What is the monster.

Richard E. Borcherds, *

Mathematics department, Evans Hall #3840, University of California at Berkeley, CA 94720-3840 U. S. A.

e-mail: reb@math.berkeley.edu

www home page www.math.berkeley.edu/~reb

When I was a graduate student, my supervisor John Conway would bring his 1 year old son into the department, who was soon known as the baby monster. A more serious answer to the question is that the monster is the largest of the (known**) sporadic simple groups. Its name comes from its size: the number of elements is

$$\begin{aligned}&8080,17424,79451,28758,86459,90496,17107,57005,75436,80000,00000\\=&2^{46}.3^{20}.5^{9}.7^{6}.11^{2}.13^{3}.17.19.23.29.31.41.47.59.71,\end{aligned}$$

about equal to the number of elementary particles in the planet Jupiter.

The monster was originally predicted to exist by Fischer and by Griess in the early 1970's. Griess constructed it a few years later in an extraordinary tour de force as the group of linear transformations on a vector space of dimension 196883 that preserved a certain commutative but non-associative bilinear product, now called the Griess product.

Our knowledge of the structure and representations of the monster is now pretty good. The 194 irreducible complex representations were worked out by Fischer, Livingstone and Thorne (before the monster was even shown to exist) They take up 8 large pages in the atlas [A] of finite groups, which is the best single source of information about the monster (and other finite simple groups). The subgroup structure is mostly known; in particular there is an almost complete list of the maximal subgroups, and the main gaps in our knowledge concern embeddings of very small simple groups in the monster. If anyone wishes to multiply elements of the monster explicitly, R. A. Wilson can supply two matrices that generate the monster. But there is a catch: each matrix takes up about 5 gigabytes of storage, and to quote from Wilson's atlas page: "standard generators have now been made as 196882×196882 matrices over GF(2)... They have been multiplied together, using most of the computing resources of Lehrstuhl D für Mathematik, RWTH Aachen for about 45 hours...". (The difficulty of multiplying two elements of the monster is caused not so much by its huge size as by the lack of "small" representations; for example, the symmetric group S_{50} is quite a lot bigger than the monster, but it only takes a few minutes to multiply two elements by hand.) Finally the modular representations of the monster for large primes were worked out by Hiss and Lux; the ones for small primes still seem to be out of reach at the moment.

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** The announcement of the classification of the finite simple groups about 20 years ago was a little over-enthusiastic, but a recent 1300 page preprint by Aschbacher and Smith should finally complete it.

In the late 1970's John McKay decided to switch from finite group theory to Galois theory. One function that turns up in Galois theory is the elliptic modular function

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots = \sum c(n)q^n$$

($q = e^{2\pi i\tau}$), which is essentially the simplest non-constant function invariant under the action $\tau \mapsto (a\tau + b)/(c\tau + d)$ of $SL_2(\mathbf{Z})$ on the upper half plane $\{\tau|\Im(\tau) > 0\}$. He noticed that the coefficient 196884 of q^1 was almost equal to the degree 196883 of the smallest complex representation of the monster (up to a small experimental error). The term "moonshine" roughly means weird relations between sporadic groups and modular functions (and anything else) similar to this. It was clear to many people that this was just a meaningless coincidence; after all, if you have enough large integers from various areas of mathematics then a few are going to be close just by chance, and John McKay was told that his observation was about as useful as looking at tea-leaves. John Thompson took McKay's observation further, and pointed out that the next few coefficients of the elliptic modular function were also simple linear combinations of dimensions of irreducible representations of the monster; for example, $21493760 = 21296876 + 196883 + 1$. He suggested that there should be a natural infinite dimensional graded representation $V = \sum_{n\in\mathbf{Z}} V_n$ of the monster such that the dimension of V_n is the coefficient $c(n)$ of q^n in $j(\tau)$, at least for $n \neq 0$. (The constant term of $j(\tau)$ is arbitrary as adding a constant to j still produces a function invariant under $SL_2(\mathbf{Z})$, and is set equal to 744 mainly for historical reasons.) Conway and Norton [C-N] followed up Thompson's suggestion of looking at the McKay-Thompson series $T_g(\tau) = \sum_n Trace(g|V_n)q^n$ whose coefficients are given by the traces of elements g of the monster on the representations V_n, and found by calculating the first few terms that they all seemed to be Hauptmoduls of genus 0. (A Hauptmodul is a function similar to j, but invariant under some group other than $SL_2(\mathbf{Z})$). Atkin, Fong, and Smith showed by computer calculation that there was indeed an infinite dimensional graded representation of the monster whose McKay-Thompson series were the Hauptmoduls found by Conway and Norton, and soon afterwards Frenkel, Lepowsky, and Meurman explicitly constructed this representation using vertex operators.

If a group acts on a vector space it is natural to ask if it preserves any algebraic structure, such as a bilinear form or product. The monster module constructed by FLM has a vertex algebra structure invariant under the action of the monster. Unfortunately there is no easy way to explain what a vertex algebra is; see [K] for the best introduction to them. Vertex algebras are a generalization of commutative rings with derivations (at least in characteristic 0). Roughly speaking they can be thought of as commutative rings with derivation where the ring multiplication is not quite defined everywhere; this is analogous to rational maps in algebraic geometry, which are also not quite defined everywhere. A more concrete but less intuitive definition of a vertex algebra is that it consists of a space with a countable number of bilinear products satisfying certain rather complicated identities. In the case of the monster vertex algebra $V = \oplus V_n$, this gives bilinear maps from $V_i \times V_j$ to V_k for all integers i, j, k, and the special case of the map from $V_2 \times V_2$ to V_2 is (essentially) the Griess product. So the Griess algebra is a sort of section of the monster vertex algebra.

Following an idea of Frenkel, the monster vertex algebra can be used to construct the monster Lie algebra by using the Goddard-Thorn no-ghost theorem from string theory.

This is a $\mathbf{Z}^2$-graded Lie algebra, whose piece of degree $(m,n) \in \mathbf{Z}^2$ has dimension $c(mn)$ whenever $(m,n) \neq (0,0)$. The monster should be thought of as a group of "diagram automorphisms" of this Lie algebra, in the same way that the symmetric group S_3 is a group of diagram automorphisms of the Lie algebra D_4. The monster Lie algebra has a denominator formula, similar to the Weyl denominator formula for finite dimensional Lie algebras and the Macdonald-Kac identities for affine Lie algebras, which looks like

$$j(\sigma) - j(\tau) = p^{-1} \prod_{\substack{m>0 \\ n \in \mathbf{Z}}} (1 - p^m q^n)^{c(mn)}$$

where $p = e^{2\pi i \sigma}$, $q = e^{2\pi i \tau}$. This formula was discovered independently in the 1980's by several people, including Koike, Norton, and Zagier. There are similar identities with $j(\tau)$ replaced by the McKay-Thompson series of any element of the monster, and Cummins and Gannon showed that any function satisfying such identities is a Hauptmodul. So this provides some sort of explanation of Conway and Norton's observation that the McKay-Thompson series are all Hauptmoduls.

So the question "What is the monster" now has several reasonable answers:

1 The monster is the largest sporadic simple group, or alternatively the unique simple group with its order.
2 It is the automorphism group of the Griess algebra.
3 It is the automorphism group of the monster vertex algebra. (This is probably the best answer.)
4 It is a group of diagram automorphisms of the monster Lie algebra.

Unfortunately none of these definitions is completely satisfactory. At the moment all constructions of the algebraic structures above seem artificial; they are constructed as sums of two or more apparently unrelated spaces, and it takes a lot of effort to define the algebraic structure on the sum of these spaces and to check that the monster acts on the resulting structure.

References.

[A] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, Atlas of finite groups, Clarendon Press, Oxford, 1985.

[C-N] J. H. Conway, S. Norton, Monstrous moonshine, Bull. London. Math. Soc. 11 (1979) 308-339.

[K] V. Kac, Vertex algebras for beginners. Second edition. University Lecture Series, 10. American Mathematical Society, Providence, RI, 1998.