Correlation functions in integrable Quantum Field Theory: Chern-Simons Research Lecture Series

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Last time we decided that

$$\langle \mathrm{vac}|\mathcal{O}_1(x)\mathcal{O}_2(0)|\mathrm{vac}\rangle = \sum \frac{1}{n!}\int \mathrm{d}\beta_1 \cdots \int \mathrm{d}\beta_n \, f_1(\beta_1,\dots,\beta_n)\, e^{-m_r \sum ?\beta_j}$$

This is for spacelike separated points ($x^2 = -r$), but it looks like imaginary time.

Harold: Can you remind me the definition of the form factors? **Fodor:** They are $\langle \mathrm{vac}|\mathcal{O}(0)|\beta_1 \dots \beta_n\rangle$.

Today we investigate our main example, the *sine-Gordon model*.

The action is:

$$\mathcal{A}^{\mathrm{sG}} = \int \left[\frac{1}{16\pi}(\partial_\mu\varphi)^2 + \frac{\mu^2}{\sin \pi\beta^2} 2\cos(\beta\varphi(x))\right] \mathrm{d}^2x$$

This action makes sense after quantization. We normalize the action so that $[\varphi(x), \partial_0\varphi(y)] = \delta(x-y)$. See, we can call $\phi = \beta\varphi$, and $\frac{1}{\beta^2}$ is Planck's constant.

A reference: L. Faddeev, L. Takhtajan, 1974 (or maybe 76). They investigate this on the classical level.

There is periodicity $\phi \mapsto \phi + 2\pi$. Then there are *solitons*: a smooth step up with height 2π. **Kolya:** "soliton" means? **Fodor:** They propagate without changing the shape.

Then there is scattering. Suppose that I have soliton, and far away antisoliton, moving in opposite direction (looking roughly like a bump). This is at time $-\infty$, and then at $+\infty$ they have moved past each other, and the only difference is that there will be some phase shift. Even in multi-soliton scattering, everything (at the classical level) is just two-particle interaction.

Ok, so the idea to quantize, is that the solitons should correspond to particles. This is somewhat unusual: usually, in quantization, particles correspond to small variations of ϕ, and so these are huge at the classical level.

In the paper by L. Faddeev and V. Korajin **?**, they found that this quantization does exist. **Kolya:** Perturbative quantization? Some formal version of the path integral?

So here, we want S-matrix, and it can by done, but there is some trouble. People naturally assumed that on the quantum level they scatter in the same way, with no reflection, but this is not true quantum mechanically. At the quantum level, there is reflection, but it cannot be observed semiclassically. It only happens as e^{-1/β^2}. This is like tunelling in quantum mechanics. To describe it classically requires taking complex solutions to equations of motion.

So this was described by A. Zamolodchikov. We take two solitons with rapidity β_1, β_2, and we combine them into one multiplet, and denote $\eta_\pm$ the particle and antiparticle. We write $B_j = e^{\beta_j}$. This is natural because it encodes energy-momentum. In addition, we denote $b_j = e^{\frac{2\nu}{1-\nu}\beta_j}$, and then $\nu = 1 - \beta^2$, so that quasiclassical is $\nu \approx 1$.

Ok, so then S-matrix:

$$S_{1,2}(\beta_1 - \beta_2) = S_0(\beta_1 - \beta_2)\, \tilde{S}_{1,2}(b_1/b_2)$$
$$S_0(\beta) = \exp\left(-i \int_0^\infty \frac{\sin(2k\nu\beta)\sinh(2\nu 1 \pi k)}{k\cosh(\pi\nu k)\sinh(\pi(1-\nu)k)} \mathrm{d}k \right)$$
$$S_{1,2}(b_1/b_2) = \frac{1}{2}(I \otimes I + \sigma^3 \otimes \sigma^3) + \frac{b_1 - b_2}{b_1 q^{-1} - b_2} \frac{1}{2}(I \otimes I - \sigma^3 \otimes \sigma^3) + \frac{\sqrt{b_1 b_2}}{b_1 q^{-1} - b_2 q}(\sigma^+ \otimes \sigma^- + \sigma^- \otimes \sigma^+)$$

This is well-known expression. It is supposed to be S-matrix, and sure enough it satisfies unitarity, and also crossing and so on from last time.

Kolya: Can you explain the logic? I have classical field theory, and I look for quantum field theory that reproduces the behavior in the classical limit? **Fodor:** Yes. For me, quantum theory *is* S-matrix. Where it comes from, I don't care. **Harold:** Where do I see the qualitative behavior of solitons? **Fodor:** This is the S-matrix for interaction of two solitons, with rapidity β_1, β_2. Since the soliton and antisoliton have the same mass, I combine them into the same multiplet. **Kolya:** We now switch: this S-matrix defines some theory, and we will show that semiclassical limit reproduces the classical limit.

The big contribution of Zamolodchikov is the introduction of the last term. Naively semiclassically S-matrix should be symmetric.

We now introduce some more notation:

$$\Phi_\alpha(x) = e^{i\alpha \frac{\nu}{2\sqrt{1-\nu}}\varphi(x)}$$

These satisfy slightly funny statistics, and so we will modify the axioms from last time slightly. One of them:

$$S(\beta_j - \beta_{j+1}) f_{\mathcal{O}_\alpha}(\beta_1 \dots \beta_j \beta_{j+1} \dots \beta_{2n}) = f(\dots \beta_{j+1}\beta_j \dots) \tag{*}$$

This does not change. See, the form factor requires even number of particles and antiparticles. But:

$$f_{\mathcal{O}_\alpha}(\beta_1 \dots \beta_n + 2\pi i) = e^{-\frac{\pi i \nu}{1-\nu}\alpha} f(\dots) \tag{**}$$

f is not a function, it is a tensor product.

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Kolya: This vector $\beta_1 \dots \beta_{2n}$ is in $(\mathbb{C}^2)^n$. The form factor is function valued in this space, multi-valued. **Fodor:** No, meromorphic.

$$2\pi i \,\mathrm{res} f = (1 - e^{-\frac{\pi i \nu}{1-\nu}\alpha} S \dots S) f_{\mathcal{O}_\alpha}(\beta_1 \dots \beta_{2n-2}) \otimes S_{2n-1,2n} \tag{***}$$

$s_{ij} = e_i^+ \otimes e_j^- + e_i^- \otimes e_j^+$

We need to solve this system of equations. We work for many years, and then:

$$f_{\mathcal{O}_\alpha}(\beta_1 \dots \beta_{2n}) = \sum_{\substack{\{1,\dots,2n\} = I^- \cup I^+ \\ \#I^- = \#I^+ = n}} w^{\epsilon_1 \cdots \epsilon_{2n}}(\beta_1 \dots \beta_{2n}) \mathcal{F}_{\mathcal{O}_\alpha}(\beta_{I^-} | \beta_{I^+})$$

Here I is the index set, and we sum over ways to break it into two pieces, and ϵ records which one. $\epsilon_j = \pm$ if $\beta_j \in I^\pm$.

$$S_{i,i+1} w^{\cdots \epsilon_i \epsilon_{i+1} \cdots}(\dots \beta_i \beta_{i+1} \dots) = w^{\cdots \epsilon_{i+1} \epsilon_i \cdots}(\dots \beta_{i+1} \beta_i \dots)$$

So let me write σ so that $S = e^\sigma$ and $s = e^{\frac{2\nu}{1-\nu}\sigma}$. And let me introduce a function satisfying the following equation:

$$\chi(\sigma + 2\pi i) p(sq^4) = \chi(\sigma) p(sq^2) \qquad p(s) = \prod_{j=1}^{2n} (s - b_j)$$

Here $q = e^{\pi i/(1-\nu)}$. This is just a difference equation, so it is easy to solve. **Kolya:** There are infinitely many solutions. **Fodor:** If I ask it to be regular for $0 > \Im(\sigma) > -\pi$, then this more or less makes it unique.

As a prize for solving this equation, you get that this function satisfies another equation:

$$\chi(\sigma + \frac{1-\nu}{\nu}\pi i)\, P(SQ) = \chi(\sigma)\, P(-S) \qquad P(S) = \prod (S - B_j)$$

$$\chi(\sigma) \underset{\sigma \to -\infty}{\simeq} e^{-2n\frac{\sigma}{1-\nu}} x^+(s) X^+(S), \qquad x^+(s) = 1 + \sum_{j=1}^\infty s^j x_j^+$$

$$\chi(\sigma) \underset{\sigma \to +\infty}{\simeq} x^-(s) X^-(S)$$

$$I_\alpha(\beta_1 \dots \beta_{2n}) = \int_{\mathbb{R} \smallsetminus 0} \chi(\sigma | \beta_1 \dots \beta_{2n}) e^{\frac{\nu\alpha}{1-\nu}\sigma} \mathrm{d}\sigma$$

This is just Laplace transform. The way it is written, it is obvious that the integral is defined for $0 < \Re(\alpha) < 2n/\nu$. This integral can be continued in α to entire complex plane, with only simple poles, at points like $\alpha = 2n + 2m + (2n + \ell)\frac{1-\nu}{\nu}$ and $\alpha = -2m - \ell\frac{1-\nu}{\nu}$, with $\ell, m \geq 0$. This is an

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important legacy that we get from our great predecessors that if you see a function, you have to continue it analytically.

So I will consider Laurant polynomials $\ell(s)$ and $L(S)$, and for such polynomials I will define a pairing $(\ell, L)_\alpha$ by two things:

- Bilinear.
- If $\ell(s) = s^m$ and $L(S) = S^k$, then $(\ell, L)_\alpha = I_{\alpha+2m+\frac{1-\nu}{\nu}k}$.

So I just put them under the integral. And this function is constructed out of this pairing.

A few words. What are you used to in usual classical mathematics? That you may pair differential forms and cycles by an integral. After the quantization, they become on the same footing: both of them become like differential forms, but in different variables. So in the classical limit, one of these guys explodes, and then it becomes an integral, so you should think of the pairing as the pairing between forms and cycles. **Kolya:** Which is forms and which is cycles? **Fodor:** This is a difficult question. There are two classical limits, one with $\nu \to 1$ and the other with $\nu \to 0$, and they switch. **Kolya:** This $\nu \to 0, 1$, this is some kind of duality, which can be interpreted at "T-duality." **Fodor:** Come on. Don't use these funny words.

SO, if we take antisymmetric polynomials $\ell_1 \wedge \dots \ell_n = \ell^{(n)}$, and similarly for L, then the definition is

$$(\ell^{(n)}, L^{(n)})_\alpha = \det(\ell_i, L_j)_\alpha$$

This is just free antisymmetric product on space of polynomials.

So then we have special functions $\ell_{I^- \sqcup I^+, j}(s)$ (explicit formula, but doesn't matter), and:

$$\mathcal{F}_{\mathcal{O}_\alpha}(\beta_{I^*}|\beta_{I^-}) = (\ell_{I^- \sqcup I^+,1} \wedge \dots \wedge \ell_{I^- \sqcup I^+,n}, L^{(n)})_\alpha$$

and they automatically satisfy the main equations (*,**) to satisfy.

So we have satisfied almost everything, but we still need the residues (***).

Then we want symmetric polyonimals $L^{(n)}_{\mathcal{O}_\alpha}(S_1, \dots, S_n | B_1, \dots, B_{2n})$, and these must satisfy:

$$L^{(n)}_{\mathcal{O}_\alpha}(S_1, \dots, S_{n-1}, B | B_1, \dots, B_{2n-2}, B, -B) = B \prod_{j=1}^{n-1} (B^2 S_p^2) L^{(n-1)}_{\mathcal{O}_\alpha}(S_1 \dots S_{n-1} | B_1 \dots B_{2n-2}) \tag{****}$$

If you know conformal field theory, you find all of the solutions to these by counting solutions to simple equations. The simplest one is:

$$L^{(n)}_{\Phi_\alpha}(S_1, \dots, S_n) = \langle \Phi_\alpha \rangle S \wedge S^3 \wedge \dots \wedge S^{2n-1}$$

And I did not say, but there are some exact forms here — the pairing vanishes on some special forms. So using that we can make all the polynomials $L^{(n)}$ to be odd in the variables. Then the rest of the polynomials correspond to some fermionc structure. The one above is: all places filled from 1 to $2n-1$. Or you create a hole somewhere and add a particle somewhere else.

Kolya: So this is like the representation of the Clifford algebra.

So, n is fixed, but you can take it huge, and then start doing a simple thing, but as it goes to small n it will all mix up.

1 Free fermions

So, there is a special case, where $\nu = \frac{1}{2}$. Then $S = -1$, and the form factor is

$$f(\beta_1 \dots \beta_{2n})_{+\dots+-\dots-} = \left(\frac{2\sin\frac{pi\alpha}{2}}{\pi}\right)^n e^{\frac{1}{2}\sum \beta^+ - \beta^-} \frac{\prod_{i<j}\sinh\frac{1}{2}(\beta_i - \beta_j)}{\prod \sinh(\beta_1^+ - \beta_1^-)}$$

Where then there are some very particular differential equations, called Penn-Levay **?**, and they have a tau function τ, and then:

$$\frac{\langle \Phi_{\alpha_1}(x)\Phi_{\alpha_1}(0)\rangle}{\langle \Phi_\alpha(0)\rangle} = \frac{\langle \phi_{\alpha_1}\rangle\langle \Phi_{\alpha_2}\rangle}{\langle \Phi_{\alpha_2}\rangle}\tau((\frac{1}{2}Mr)^2)$$

and from this we can satisfy the short time asymptotics:

$$\frac{\langle \Phi_{\alpha_1}(x)\Phi_{\alpha_1}(0)\rangle}{\langle \Phi_\alpha(0)\rangle} = r^{\frac{\alpha_1\alpha_2}{2}}\left(1 + \frac{\alpha_1\alpha_2}{4\alpha^2}\left\{(Mr)^2 + \frac{4(Mr)^{2(1+\alpha)}}{(\alpha+2)^2}s - \frac{4(Mr)^{2(1-\alpha)}}{(2-\alpha)^2}s^{-1} + \dots\right\}\right)$$

where s has some explicit description in gamma functions. I will show in my talks how to do this in general.

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