Integration over Local Fields via Hrushovski-Kazhdan Motivic Integration

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0 Introduction

The goal of this document is to give a reasonably self-contained description of Hrushovski-Kazhdan's proof of the uniformity of integration over local fields through their theory of motivic integration over $\mathrm{ACVF}_{(0,0)}$, the theory of nontrivially-valued algebraically closed fields of characteristic and residue characteristic 0. As a result, as far as this paper is concerned there is nothing new under the sun: the bulk of what I've done is repeat, reorganize, and motivate certain aspects of their very deep paper *Integration in Valued Fields*. I assume familiarity with the basic definitions and techniques of model theory and state without proof some of the important results about the model theory of ACVF and its completions.

Except when otherwise specified, I use ACVF to mean the completion $\mathrm{ACVF}_{(0,0)}$, and denote assume that we work over parameters from a fixed model $E \models \mathrm{ACVF}_{(0,0)}$. We work in the multisorted language $\mathcal{L}_{K,k,\Gamma}$ given by

$$((+_K, \times_K, -_K, 1_K, 0_K), (+_k, \times_k, -_k, 1_k, 0_k), (0_\Gamma, +_\Gamma, -_\Gamma, <_\Gamma), (v : K^\times \to \Gamma); (res : \{x \in K \mid v(x) \geq 0\} \to k);$$

occasionally we will work with parameters over a model $E \models \mathrm{ACVF}_{(0,0)}$. We will also later introduce the sort RV in the course of the paper.

Thanks are due to Silvain Rideau and Thomas Scanlon for stimulating discussion on the topic.

1 Geometry in $\mathrm{ACVF}_{(0,0)}$

In this section we review some of the basic facts about $\mathrm{ACVF}_{(0,p)}$, with special attention paid to the case of $\mathrm{ACVF}_{(0,0)}$.

1.1 Basics of ACVF: Quantifier Elimination and its Consequences

The main early result in the model theory of valued fields was Robinson's quantifier elimination theorem:

Theorem 1.1. For all primes p and $p = 0$, $\mathrm{ACVF}_{(0,p)}$ is complete and eliminates quantifiers in the language $\mathcal{L} = \mathcal{L}_{K,k,\Gamma}$

Quantifier elimination, combined with a little geometric insight, gives a characterization of the definable subsets of K^1 in this language:

Proposition 1.2. Every definable subset $X \subseteq \mathrm{VF}$ is a finite union of *swiss-cheeses*, i.e. sets of the form $B \setminus \bigcup_{i=1}^{m} B_i$.

Proof. This follows from quantifier elimination, the fact that every polynomial $p(x) \in K[x]$ splits by algebraic closedness, and the easy fact that if $B, B' \subseteq \mathrm{VF}$ are balls, then either $B \subseteq B'$, $B' \subseteq B$, or $B \cap B' = \varnothing$ □

A fact that will be crucial for the proof of uniformity of integration over local fields is the fact that $\mathrm{ACVF}_{(0,0)}$ is a certain "limit theory" of the various $\mathrm{ACVF}_{(0,p)}$ as p ranges over all primes. More technically:

Theorem 1.3. (Transfer Principle) For all sentences ϕ, $\mathrm{ACVF}_{(0,0)} \models \phi$ if and only if $\{p \mid \mathrm{ACVF}_{(0,p)} \models \phi\}$ is cofinite.

Proof. For any nonprincipal ultrafilter $\mathcal{D}$ on the set of primes and any choice $K_p \models \mathrm{ACVF}_{(0,p)}$, $\prod_{D} K_p \models \mathrm{ACVF}_{(0,0)}$. But since $\mathrm{ACVF}_{(0,0)}$ and $\mathrm{ACVF}_{(0,p)}$ are all complete, the actual choice of nonprincipal ultrafilter $\mathcal{D}$ is immaterial. But since the intersection of all nonprincipal ultrafilters on the set of primes $\bigcap \mathcal{D}$ is simply the *Fréchet filter* (also called the *cofinite filter*) $\mathcal{F}$, we have that $\mathrm{ACVF}_{(0,0)} \models \phi$ if and only if $\{p \mid \mathrm{ACVF}_{(0,p)} \models \phi\} \in \mathcal{F}$, as desired. □

1.2 Orthogonality

Definition 1.4. Two definable sets X and Y are said to be *strongly orthogonal* if every definable subset $D \subseteq X^n \times Y^m$ is a finite union of definable sets of the form $E \times F$, with $E \subseteq X^n$ and $F \subseteq Y^m$ definable.

Note that if X and Y are strongly orthogonal, then so are X^n and Y^m, for all $n, m \in \mathbb{N}$. One of the main uses of this notion is that functions between strongly orthogonal sets are trivial:

Proposition 1.5. Let X and Y be strongly orthogonal and $f : X \to Y$ a definable map. Then $f(X)$ is finite.

Proof. Let $f : X \to Y$ be definable, and consider its graph $\Gamma_f = \{(x, y) \mid f(x) = y\} \subseteq X \times Y$. By strong orthogonality, $\Gamma_f = \bigcup_{i=1}^{m} E_i \times F_i$ for $E_i \subseteq X$, $F_i \subseteq Y$ definable. In order to be the graph of a function, each F_i is forced to be a *singleton* $\{y_i\}$, so that

$$\Gamma_f = \bigcup_{i=1}^{m} E_i \times \{y_i\}$$

and so we may decompose $X = \bigcup_{i=1}^{n} E_i$, and on each E_i, $f \restriction_{E_i} (x) = y_i$ is constant. Thus $f(X)$ is finite. □

This definition has an equivalent formulation in terms of definable families that we will find useful:

Proposition 1.6. X and Y are strongly orthogonal if and only if for all tuples $\overline{a}$ in Y and for all substructures B, if $D \subseteq X^n$ is definable over $B\overline{a}$ then D is definable over B.

Proof. Suppose that X and Y are strongly orthogonal, that B is a substructure and $\overline{a}$ is a tuple of elements of Y, and that $D \subseteq X^n$ is definable over $B\overline{a}$. The proof works the same for B as it does for $\varnothing$, so we assume that $B = \varnothing$. Then $D = \phi(\overline{x}, \overline{a})$ Consider the formula $\phi(\overline{x}, \overline{y})$, defining the set $\tilde{D} \subseteq X^n \times Y^m$. By strong orthogonality we have that $\tilde{D} = \bigcup_{i=1}^{k} E_i \times F_i$ with $E_i \subseteq X^n$ and $F_i \subseteq Y^m$ definable. Consider the sets

$$\tilde{D}_{\overline{a}} = \{\overline{x} \,|\, (\overline{x}, \overline{a}) \in \tilde{D}\}$$

for any $\tilde{a} \in \overline{y}$. Let $\mathrm{supp}(\overline{a}) = \{i \,|\, \overline{c} \in F_i\}$. By construction

$$D = \tilde{D}_{\overline{a}} = \bigcup_{i \in \mathrm{supp}(\overline{a})} E_i$$

which is definable over $\varnothing$, as desired.

Conversely, suppose that for all tuples $\overline{a}$ in Y and for all substructures B, if $D \subseteq X^n$ is definable over $B\overline{a}$ then D is definable over B. We wish to show that X and Y are strongly orthogonal. Let $\tilde{D} \subseteq X^n \times Y^m$ be definable; we wish to show that it is a union of rectangles. By assumption, the fibers$\tilde{D}_{\overline{c}}$ as defined above are in fact defined over $\varnothing$. By compactness, there is a finite family $\{E_i\}_{i \in [k]}$ such that for all $\overline{c}$ there is some i with $\tilde{D}_{\overline{c}} = E_i$. Now define $F_i := \{\overline{a} \,|\, \tilde{D}_{\overline{a}} = E_i\} \subseteq Y^n$. Then

$$\tilde{D} = \bigcup_{i=1}^{k} E_i \times F_i$$

with E_i, F_i definable. □

In the context of $\mathrm{ACVF}_{(0,0)}$, all stable sets Y are strongly orthogonal to Γ:

Proposition 1.7. (HK- 3.10) Let Y be a stable definable set in $\mathrm{ACVF}_{(0,0)}$. Then Y is strongly orthogonal to Γ.

Proof. Let Y be a stable definable set in ACVF. As a first approximation to strong orthogonality, we claim that that if $f : Y \to \Gamma$ is definable then $f(Y)$ is finite. If $f(Y)$ were infinite then its elements are ordered by Γ's ordering $<$; if $(f(Y), <)$ has order type I, list $f(Y) = \{c_i\}_{i \in I}$ with $c_i < c_j$ if and only if $i < j$. The formula $\phi(x; y) : f(x) < f(y)$ orders an infinite subset of Y as follows: pick for each c_i an element $z_i \in f^{-1}(c_i)$; then $\phi(z_i, z_j)$ holds if and only if $i < j$, contradicting the stability of Y.

By the previous proposition It suffices to show that any subset of Γ defined over a is in fact defined over $\varnothing$. By the o-minimality of DOAG and stable embeddedness of Γ, we have that any a-definable subset X of Γ is a finite union of points and intervals, so $X = \bigcup_{i=1}^{m} I_i$ where I_i is a point or an open interval. Now, given a the left and right endpoints of each interval I_i are definable, defined by functions $f_{i,+}, f_{i,-} : Y \to \Gamma$. But each such function has finite image, say $f_{i,\pm} = \{c_1, \cdots, c_m\}$. But then each c_j is in fact definable, so that X itself was already definable in Γ because an interval is interdefinable with its endpoints. Hence Y and Γ are strongly orthogonal. $\square$

2 Grothendieck Semirings

2.1 Abstract Grothendieck Semirings

In order to study motivic measures and generalized Euler characteristics, the abstract notion of the Grothendieck semiring of a category $\mathcal{C}$ is a very useful tool.

Definition 2.1. Let $\mathcal{C}$ be a category with finite products $\times$, coproducts $\oplus$, an initial element $0_{\mathcal{C}}$, and terminal object $1_{\mathcal{C}}$. Then the *Grothendieck semiring* of $\mathcal{C}$, $\mathrm{K}_+(\mathcal{C})$, is the free semiring $\mathbb{N}\left[\left\{[X]\right\}_{X \in \mathrm{Ob}(\mathcal{C})}\right]$ generated by formal classes $[X]$ for each $X \in \mathrm{Ob}(\mathcal{C})$, subject to the following relations:

- $[X] = [Y]$ if there is an isomorphism $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$.
- $[X] + [Y] = [X \oplus Y]$
- $[X] \cdot [Y] = [X \times Y]$.
- $[0_{\mathcal{C}}] = 0$
- $[1_{\mathcal{C}}] = 1$

The main technical results needed by Hrushovski and Kazhdan to prove the uniformity of p-adic integration were results about the *decomposition* of certain Grothendieck semirings associated to certain categories of definable sets. We outline the construction of some relevant categories here:

Definition 2.2. Let T be a complete theory naming at least two distinct constants c and d. The category $\mathrm{Def}(T)$ of definable sets has definable sets as objects and definable functions as morphisms. As a category, $\mathrm{Def}(T)$ admits

- An initial object $\varnothing$ and a terminal object $\{c\}$.
- Finite products, where $A \times B$ is simply the cartesian product.
- Finite coproducts, where $A \oplus B = (A \times \{c\}) \cup (\{d\} \times B)$ is simply the disjoint union of A and B.

It is straightforward to verify that the category $\mathrm{Def}(T)$ defined actually has all the structure alluded to. We will often have occasion to consider subcategories $\mathcal{C} \subset \mathrm{Def}(T)$; our convention will be that $\mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathrm{Def}(T))$ and that all initial and terminal morphisms, as well as the canonical product and coproduct maps exist in $\mathcal{C}$.

2.2 Grothendieck Semirings Coming from $\mathrm{ACVF}_{(0,0)}$

The main structure whose Grothendieck semiring we wish to understand is that of $\mathrm{ACVF}_{(0,0)}$. In line with typical approaches to studying valued fields, one may try to reduce problems about a valued field $K \models \mathrm{ACVF}_{(0,0)}$ to problems about the residue field k (which models ACF_0) and the value group Γ (which models DOAG). One way that this can be done is by considering the auxiliary structure RV in the language of groups $(\times, 1)$, interpreted in ACVF by the identification $\mathrm{RV}(K) = K^\times/1+\mathcal{M}$. RV encodes a lot of valuative information, since the valuation map $v : K^\times \to \Gamma$ descends to a map $v_{\mathrm{rv}} : \mathrm{RV} \to \Gamma$ as $v(1+\mathcal{M}) = \{0\}$. Moreover, there is a natural embedding $k^\times \to \mathrm{RV}$ since the cosets of $\mathcal{O}_K^\times/(1+\mathcal{M})$ correspond exactly to elements in $k^\times$.

These ACVF-interpretable structures fit into a short exact sequence

$$0 \longrightarrow k^\times \xrightarrow{\ i\ } RV \xrightarrow{\ v_{\mathrm{rv}}\ } \Gamma \longrightarrow 0$$

since $\mathcal{O}_K^\times/(1+\mathcal{M})) \cong k^\times$ is precisely the kernel of v_{rv}. This short exact sequence allows us to decompose definable sets of RV into definable sets in $k^\times$ and Γ. Towards this direction we need the following technical lemma:

Lemma 2.3 (HK 3.21)**.** Let T be a theory with a short exact sequence of definable abelian groups

$$0 \longrightarrow A \xrightarrow{\ i\ } B \xrightarrow{\ \theta\ } C \longrightarrow 0$$

with i and θ definable functions. Suppose further that A and B are such that

- A^{eq} and C^{eq} are stably embedded and strongly orthogonal.
- For all $n \in \mathbb{N}$, every definable subgroup of A^n is defined by finitely many $\mathbb{Z}$-linear equations.
- If $P \subseteq B^n$ is such that the induced projection $\theta : P \to C^n$ has each fiber $\theta^{-1}(\bar{c})$ finite for all $\bar{c} \in C^n$, then P is finite.

Then for every definable set $Z \subseteq B^n$ there is a finite number m such that for all $i \le m$ there are $Y_i \subseteq \overline{b_i'} + A^{k_i}$ definable subsets of a *single* coset of A^{k_i}, $N_i \in M_{n,k_i}(\mathbb{Z})$, and $W_i \subseteq C^n$ definable such that

$$Z = \bigcup_{i=1}^{m} \{\bar{b} \in B^n \mid \theta(\bar{b}) \in W_i \wedge N_i\bar{b} \in Y_i\}$$

Proof. By replacing B^n with B and $\mathbb{Z}$ with $M_n(\mathbb{Z})$, we may assume that $n = 1$ for the purposes of the argument. Let $Z \subseteq B$ be definable. To decompose Z into a finite union of sets of the above form, first decompose Z as

$$Z = \bigcup_{c \in C} \left(Z \cap \theta^{-1}(c)\right)$$

Each fiber $\theta^{-1}(c)$ is a coset of A, so if $b \in \theta^{-1}(c)$ then $\theta^{-1}(c) - b \subseteq A$, and hence $(\theta^{-1}(c) \cap Z) - b \subseteq A$. For definable subsets $X \subseteq b + A$ and $X' \subseteq b' + A$, write $[X] = [X']$ just in case there is an $a \in A$ such that $(X - b) + a = X' - b'$.

As $(\theta^{-1}(c) \cap Z) - b$ is parameter-definable, by the stable-embeddedness of A there is a definable set $X(\overline{t}) \subseteq A \times A^{|\overline{t}|}$ such that for all $c \in C$ and $b \in \theta^{-1}(c)$ there is an $\overline{a} \in A^{|\overline{t}|}$ such that

$$(\theta^{-1}(c) \cap Z) - b = X(\overline{a}).$$

Define an equivalence relation E on $A^{|\overline{t}|} \times A^{|\overline{t}|}$ by setting

$$\overline{a} E \overline{a'} \iff (\exists t \in A)[X(\overline{a}) = t + X(\overline{a'})].$$

This relation allows us to construct a well-defined definable map $\alpha : C \to (A^{|\overline{t}|})^2/E$ given by setting $c \mapsto [\overline{a}]_E$ for the *unique* E-class such that $[(\theta^{-1}(c) \cap Z)] = [X(\overline{a})]$ for all $\overline{a} \in [\overline{a}]_E$. This map is well-defined in spite of the fact that *a priori* different choices of $b \in (\theta^{-1}(c) \cap Z)$ lead to different $X(\overline{a})$, since they differ by at most a translation by an element of A.

By the orthogonality of A and C, (**here we are using the assumption about** A^{eq} **and** C^{eq}) α must have finite image. Since the image of α is finite, we can break up C into finitely many (parameter) definable sets $C = \bigcup_{i=0}^{k} C_i$ on which $\alpha \restriction_{C_i}$ is constant. Without loss of generality we assume that α is constant on C. This means that for all $c \in C$, $\alpha(c) = [\overline{a}]_E$ and that for all c there is a $b \in B$ with

$$(\theta^{-1}(c) \cap Z) - b = X([\overline{a}]_E) := X.$$

Consider the stabilizer

$$S = \{a \in A \mid a + X = X\} = \{a \in A \mid (\forall c \in C_i)\ a + (\theta^{-1}(c) \cap Z) = \theta^{-1}(c) \cap Z\} = \{a \in A \mid a + Z = Z\}$$

with the last equality coming from the fact that $Z = \bigcup_{c \in C} (\theta^{-1}(c) \cap Z)$ as well as the constancy of α. Note that for *some* $b \in \theta^{-1}(c)$,

$$(\theta^{-1}(c) \cap Z) - b = X.$$

While b need not be well defined, its class $b + S \in B/S$ *is* well defined, yielding a definable function $\beta : C \to B/S$ mapping $c \mapsto b+S$ for the unique class $b+S$ yielding $(\theta^{-1}(c) \cap Z) - b = X$.

By construction $S \subseteq A$ is a definable group, and so $S = \bigcap_{j=1}^{\ell} \ker(T_j : A \to A)$ for some $T_j \in M_n(\mathbb{Z})$. As $S \subseteq \ker(T_j)$, this implies that the maps $\beta_j(c) := T_j(\beta(c)) \in B$ is well-defined, despite *a priori* only being a well-defined element of B/S. If $d \in \ker(T_j : C \to C)$ then $(\theta \circ \beta_j)(d + c) = T_j(c)$ and so $\beta_j(d + c) - \beta_j(c) \in A$. Thus the function $F_j : C \times \ker(T_j : C \to C) \to A$ given by $F_j(x, y) = \beta_j(x + y) - \beta_j(x)$ has finite image by the orthogonality of C and A. But this implies that the set

$$\{\beta_j(c) \mid c \in \theta(Z)\} \subseteq B$$

has finite-to-1 projection and therefore, by the hypothesis of the lemma, is itself finite for all j. Take $N = (T_1, \cdots, T_\ell)$ and let $U = N(Z)$; by construction, this implies that $\theta^\ell(U)$ is finite so that U is contained in a finite union $\bigcup_{k=1}^{m} b_k + A$. But then C and U are orthogonal as C and $b_k + A$ are orthogonal for all (finitely many) k. But then

$$\{(\theta(z), Nz) \mid z \in Z\} = \bigcup W_h \times U_h$$

is a union of rectangles. Now if $b \in B$ is such that $\theta(b) \in W_h$ and $N(b) \in U_h$ then by the above decomposition there is some $z \in Z$ with $\theta(z) = \theta(b)$ and $N(z) = N(b)$. But then $b - z \in S$ and so $b \in S + z \subseteq S + Z = Z$ and so $b \in Z$. Thus

$$Z = \bigcup \{z \mid \theta(z) \in W_h \wedge Nz \in U_h\}$$

as desired. □

This lemma allows us to decompose the Grothendieck semiring of B in terms of those of A and C:

Corollary 2.4. (HK 3.25) Let T be a theory as in the previous lemma with an exact sequenence of abelian groups

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\theta} C \longrightarrow 0$$

with C linearly ordered. Then every definable set Z of B^n is a disjoint union of images under the action of $GL_n(\mathbb{Z})$ of definable sets of the form $X \times \theta^{-1}(Y)$ with $X \subseteq A^m$ with $\#\theta(X) = 1$ and $Y \subseteq C^{m-n}$.

Moreover, the Grothendieck semiring $\mathrm{K}_+(\mathrm{Def}(B))$ is generated by the classes $[Y]$ for $Y \subset B^n$ with $\#\theta(Y) = 1$ and pullbacks $\theta^{-1}(W)$ for $W \subseteq C^n$ definable, ranging over all $n \in \mathbb{N}$.

Proof. By the above lemma we may decompose any definable $Z \subseteq B^n$ as a finite union of sets of the form

$$X = \{z \in B^n \mid \theta(z) \in W \wedge N(z) \in Y\}$$

with $N \in M_{n,k}(\mathbb{Z})$. By performing elementary matrix operations we may assume that N is the composition of a projection $\pi : \mathbb{Z}^n \to \mathbb{Z}^k$ with a diagonal $\tilde{N} \in M_k(\mathbb{Z})$ with *nonzero* determinant. This means that after composing with the projection, we have that $\theta((\pi(X)))$ is *finite* as all its fibers are finite. As C is linearly ordered, every element of $\theta((\pi(X))) = \{c_1, \cdots, c_k\}$ is in fact definable, and so we may write W above as $\bigcup_{i=1}^{k} \{c_i\} \times W_i$, yielding

$$X = \bigcup_{i=1}^{k} [\theta^{-1}(c_i) \cap \pi(X)] \times \theta^{-1}(W_i)$$

as desired. Moreover, this union is clearly a disjoint union.

The statement about $\mathrm{K}_+(\mathrm{Def}(B))$ being generated by the the classes $[Y]$ for $Y \subset B^n$ with $\#\theta(Y) = 1$ and pullbacks $\theta^{-1}(W)$ for definable $W \subseteq C^n$ follows immediately from the above decomposition theorem. □

To apply this lemma to the exact sequence

$$0 \longrightarrow k^\times \xrightarrow{i} RV \xrightarrow{v_{\mathrm{rv}}} \Gamma \longrightarrow 0$$

we simply have to verify the hypotheses.

Proposition 2.5. 1. $k^\times$ and Γ are stably embedded and $(k^\times)^{eq}$ and Γ^{eq} are strongly orthogonal.

2. For all $n \in \mathbb{N}$, every definable subgroup of $k^\times$ is defined by finitely many $M_n(\mathbb{Z})$-linear equations.

3. If $P \subseteq \mathrm{RV}^n$ is such that the induced projection $\theta : P \to \Gamma^n$ has each fiber $\theta^{-1}(\overline{\gamma})$ finite for all $\overline{\gamma} \in \Gamma^n$, then P is finite.

Proof. 1. By the quantifier elimination of ACVF, $(k, +, -, \times, 0, 1)$ and $(\Gamma, +, 0)$ are pure stably embedded. As $(k, +, -, \times, 0, 1)$ is stable, any definably interpretable structure is stably embedded in it ([Poizat 12.31: "Parameter Separation Theorem"]), and so $(k^\times)$ with its induced structure from ACF is stably embedded.

I claim that $(k^\times)^{eq}$ and $(\Gamma)^{eq}$ are strongly orthogonal; since both structures eliminate imaginaries it suffices to show that $k^\times$ and Γ are strongly orthogonal. We showed this in the previous section, in Proposition 1.7.

2. Since $k^\times$ has Morley rank and Morley degree 1, its proper definable subgroups must be rank 0, i.e. finite. It is well known that the finite subgroups of $k^\times$ are defined by finitely many $\mathbb{Z}$-linear equations, as they are defined by equations of the form $x^k = 1$. The case for general $(k^\times)^n$ follows by induction.

3. (HK; 3.7 + 3.11) We prove the result by induction on the ambient dimension of P:

   (a) ($n = 1$) Suppose that $P \subseteq \mathrm{RV}$ is definable with $v_{\mathrm{rv}} : P \to \Gamma$ finite-to-1. Consider the pullback $\mathrm{rv}^{-1}(P) \subseteq K^\times$. This is a definable set, and so by quantifier elimination for ACVF, $\mathrm{rv}^{-1}(P)$ is a finite union of disjoint swiss cheeses, that is, $\mathrm{rv}^{-1}(P) = \bigcup_{i=1}^{m} C_i$ where each slice of cheese C_i is given by $C_i = B_i \setminus (\bigcup_{j=1}^{\ell} B_{ij})$ where B and the B_i are balls. Suppose for contradiction that P is infinite. Then, in particular, $\mathrm{rv}(C) \subseteq P$ is infinite for one of the disjoint swiss cheeses decomposing $\mathrm{rv}^{-1}(P)$. Since balls are either nested or disjoint in ACVF, there are two cases to consider:

      i. If $C = B \setminus \bigcup B_i$ with B closed and $\mathrm{rad}(B) = \mathrm{rad}(B_i)$ for some i, then C is an infinite subset of a thin annulus, and therefore $\mathrm{rv}(C)$ contains a cofinite subset of the k-affine space $\mathrm{RV}_\gamma = \mathrm{rv}^{-1}(\gamma)$, contradicting the assumption that P had finite fibers.

      ii. If $C = B \setminus \bigcup B_i$ with $\mathrm{rad}(B) < \mathrm{rad}(B_i)$ for all i strictly, then C contains a ball, so that $\mathrm{rv}(C)$ contains a full fiber $v_{\mathrm{rv}}(W)$ for some nonempty $W \subseteq \Gamma$, contradicting the assumption that P has finite fibers.

Thus P cannot contain any infinite swiss cheeses, and so P is finite.

(b) $(n > 1)$ By induction, for each projection $\pi_i : P \to \mathrm{RV}^{n-1}$ for $1 \leq i \leq n$ we have that $v_{\mathrm{rv}} : \pi_i(P) \to \Gamma^{n-1}$ is finite-to-1 and hence, by inductive hypothesis, $\pi_i(P)$ is finite. But then P itself must be finite by the Pidgeonhole Principle.

□

Motivated by the decomposition lemma describing subsets of RV^n we define the following subcategories of definable sets sufficient to "see" all of RV^n.

Definition 2.6.

- The category $\Gamma[n]$ has as its objects definable subsets of Γ^n and a definable function $f : X \to Y$ for $X, Y \in \Gamma[n]$ is in $\mathrm{Mor}_{\Gamma[n]}(X, Y)$ if and only if f is a bijection and there exists a partition $X = \bigcup_{i=1}^{m} X_i$ into definable sets such that on each X_i,

$$f \restriction_{X_i} (x) = Tx$$

for some $T \in GL_n(\mathbb{Z}) \ltimes A^n = Aff_n^{\mathbb{Z}}(A)$ an integral affine translation.

- The category $\Gamma[*]$ has

$$\mathrm{Ob}(\Gamma[*]) = \mathrm{Ob}(\mathrm{Def}(\mathrm{DOAG})) = \bigcup_{i \in \omega} \Gamma[i]$$

and

$$\mathrm{Mor}_{\Gamma[*]}(X, Y) = \bigcup_{i \in \omega} \mathrm{Mor}_{\Gamma[i]}(X, Y).$$

- Let $\Gamma^{fin}[n]$ be the subcategory of $\Gamma[n]$ consisting of only the finite subsets with *all* $\Gamma[n]$ morphisms betweent them. Define $\Gamma^{fin}[*]$ by setting $\mathrm{Ob}(\Gamma^{fin}[*]) = \bigcup_{i \in \omega} \Gamma^{fin}[i]$ and by setting $\mathrm{Mor}_{\Gamma^{fin}[*]}(X, Y) = \bigcup_{i \in \omega} \mathrm{Mor}_{\Gamma^{fin}[i]}(X, Y)$.

- The category $\mathrm{RES}[n]$ will be the defined as the category of *generalized algebraic varieties* whose structure is given by definable subsets of products of $\mathrm{RV}_\gamma = v_{\mathrm{rv}}^{-1}(\gamma)$. Concretely, a *generalized algebraic variety* X is a subset $X \subseteq \prod_{i=1}^{n} \mathrm{RV}_{\gamma_i} := \mathrm{RV}_{\overline{\gamma}}$ that is the intersection of finitely many zero sets of $\overline{\gamma}$*-polynomials* F, i.e. a polynomial $F(\overline{x}) = \sum_\eta c_\eta x^\eta \in K[\overline{x}]$ (for $\eta = (\eta_1, \cdots, \eta_n)$ a multi-index) that defines a function

$$F(x) : \mathrm{RV}_{\overline{\gamma}} \to k.$$

In other words, F is a $\overline{\gamma}$-polynomial just in case, for every nonzero monomial term $c_\eta X^\eta$ occurring in F,

$$v(c_\eta) + \sum_{i=1}^{n} \eta_i \cdot \gamma_i = 0.$$

The objects of $\mathrm{RES}[n]$ are given by

$$\mathrm{Ob}(\mathrm{RES}[n]) = \{X \subseteq \mathrm{RV}_{\overline{\gamma}} \mid X \text{ is a finite Boolean combination of generalized algebraic varieties}\}$$

and $f : X \to Y$ is a morphism in RES$[n]$ just in case f is a definable bijection between them.

- The category RES$[*]$ has
$$\mathrm{Ob}(\mathrm{RES}[*]) = \bigcup_{i\in\omega} \mathrm{RES}[i]$$
and
$$\mathrm{Mor}_{\Gamma[*]}(X,Y) = \bigcup_{i\in\omega} \mathrm{Mor}_{\mathrm{RES}[i]}(X,Y).$$

- The category RV$[*]$ has as its objects all definable subsets of RV^n for all $n \in \omega$, and has as morphisms all definable bijections between sets $X, Y \subseteq \mathrm{RV}^n$.

- The category VF$[*]$ has as its objects all definable subsets of RV^n for all $n \in \omega$, and has as morphisms all definable bijections between sets $X, Y \subseteq \mathrm{VF}^n$.

The motivation for defining $\Gamma[*]$ the way we did has to do with the fact that maps $f \in \mathrm{Mor}_{\Gamma[*]}(X,Y)$ are precisely the DOAG-definable maps able to be *lifted* to definable maps $\tilde{f} : v_{\mathrm{rv}}^{-1}(X) \to v_{\mathrm{rv}}^{-1}(Y)$:

Proposition 2.7. ([HK, 3.26, 3.28]) Let $Z \subseteq \Gamma^n$ and $f : Z \to \Gamma$ definable such that there is a base E and E-definable $\tilde{Z} \subseteq \mathrm{RV}^n$ and $\tilde{f} : \tilde{Z} \to \mathrm{RV}$ with $v_{\mathrm{rv}}(\tilde{f}) = f(v_{\mathrm{rv}}(x))$. Then there exists a partition $Z = \bigcup_{i=1}^{m} Z_i$ such that for each Z_i, $f|_{Z_i}(x) = (\sum_{j=1}^{n} \ell_j z_j) + \gamma$ for some $(\ell_1, \cdots, \ell_n) \in \mathbb{Z}^n$ and $\gamma \in \Gamma$. Thus, every *definable* bijection lifting f is piecewise given by an element $T \in GL_n(\mathbb{Z}) \ltimes E^n$.

Proof. Clearly for any $f \in \mathrm{Mor}_{\Gamma[*]}(X,Y)$ there is an $\tilde{f} \in \mathrm{Mor}_{\mathrm{Def(RV)}}(v_{\mathrm{rv}}^{-1}(X), v_{\mathrm{rv}}^{-1}(Y))$ with $v(\tilde{f})(x) = f(v(x))$, since any function of the form $f(x) = Tx = Mx + \gamma$ for $M \in GL_n(\mathbb{Z})$ and $\gamma \in v(E) \subseteq \Gamma^n$ can be lifted to some $\tilde{f} = \tilde{T}$ given by $\tilde{T}(x) = Mx + c$ for some $c \in \mathrm{rv}^{-1}(\gamma)$.

The converse of this is covered in the proofs of Lemma 3.26 and 3.28 in [HK]. □

We are now in the position to construct a map $\mathrm{K}_+(\mathrm{RES}[*]) \otimes_{\Gamma^{fin}[*]} \mathrm{K}_+(\Gamma[*]) \to \mathrm{K}_+(\mathrm{RV}[*])$ which will take center stage in our construction of motivic integration.

Proposition 2.8.
1. The natural embedding functor $i : \Gamma^{fin}[*] \to \Gamma[*]$ yields an embedding of Grothendieck semirings $i_* : \mathrm{K}_+(\Gamma^{fin}[*]) \to \mathrm{K}_+(\Gamma[*])$
2. The functors $v_{\mathrm{rv}}^{-1} : \Gamma[*] \to \mathrm{RV}[*]$ and $i : \mathrm{RES}[*] \to \mathrm{RV}[*]$ yield embeddings $(v_{\mathrm{rv}}^{-1})_* : \mathrm{K}_+(\Gamma[*]) \to \mathrm{K}_+(\mathrm{RV}[*])$ and $i_* : \mathrm{K}_+(\mathrm{RES}[*]) \to \mathrm{K}_+(\mathrm{RV}[*])$.
3. The functor $v_{\mathrm{rv}}^{-1} : \Gamma[*] \to \mathrm{RV}[*]$ induces an embedding of $\mathrm{K}_+(\Gamma^{fin}[*]) \to \mathrm{K}_+(\mathrm{RES}[*])$.

Proof.
1. This is immediate from the fact that $\Gamma^{fin}[*]$ is a fully faithful subcategory of $\Gamma[*]$.

2. Note that to get an well-defined functor $v_{\mathrm{rv}}^{-1} : \Gamma[*] \to \mathrm{RV}[*]$, one must choose beforehand a canonical set-theoretic section $s : \Gamma \to \mathrm{RV}$ in order to lift affine transformations; however, the choice of this section clearly does not matter at the level of Grothendieck semirings. By the previous lemma on liftings of definable functions from Γ to RV, the functor $v_{\mathrm{rv}}^{-1} : \Gamma[*] \to \mathrm{RV}[*]$ is "essentially" full and faithful and therefore induces an embedding of Grothendieck semirings $\mathrm{K}_+(\Gamma[*]) \to \mathrm{K}_+(\mathrm{RV}[*])$. Likewise, our definition of RES[∗] makes it apparent that the objects of RES[∗] are Boolean combinations of definable subsets of single cosets of $(k^\times)^n$ and that the embedding $i : \mathrm{RES}[*] \to \mathrm{RV}[*]$ is fully faithful by definition, yielding an injection $i_* : \mathrm{K}_+(\mathrm{RES}[*]) \to \mathrm{K}_+(\mathrm{RV}[*])$.

3. The natural map $v_{\mathrm{rv}}^{-1} : \Gamma[*] \to \mathrm{RV}[*]$ maps $\Gamma^{fin}[*]$ to RES[∗], since if $Z = \{\overline{\gamma}_1, \cdots, \overline{\gamma}_n\} \in \Gamma^{fin}[*]$ then
$$v_{\mathrm{rv}}^{-1}(Z) = \bigcup_{i=1}^{n} \mathrm{RV}_{\overline{\gamma}_i} \in \mathrm{RES}[*]$$
thus yielding the desired embedding $\mathrm{K}_+(\Gamma^{fin}[*]) \to \mathrm{K}_+(\mathrm{RES}[*])$ as in the above arguments.

□

We now define the notion of the *tensor product* of two semirings over another semiring:

Definition 2.9. Let R be a commutative semiring and let S and S' be commutative semirings equipped with morphisms $\alpha : R \to S$ and $\alpha' : R \to S'$, define the tensor product $S \otimes_R S'$ via the usual universal property: $S \otimes_R S'$ is the unique (up to isomorphism) object such that given maps $\beta : S \to T$ and $\beta' : S' \to T$ such that $\beta \circ \alpha = \beta' \circ \alpha'$, there exists a unique map $\beta \otimes \beta' : S \otimes_R S' \to T$ and canonical maps $\mathrm{id}_S \otimes 1 : S \to S \otimes_R S'$ and $1 \otimes \mathrm{id}_{S'} : S' \to S \otimes_R S'$ so that $(\beta \otimes \beta') \circ (\mathrm{id}_S \otimes 1) = \beta$ and $(\beta \otimes \beta') \circ (1 \otimes \mathrm{id}_{S'}) = \beta'$.

By usual abstract nonsense, it is not difficult to check that such a semiring always exists and is well-defined up to isomorphism. It behaves in many ways exactly as the tensor product defined for commutative R-algebras; for instance, it is generated by simple tensors of the form $s \otimes s'$ for $s \in S, s' \in S'$.

With this notion fixed, we note that the map
$$\mathrm{K}_+(\mathrm{RES}[*]) \times \mathrm{K}_+(\Gamma[*]) \to \mathrm{K}_+(\mathrm{RV}[*])$$
given by mapping $[X] \times [Y] \mapsto [X \times v_{rv}^{-1}(Y)]$ descends to a well-defined map
$$\mathrm{K}_+(\mathrm{RES}[*]) \otimes \mathrm{K}_+(\Gamma[*]) \to \mathrm{K}_+(\mathrm{RV}[*])$$
given on simple tensors by mapping $[X] \otimes [Y] \mapsto [X \times v_{rv}^{-1}(Y)]$. The key fact that we use in the proof of uniformity is simply a restatement of the decomposition lemma for definable sets in RV^n

Theorem 2.1. The natural map $\mathrm{K}_+(\mathrm{RES}[*]) \otimes \mathrm{K}_+(\Gamma[*]) \to \mathrm{K}_+(\mathrm{RV}[*])$ given on simple tensors by mapping $[X] \otimes [Y] \mapsto [X \times v_{rv}^{-1}(Y)]$ is surjective.

Proof. This is simply given by unwinding the definitions of RES[∗], Γ[∗], and RV[∗] together with the decomposition lemma; for each $n \in \omega$ we have that the map

$$\bigoplus_{i=1}^{n} \left[(\mathrm{K}_+(\mathrm{RES}[n-i]) \otimes_{\mathbb{N}} \mathrm{K}_+(\Gamma[i]))\right] \to \mathrm{K}_+(\mathrm{RV}[n])$$

induced by the mapping

$$([X_1] \otimes [Y_1], \cdots, [X_n] \otimes [Y_n]) \mapsto \left[\bigsqcup_{i=1}^{n} (X_i \otimes v_{\mathrm{rv}}^{-1}(Y_i))\right] = \sum_{i=1}^{n} [X_i \otimes v_{\mathrm{rv}}^{-1}(Y_i)]$$

is well-defined and surjective by the decomposition lemma applied to the objects of RV$[n]$. It immediately follows that the map

$$(\mathrm{K}_+(\mathrm{RES}[*]) \otimes_{\Gamma^{fin}[*]} \mathrm{K}_+(\mathrm{RES}[*])) \to \mathrm{K}_+(\mathrm{RV}[*])$$

is surjective as each element of $\mathrm{K}_+(\mathrm{RV}[*])$ is hit by some sum of tensors from the domain $\mathrm{K}_+(\mathrm{RES}[*]) \otimes_{\Gamma^{fin}[*]} \mathrm{K}_+(\mathrm{RES}[*])$. □

The map constructed is in fact an isomorphism, but injectivity is far more difficult to establish and is not required for the proof of uniformity.

3 Uniformity in Integration Over Local Fields

At this point it would be quite reasonable to ask how all this discussion of Grothendieck semirings relates to evaluating integrals over local fields. The crucial notion is that of *definable Γ-valued volume forms* on objects of some category of definable sets $\mathcal{C}$, and appropriate categories of sets $X \in \mathrm{Ob}(\mathcal{C})$ equipped with volume forms ω: $\mu_\Gamma \mathcal{C}$. The intutive idea is that given such a "volume form" ω on some definable X in the chosen category $\mathcal{C}$, "motivic integration" *should* be map of Grothendieck semirings $\int : \mathrm{K}_+(\mu_\Gamma \mathcal{C}) \to$ "$\mathrm{K}_+(\mathcal{C})(T)$" for a formal variable T. For technical reason this intuition does not quite work, but when working over local field, where the value group is $\mathbb{Z}$, we are able to take geometric sums uniformly for $p >> 0$. In other words, the integral of some volume form should return a *geometric object*- a formal sum of classes $[Y] \in \mathcal{C}$- rather than a number. The numerical integration then takes place at some later stage, given as an "evaluation" map $\mathrm{ev} : \mathrm{K}_+(\mathcal{C}) \to R$. The main example we have in mind are evaluation maps arising from taking the Haar measure of certain subsets of (powers of) a local field L.

3.1 Γ-Valued Volume Forms

In this section we define the notion of a Γ-valued definable volume form, which we will use to define appropriate categories of definable sets X equipped with volume forms ω and measure-preserving isomorphisms.

Definition 3.1.
- A *definable Γ-valued volume form* on a definable set X is simply a definable function $\omega : X \to \Gamma$.

- Define the category $\mu\Gamma[*]$ to have objects

$$\mathrm{Ob}(\mu\Gamma[*]) = \{(X,\omega)\,|\,X \in \mathrm{Ob}(\Gamma[*]) \text{ and } \omega : X \to \Gamma \text{ definable}\}$$

  with morphisms that are Γ-measure-preserving, i.e. a map $f : (X, \omega_X) \to (Y, \omega_Y)$ is in $\mathrm{Mor}_{\mu\Gamma[*]}((X,\omega),(Y,\omega)$ just in case

$$\left[\sum_{i=1}^{n} \pi_i(x)\right] + \omega_X(x) = \left[\sum_{i=1}^{n} \pi_i(f(x))\right] + \omega_Y(f(x))$$

  where $\pi_i : \Gamma^n \to \Gamma$ is the projection onto the i^{th} coordinate. Let $\mu\Gamma^{fin}[*]$ be the full subcategory of $\mu\Gamma[*]$ whose objects are pairs (X,ω) with $X \in \Gamma^{fin}[*]$.

- Define the category $\mu_\Gamma\,\mathrm{RV}[*]$ to have objects

$$\mathrm{Ob}(\mu\,\mathrm{RV}[*]) = \{(X,\omega)\,|\,X \in \mathrm{Ob}(\mathrm{RV}[*]) \text{ and } \omega : X \to \Gamma \text{ definable}\}$$

  and $f : (X,\omega_X) \to (Y,\omega_Y)$ is in $\mathrm{Mor}_{\mu_\Gamma\,\mathrm{RV}[*]}((X,\omega),(Y,\omega)$ just in case

$$\left[\sum_{i=1}^{n} v_{\mathrm{rv}}(\pi_i(x))\right] + \omega_X(x) = \left[\sum_{i=1}^{n} v_{\mathrm{rv}}(\pi_i(f(x)))\right] + \omega_Y(f(x))$$

- Set $\mu\,\mathrm{RES}[*]$ to be the full subcategory of $\mu_\Gamma\,\mathrm{RV}[*]$ whose objects are pairs (X,ω) with $X \in \mathrm{RES}[*]$.

Remark 3.2. I gave here slightly different definitions for the categories $\mu_\Gamma\,\mathrm{RES}[*]$ and $\mu_\Gamma\,\mathrm{RV}[*]$ that are, as far as I can tell, sufficient to carry out the proof of uniformity of integration over local fields.

As before, we construct a map

$$\mathrm{K}_+(\mu_\Gamma\,\mathrm{RES}[*]) \otimes_{\mu\Gamma^{fin}[*]} \mathrm{K}_+(\mu\Gamma[*]) \to \mathrm{K}_+(\mu\,\mathrm{RV}[*])$$

given by defining $[(X,\omega_X] \otimes [Y,\omega_Y] \mapsto [X \times v_{\mathrm{rv}}^{-1}(Y), \omega_X + (\omega_Y \circ v_{\mathrm{rv}})]$ where

$$\omega_X + (\omega_Y \circ v_{\mathrm{rv}})](x,z) = \omega_X(x) + \omega_Y(v_{\mathrm{rv}}(z)).$$

which is well-defined as it is induced by the functors $(\mathrm{id}, \mathrm{id}^*) : \mu_\Gamma\,\mathrm{RES}[*] \to \mu_\Gamma\,\mathrm{RV}[*]$ and $(v_{\mathrm{rv}}^{-1}, v_{\mathrm{rv}}^*) : \mu\Gamma[*] \to \mu\,\mathrm{RV}[*]$ where id^* and v_{rv}^* are the obvious pullback maps, and by identifying $\mu\Gamma^{fin}[*]$ with its image in $\mu_\Gamma(\mathrm{RES}[*]) \subseteq \mu_\Gamma(\mathrm{RV}[*])$ under the functor $(v_{\mathrm{rv}}^{-1}, v_{\mathrm{rv}}^*) : \mu\Gamma[*] \to \mu\,\mathrm{RV}[*]$.

Theorem 3.3. The map

$$(\mathrm{id}, v_{\mathrm{rv}}^{-1}) : \mathrm{K}_+(\mu_\Gamma\,\mathrm{RES}[*]) \otimes_{\mu\Gamma^{fin}[*]} \mathrm{K}_+(\mu\Gamma[*]) \to \mathrm{K}_+(\mu\,\mathrm{RV}[*])$$

is surjective.

Proof. By the surjectivity of the map $\mathrm{K}_+(\mathrm{RES}[*]) \otimes_{\Gamma^{fin}[*]} \mathrm{K}_+(\Gamma[*]) \to \mathrm{K}_+(\mathrm{RV}[*])$ we need only show that every $[(X \times v_{\mathrm{rv}}^{-1}(Y), \omega)] \in \mathrm{K}_+(\mu_\Gamma \mathrm{RV}[*])$ is in the image of $\mathrm{K}_+(\mu_\Gamma \mathrm{RES}[*]) \otimes_{\mu\Gamma^{fin}[*]} \mathrm{K}_+(\mu\Gamma[*])$. We argue by induction on m, where $Y \in \Gamma[m]$.

The case $m = 0$ is immediate. For the case of $m > 0$, suppose that for all $n < m$, any class $[(X \times v_{\mathrm{rv}}^{-1}(Y), \omega)]\mu_\Gamma \mathrm{RV}[*]$ with $Y \in \Gamma[m]$ is the image of some element in $\mathrm{K}_+(\mathrm{RES}[*]) \otimes_{\Gamma^{fin}[*]} \mathrm{K}_+(\Gamma[*])$. For $a \in \Gamma$ and $v_{\mathrm{rv}}^{-1}(Y)_a = \{(a_1, \cdots, a_{m-1}) \,|\, (a_1, \cdots, a_{m-1}, a_m) \in v_{\mathrm{rv}}^{-1}(Y)\}$ we consider the restricted volume form

$$\omega_a : X \times v^{-1}(Y)_a \to \Gamma$$

such that $\omega_a(x, y) = \omega(x, y, a)$. By the inductive hypothesis, each $[(X \times v_{\mathrm{rv}}^{-1}(Y)_a)]$ is in the image of $\mathrm{K}_+(\mu_\Gamma \mathrm{RES}[*]) \otimes_{\mu\Gamma^{fin}[*]} \mathrm{K}_+(\mu\Gamma[*])$. By compactness, since being the v_{rv}-pullback of a Γ-definable form is definable, there is a partition of $X \times v_{\mathrm{rv}}^{-1}(Y)$ into finitely many definable sets Z_i on which the class $[(Z_i, \omega \restriction_{Z_i})]$ is in the image of $\mathrm{K}_+(\mu_\Gamma \mathrm{RES}[*]) \otimes_{\mu\Gamma^{fin}[*]} \mathrm{K}_+(\mu\Gamma[*])$. But then $[(X \times v_{\mathrm{rv}}^{-1}(Y), \omega)] = \sum_{i=1}^{\ell} [(Z_i, \omega \restriction_{Z_i})]$ and so the map

$$(\mathrm{id}, v_{\mathrm{rv}}^{-1}) : \mathrm{K}_+(\mu_\Gamma \mathrm{RES}[*]) \otimes_{\mu\Gamma^{fin}[*]} \mathrm{K}_+(\mu\Gamma[*]) \to \mathrm{K}_+(\mu \,\mathrm{RV}[*])$$

is surjective. □

3.2 Relating RV and VF

So far we have only considered the structure of definable subsets of RV^n, altogether neglecting the definable subsets of VF^n. One of the central results of the Hrushovski-Kazhdan paper relates the Grothendieck rings of these two structures.

Theorem 3.4 (HK 4.12)**.** Let $X \subseteq \mathrm{VF}^n$ be definable. Then the element $[X] \in \mathrm{K}_+(\mathrm{VF}[*])$ can be decomposed as

$$[X] = \left[\bigsqcup_{i=1}^{n} \pi_i(\mathrm{rv}^{-1}(H_i))\right]$$

Where the $H_i \subseteq (\mathrm{RV}^n \cup \infty) \times \mathrm{RV}^{\ell_i}$ and $\pi_i : (\mathrm{RV}^n \cup \infty) \times \mathrm{RV}^{\ell_i} \to (\mathrm{RV}^n \cup \infty)$ is the projection onto the first n coordinates. Moreover, the maps witnessing the equality $[X] = \left[\bigsqcup_{i=1}^{n} \pi_i(\mathrm{rv}^{-1}(H_i))\right]$ may be taken to be Haar-measure preserving at the level of L points for L of residue characteristic $p >> 0$.

3.3 Integration in Local Fields

In this section we prove the main uniformity result of Hrushovski and Kazhdan for integration over local fields. In this section, for all the categories $\mathcal{C}$ of definable sets in ACVF, we write $\mathcal{C}_0$ to signify that we work in $\mathrm{ACVF}_(0,0)$ and $\mathcal{C}_p$ to signify that we work in $\mathrm{ACVF}_{(0,p)}$. We may now state and prove the main uniformity result:

Theorem 3.5. Let $f = (f_1, \cdots, f_k) : X \to \Gamma^k$ be a definable function with $X \subseteq \mathrm{VF}^n$ such that for all local fields L and $s \in (\mathbb{R}^{\geq 1})^k$,

$$\int_{X(L)} |f|^s \text{ converges.}$$

Then there is a finite collection of generalized algebraic varieties $X_i \in \mathrm{RES}_{\mathbb{Q}}[*]$, objects $\Delta_j \in \Gamma[*]$, $n_k \in \mathbb{N}$, $\gamma_i \in \mathbb{Q}^{\geq 0}$, linear functions $\{h_1^i, \cdots, h_k^i\}$ with $\mathbb{Q}$-coefficients so that if L is a local field of residue characteristic $p >> 0$ with $v(L^\times) = \frac{1}{r}\mathbb{Z}$, $v(p) = 1$, and residue field $\mathbb{F}_q$ and $s \in \mathbb{R}^{\geq 1}$ then

$$\int_{X(L)} |f|^s = \sum_{i=1}^{\ell} \left(q^{r\gamma_i} (q-1)^{n_i} |X_i(L)| \sum_{\overline{\delta} \in \Delta(L)} q^{r(h_i(\overline{\delta}) + \sum s_j \delta_j} \right)$$

Proof. Let $v(X) = \{(v(x_1), \cdots, v(x_n)) \,|\, (x_1, \cdots, x_n) \in X\}$. For any local field (L, s), we may expand the integral

$$\int_{X(L)} |f|^s = \sum_{\overline{\gamma} \in [v(X)](L)} (q^{rs_i}) \operatorname{vol}_L(Z_f(\overline{\gamma}))$$

where

$$Z_f(\overline{\gamma}) = \Big\{ \overline{x} \in X \mid \bigwedge_{\ell=1}^{k} v(f_\ell(x)) = \gamma_\ell \Big\}.$$

By the theorem relating definable subsets of VF_0^n and RV_0^n, as well as the surjectivity of the natural map

$$\mathrm{K}_+(\mathrm{RES}[*]) \otimes_{\Gamma^{fin}[*]} \mathrm{K}_+(\Gamma[*]) \to \mathrm{K}_+(\mathrm{RV}[*])$$

we can decompose $Z_f(\overline{\gamma})$ as

$$[Z_f(\overline{\gamma})]_0 = \left[\bigsqcup_{i=1}^{m} \pi_i \left(\mathrm{rv}^{-1}(X_i \times v_{\mathrm{rv}}^{-1}(\Delta_i(\overline{\gamma}))) \right) \right]_0.$$

But then for $p >> 0$ we have that

$$[Z_f(\overline{\gamma})]_p = \left[\bigsqcup_{i=1}^{m} \pi_i \left(\mathrm{rv}^{-1}(X_i \times v_{\mathrm{rv}}^{-1}(\Delta_i(\overline{\gamma}))) \right) \right]_p.$$

and that

$$\operatorname{vol}_L(Z_f(\overline{\gamma})) = \operatorname{vol}_L(\bigsqcup_{i=1}^{m} \pi_i \left(\mathrm{rv}^{-1}(X_i \times v_{\mathrm{rv}}^{-1}(\Delta_i(\overline{\gamma})))) \right) = \sum_{i=1}^{m} \operatorname{vol}_L(\mathrm{rv}^{-1}(X_i)(L)) \operatorname{vol}_L(v^{-1}(\Delta_i(\overline{\gamma})(L)))$$

inside $\mathrm{K}_+(\mathrm{VF}_0[*])$, with $X_i \in \mathrm{RES}[*]$ and $\Delta_i \in \Gamma[*]$.

We can consider f as giving a definable Γ-valued volume form on $\bigsqcup_{i=1}^{m} \pi_i\left(\mathrm{rv}^{-1}(X_i \times v_{\mathrm{rv}}^{-1}(\Delta_i(\overline{\gamma}))\right)$ by our decomposition of the ring $\mathrm{K}_+(\mu_\Gamma \, \mathrm{RV}_0)$. As $X_i \in \mathrm{RES}[*]$, by further decomposing $Z_f(\overline{\gamma})$ we may assume that $v((\prod f_j) \restriction_{X_i}) = \gamma_i$ is constant, so that

$$\mathrm{vol}_L((X_i(L))) = q^{r\gamma_i}|X_i(L)|$$

and so

$$\int_{X(L)} |f|^s = \sum_{i=1}^{m} \left(|X_i(L)| q^{r\gamma_i} \sum_{\overline{\gamma} \in v(X)(L)} q^{r \sum (s \cdot \overline{\gamma})} \mathrm{vol}_L(v^{-1}\Delta_i(\overline{\gamma})) \right).$$

But then the volume $\mathrm{vol}_L(v^{-1}\Delta_i(\overline{\gamma}))$ can be expressed as the sum

$$\mathrm{vol}_L(v^{-1}\Delta_i(\overline{\gamma})) = \sum_{\overline{\delta} \in \Delta_i(L),\, \pi_i(\delta) = \overline{\gamma}} (q-1)^{n_i} q^{h_i(\overline{\delta})}$$

for $h_i : \Delta_i \to \Gamma$ given by $h(\overline{\delta}) := \sum_{j=n+1}^{n+n_i} \delta_j$ is the sum of the fibers of the projection map over δ; this equality holds because it is simply counting disjoint *residue* fibers of the polyhedron Δ_i. But then

$$\int_{X(L)} |f|^s = \sum_{i=1}^{m} \left(|X_i(L)| q^{r\gamma_i} \sum_{\overline{\gamma} \in v(X)(L)} q^{r \sum (s \cdot \overline{\gamma})} \mathrm{vol}_L(v^{-1}\Delta_i(\overline{\gamma})) \right)$$

$$= \sum_{i=1}^{m} \left(|X_i(L)| q^{r\gamma_i} \sum_{\overline{\delta} \in \Delta_i(L)} q^{r(h_i(\overline{\delta}) + \sum s_j \delta_j)} (q-1)^{n_i} \right)$$

$$= \sum_{i=1}^{\ell} \left(q^{r\gamma_i} (q-1)^{n_i} |X_i(L)| \sum_{\overline{\delta} \in \Delta(L)} q^{r(h_i(\overline{\delta})) + \sum s_j \delta_j)} \right)$$

as desired. □

This recovers Hrushovski and Kazhdan's theorem:

Corollary 3.6 (HK 1.3)**.** Let $f \in \mathbb{Q}[x_1, \cdots, x_n]^k$. Then there exists a finite collection of generalized algebraic varieties $X_i \in \mathrm{RES}_{\mathbb{Q}}[*]$, objects $\Delta_j \in \Gamma[*]$, $n_k \in \mathbb{N}$, $\gamma_i \in \mathbb{Q}^{\geq 0}$, linear functions h_i with $\mathbb{Q}$-coefficients so that if L is a local field of residue characteristic $p >> 0$ with $v(L^\times) = \frac{1}{r}\mathbb{Z}$, $v(p) = 1$, and residue field $\mathbb{F}_q$ and $\overline{s} \in (\mathbb{R}^{\geq 1})^k$ then

$$\int_{\mathcal{O}_L^k} |f|^s = \sum_{i=1}^{\ell} \left(q^{r\gamma_i} (q-1)^{n_i} |X_i(L)| \sum_{\overline{m} \in \Delta(\mathbb{Z})} q^{r(h_i(\overline{\delta})) + \sum s\delta_i)} \right)$$

Proof. Apply the above theorem to the function $v \circ \left(\prod_{i=1}^{k} f_k(\overline{x}) \right)$. □

Remark 3.7. On the face of it, this result as applied to uniform integration over p-adic does not recover the full strength of the Cluckers-Loeser theory of motivic integration, because in the setting of Hrushovski-Kazhdan the only sets that we can integrate volume forms over in a local field L are precisely the *traces* of definable subsets of ACVF, and not over *all* definable subsets of local fields L. It seems to me that a more natural setting to consider the questions of uniformity in the local field case would be the setting of Henselian fields equipped with an angular component map, where there *is* a relative quantifier elimination theorem given by Ax-Kochen-Ershov type results; however, doing a Hrushovski-Kazhdan style analysis of the Grothendieck ring in this setting seems like it would be far more difficult.

Also, the version of uniformity presented here does not explicitly return a *rational* expression for the integral, though it is easy to conclude rationality from the methods of Cluckers, Loeser, and Denef in summing expressions of the form $\sum_{\overline{m}\in\Delta(\mathbb{Z})} q^{r(h_0(\overline{\delta})+\sum s_j\delta_j)}$.

Corollary 3.8. Let $\mathrm{I}_+^{\mathrm{loc}}(\mu_\Gamma\,\mathrm{VF}_0[*])$ be the subring of classes of equivalent Γ-volume forms $[(X,\omega)]$ whose integral $\int_{X(L)}\omega^s$ exists and is independent of choice of representative for all local fields L of sufficiently large residual characteristictic $p >> 0$ and ramification index r and $s\geq 1$ in $\mathbb{R}$. Then there is a function

$$_{\mathrm{mot}}\int : \mathrm{I}_+^{\mathrm{loc}}(\mu_\Gamma\,\mathrm{VF}_0[*]) \to (\mathrm{K}_+(\mathrm{RES}_0[*])\otimes \mathrm{K}_+(\Gamma_0[*]))[[T^{\mathbb{Q}},T^R,S]]$$

so that evaluating the integral over the local field is an instance of specializing variables:

$$\mathrm{ev}_{L,s}([(X,\omega)]) = \mathrm{ev}_{T=|k_L|,R=r,S=s}\left({}_{\mathrm{mot}}\int([X,\omega])\right)\in\mathbb{R}$$

Proof. The function ${}_{\mathrm{mot}}\int : \mathrm{I}_+^{\mathrm{loc}}(\mu_\Gamma\,\mathrm{VF}_0[*]) \to (\mathrm{K}_+(\mathrm{RES}_0[*])\otimes \mathrm{K}_+(\Gamma_0[*]))[[T^{\mathbb{Q}},T^R,S]]$ is defined on the level of *representatives* and not on *classes*: If (X,ω) represents $[(X,\omega)]$ then decompose X as in the proof of the uniformity theorem so that

$$\int_{X(L)}|f|^s = \sum_{i=1}^{\ell}\left(q^{r\gamma_i}(q-1)^{n_i}|X_i(L)|\sum_{\overline{\delta}\in\Delta(L)} q^{r(h_i(\overline{\delta})+s\delta)}\right)$$

for all L of sufficiently large residue characteristic and $s\in\mathbb{R}^{\geq 1}$. Then set

$$_{\mathrm{mot}}\int([X,\omega]) := \sum_{i=1}^{\ell}\left(T^{R\gamma_i}(T-1)^{n_i}[X_i]\sum_{\overline{\delta}\in[\Delta_i]} T^{R(h_i(\overline{\delta})+S\delta)}\right)$$

By construction and the uniformity theorem, for large enough residue characteristics $p >> 0$ we have that $\mathrm{ev}_{L,s}([(X,\omega)]) = \mathrm{ev}_{T=|k_L|,R=r,S=s}\left({}_{\mathrm{mot}}\int([X,\omega])\right)$ □

4 References

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