{\underset{1}{\cap}} q^{-\frac{9}{4}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & \genfrac{}{}{0pt}{}{\overset{1}{\cup}}{\underset{1}{\cap}} & 0 & q^{-\frac{9}{4}} \genfrac{}{}{0pt}{}{\overset{1}{\cup}}{\underset{2}{\cap}} & 0 & \genfrac{}{}{0pt}{}{\overset{1}{\cup}}{\underset{1}{\cap}} q^{-\frac{9}{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) \qquad (7.17)$$

where ${}_{-1}\overset{1}{\frown}{}_{1} = q^{-\frac{1}{4}}$, ${}_{1}\overset{1}{\frown}{}_{-1} = -q^{\frac{1}{4}}$, ${}_{-2}\overset{2}{\frown}{}_{2} = q$, ${}_{0}\overset{2}{\frown}{}_{0} = 1$, ${}_{2}\overset{2}{\frown}{}_{-2} = -q^{-1}$.

Substituting (7.15)-(7.17) in (7.10) and (7.11) we obtain an explicit expression for $R^{\omega_1\omega_1}$ though $s\ell(2)$-CGC.

Appendix

Here we give some elementary facts about Hopf algebras. The details are given in [14].

DEFINITION 1. An associative algebra A with unit e and multiplication $m: A\otimes A \to A$ is a bialgebra if there is a homomorphism of algebras $\Delta: A \to A\otimes A$ satisfying the coassociativity condition. The coassociativity implier the commutativity of the following diagram:

$$\begin{tikzcd} & A\otimes A \arrow[dr, "\Delta\otimes id"] & \\ A \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & A\otimes A\otimes A \\ & A\otimes A \arrow[ur, "id\otimes\Delta"'] & \end{tikzcd} .$$

The homomorphism Δ is called a comultiplication.

PROPOSITION A.1. On the dual space of a bialgebra A the structure of bialgebra with multiplication $\Delta: A^*\otimes A^* \longrightarrow A^*$ comultiplication $m: A^* \longrightarrow A^*\otimes A^*$ and with counint ε; $\varepsilon(a) = \delta_{a,e}$ is also defined.

DEFINITION 2. Bialgebra A is a Hopf algebra if on there is an antiautomorphism γ such that the following diagram is commutative

$$\begin{tikzcd} & A\otimes A \arrow[r, shift left, "id\otimes\gamma"] \arrow[r, shift right, "\gamma\otimes id"'] & A\otimes A \arrow[dr, "m"] & \\ A \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"'] & & \mathbb{C} \arrow[r, "e"'] & A \end{tikzcd}$$

This antiautomorphism is called the antipode.

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