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Contents.

1. Introduction.

In this paper we try to define the higher dimensional analogues of vertex algebras. In other words we define algebras which we hope have the same relation to higher dimensional quantum field theories that vertex algebras have to one dimensional quantum field theories (or to "chiral halves" of two dimensional conformal field theories).

The ideas in this paper are not really in final form. On the other hand, I have been rewriting versions of this paper for about 12 years, and it has by now become obvious that it never will attain a satisfactory final form. I apologize in advance for the resulting chaotic and incomplete nature of some of the sections.

The main ideas of this paper are as follows. We first define vertex groups, which can be thought of roughly as groups together with some allowable singularities for functions on the group. We look at the category of representations of the group, and redefine multilinear maps between representations to allow maps with certain sorts of singularities. The result of this is that the composition of multilinear maps is no longer a multilinear map, but it is sufficiently close to a multilinear map that we can still sensibly define analogues of associativity, G-algebras, commutative G-algebras, and so on. We will refer to the analogues of G algebraic structures in this relaxed multilinear category as G vertex algebraic structures; for example we can define G vertex (associative) algebras, G vertex Lie algebras, and so on. We will mostly be interested in commutative G vertex algebras, which are the analogues of commutative rings acted on by the "group" G. We show that vertex algebras are exactly the same as commutative associative G vertex algebras, where

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G is the simplest nontrivial example of a vertex group. It is now obvious how to define higher dimensional analogues of vertex algebras: they are just the commutative associative G vertex algebras for a higher dimensional vertex group G.

Many examples of commutative G vertex algebras come from quantum field theories. Very roughly, a quantum field theory ought to give a G algebra if and only if it has "good" operator product expansions, and the quantum field theory is then determined by its G vertex algebra. The underlying group of G will be the group of translations of spacetime, or some larger group such as the Poincare group, and the "allowable" singularities on G are the singularities that appear in the operator product expansions. Any G vertex algebra automatically has "good" operator product expansions; in fact, G vertex algebras are just a formalization of what physicists do with operator product expansions. A simple example of a higher dimensional commutative G vertex algebra is the space of Wick polynomials in a free field, which is the G vertex algebra of a free quantum field. The commutativity of the G algebra is of course very closely related to the "locality" property of a quantum field theory. On the other hand not all G vertex algebras come from quantum field theories, because there is usually no sensible way to construct a Hilbert space from a G vertex algebra.

G vertex algebras are also closely related to operator product expansion (OPE) algebras. The latter do not seem to be rigorously defined anywhere, and G vertex algebras can be thought of as an attempt to give a rigorous definition.

The simplest examples of G vertex algebras are associative algebras acted on by the underlying group of the vertex group G. In general a G vertex algebra can be thought of as a sort of associative algebra acted on by a group G, except that the ring multiplication has singularities and is not defined everywhere. G vertex algebras have most of the formal properties of associative algebras: for example, we can define left and right ideals, homomorphisms, tensor products, commutative G vertex algebras, and so on. We can also define left and right modules over G vertex algebras and multilinear maps between them, and these multilinear maps are sometimes represented by "tensor products". From this viewpoint quantization can be described as follows: quantization means deforming an honest commutative algebra acted on by the underlying group of G into a commutative G vertex algebra over a vertex group G. The commutative algebra we start with will usually be an associative algebra generated by classical fields and their derivatives (of all orders). The correlation functions of the quantized theory can be recovered from the G vertex algebra if we are given a trace on the G vertex algebra.

Another way to think of G vertex algebras is as follows. If V is an associative ring acted on by a group G, then for any fixed $v_i \in V$ and any $g_i \in G$ we can think of $v_1^{g_1} v_2^{g_2} \cdots v_i^{g_i}$ as being a function on G^i with values in V, and it is not difficult to write down a set of axioms for these functions equivalent to the axioms for an associative algebra acted on by G. The definition of a G vertex algebra is now in principle very easy: we use these axioms, except that we allow the functions from G^i to V to have some sort of singularities.

In practice it is necessary to allow G to be slightly more general than a group: for example it could be a formal group or Lie algebra. To include all these cases together we use cocommutative Hopf algebras H. A typical example of a cocommutative Hopf algebra is the group ring of a group, and representations of the group are the same as representations of its group ring. Similarly formal groups and Lie algebras can also be considered as special cases of Hopf algebras. In most of this paper G will not be a group, but will be a cocommutative Hopf algebra together with a "vertex structure", as described below.

Informally we can think of any Hopf algebra H as being something like the group ring of a group, and can think of its dual H^* as being something like the ring of regular functions on this group. In order to define G vertex algebras we will need to know what is meant by a "singular function on G", and to specify these we need to give a vertex structure on the Hopf algebra H. This means that we are given an algebra K over the ring H^* which behaves as if it were the ring of singular functions on some group G.

The simplest nontrivial example of a vertex structure on a Hopf algebra is given by taking the Hopf algebra H to be the Hopf algebra of the one dimensional formal group, and by defining the ring of singular functions to be the quotient field of the ring of regular functions. Then the commutative G vertex algebras for this G turn out to be exactly the same as vertex algebras (as defined in [B] or [K]). (The (possibly non-commutative) G vertex algebras for this G are also equivalent to previously defined algebraic structures: they are more or less the same as "field algebras" defined in [K] and to "quantum algebras" defined in [L-Z].) In other words vertex algebras are just commutative G vertex algebras for the simplest possible nontrivial example G of a Hopf algebra with a vertex structure.

It seems to be hard to construct interesting examples of G vertex algebras for higher dimensional G, possible because they tend to be closely related to higher dimensional quantum field theories, which are notoriously difficult to construct. However we can construct G vertex algebras corresponding to generalized free field theories, which we do in section 8.

In section 9 we prove the main theorem of this paper, which is a generalization of the vertex algebra identity [K, 4.8] to G vertex algebras for higher dimensional G. The main result is that (roughly speaking) the integral of a vertex operator over an n-dimensional cycle is a vertex differential operator of order n.

Section 10 and 11 give a few miscellaneous thoughts about G vertex algebras.

There have been several previous definitions of mathematical structures inspired by quantum field theory; in particular there are various rigorous definitions of a quantum field theory based on operators on Hilbert spaces (see [S-W] or [G-J] for example), and there is Atiyah and Segal's notion of a topological field theory [Se]. In section 12 we discuss the relation between G vertex algebras and these other concepts.

Beilinson and Drinfeld [B-D] have another approach to defining vertex algebras, as Lie algebra objects in a suitable category. They remark that their definition extends to higher dimensions, but the main emphasis is on the extension to higher genus curves. I do not know what the relation of this is to this paper. The pseudo tensor categories in [B-D] are essentially the same as the multilinear categories of this paper. Soibelman [So] has some unpublished notes which overlap with the present paper. In particular he defines "meromorphic pseudo tensor categories" which are very similar to relaxed multilinear categories, and considers associative algebras in them. His notes include a generalization to the "braided" case, which we largely ignore in this paper.

This is based on a lecture that I promised but did not give at the 1996 Taniguchi symposium. I would like to thank the Taniguchi foundation for inviting me to Japan, and C. Snydal and N. R. Scheithauer and the referee for suggesting many corrections and improvements.

2. Formal groups and Hopf algebras.

As well as considering groups acting on modules, we also need to consider various group-like things such as Lie algebras or formal groups acting on spaces. The most convenient way of doing these is to use cocommutative Hopf algebras, which are a common generalization of groups, Lie algebras, and formal groups. In this section we recall some standard results about formal groups and Hopf algebras, most of which can be found in any standard references such as [A] for Hopf algebras and [S] for formal groups.

As motivation for the definition of Hopf algebras we consider the following example.

Example 2.1. Suppose G is any discrete group and H is its group ring over some commutative ring R. Then H is an associative algebra with identity, and also has a map Δ (called the comultiplication) from H to $H \otimes_R H$ given by $\Delta(g) = g \otimes g$ for $g \in G$, and a map ϵ (called the counit) from H to R given by $\epsilon(g) = 1$, and a map S (called the antipode) from H to H given by $S(g) = g^{-1}$. Then H has the following properties:

1. H is an associative algebra with identity over R.
2. Δ is an algebra homomorphism from H to $H \otimes H$, which is coassociative, and ϵ is a homomorphism of algebras from H to R, which is a counit for the comultiplication.
3. $\mu(S \otimes 1(\Delta(h))) = \mu(1 \otimes S(\Delta(h))) = \epsilon(h)$ for all h.

Definition 2.2. *A* **Hopf algebra** *H is defined to be a module over R with multiplication, comultiplication, identity, counit, and antipode S, satisfying the axioms above. If in addition the comultiplication is cocommutative then the Hopf algebra is called* **cocommutative**.

So the group ring of any group is a cocommutative Hopf algebra. Conversely any cocommutative Hopf algebra behaves in many ways as if it were the group ring of some group.

The comultiplication is rather difficult to handle in notation. We assume that for each $h \in H$ we have chosen finite sets of elements h_{1i} and h_{2i} such that $\Delta(h) = \sum_i h_{1i} \otimes h_{2i}$. If H is a Hopf algebra then the comultiplication on H induces a multiplication on the dual $H^* = \mathrm{Hom}_R(H, R)$ which makes H^* into an associative algebra over R with unit η. If H is cocommutative then H^* is commutative. If H is a finitely generated free module over R then H^* also has a coalgebra structure making it into a Hopf algebra induced by the product of H, but in general H^* does not have a coalgebra structure, because the natural map from $H^* \otimes_R H^*$ to $(H \otimes_R H)^*$ is not usually an isomorphism.

Example 2.3. If L is a Lie algebra over R then the universal enveloping algebra $U(L)$ of L is a Hopf algebra, with the comultiplication induced by $\Delta(g) = g \otimes 1 + 1 \otimes g$ and S induced by $S(g) = -g$ for $g \in L \subset U(L)$. If R is a field of characteristic 0 then Lie algebras over R are essentially equivalent to formal groups, and the universal enveloping algebra of g is just isomorphic to the Hopf algebra of the formal group corresponding to g (defined below). If R is not an algebra over $\mathbf{Q}$ then the Hopf algebra $U(L)$ usually does not have particularly good properties, and the Hopf algebras of formal groups behave much better (at least for the purposes of this paper).

A formal group of dimension n over a commutative ring R (see [S, II.4.6]) is an n-tuple $F(x,y) = (F_1(x,y),\dots,F_n(x,y))$ of formal power series in $2n$ variables $x = (x_1,\dots,x_n), y = (y_1,\dots,y_n)$ such that

(1) $F(0,y) = y$, $F(x,0) = x$.

(2) $F(x,F(y,z)) = F(F(x,y),z)$.

Axiom 1 implies the existence of unique power series $\phi(x) = (\phi_1(x),\dots,\phi_n(x))$ with $\phi(0) = 0$ and

(3) $F(x,\phi(x)) = 0 = F(\phi(x),x)$.

We will abbreviate $F(x,y)$ to xy and $\phi(x)$ to x^{-1}, so that the axioms above become the same as the usual group axioms (except that the multiplicative identity is rather confusingly written as 0).

Example 2.4. The formal group of GL_n. This formal group has n^2 variables $x = (x_{ij})$ which we think of as the n^2 entries of an $n \times n$ matrix. The power series F and ϕ are defined by $I_n + F(x,y) = (I_n + x)(I_n + y)$ and $I_n + \phi(x) = (I_n + x)^{-1}$ where I_n is the identity matrix. So for example $F_{ik}(x,y) = x_{ik} + y_{ik} + \sum_j x_{ij}y_{jk}$.

We can also define formal groups of countable infinite dimension in a countable infinite number of variables x_i, $i \in N$, provided we are careful to use the correct definition of formal power series in an infinite number of variables: a formal power series in variables x_i is defined as a formal infinite sum $\sum_{0<i_1\le i_2\le\cdots\le i_n} a_{i_1 i_2\dots i_n} x_{i_1}x_{i_2}\cdots x_{i_n}$. In particular the ring of formal power series in an infinite number of variables is strictly larger than the completion of the ring of polynomials in the x_i's at the ideal generated by the x_i's, because in the latter there would only be a finite number of nonzero a's for any given value of n.

Example 2.5. Suppose G is a formal group of dimension n. We can construct a cocommutative Hopf algebra RG, called the formal group ring of G, as follows. We let RG be a free module over R with a basis of elements $D^{(i)}$, where $i = (i_1,\dots,i_n)$ is an n-tuple of nonnegative integers. (If the dimension n is infinite we should add the condition that all but a finite number of the i_j's are zero.) We define the comultiplication by $\Delta(D^{(i)}) = \sum_j D^{(j)} \otimes D^{(i-j)}$, and the antipode S by $S(D^{(i)}) = (-1)^{i_1+\cdots i_n} D^{(i)}$. We can identify RG with the topological dual of the ring of formal power series $R[[x]]$ by letting $D^{(i)}$ be a dual basis to the elements $x^i = x_1^{i_1}\cdots x_n^{i_n}$ in the obvious way. Then the continuous algebra homomorphism from $R[[x]]$ to $R[[x,y]]$ taking x to $F(x,y)$ induces a dual map from $RG \otimes RG$ to RG which we use as the algebra structure on RG. (Alternatively we could also define RG to be the algebra of "continuous left invariant differential operators on G".) If we work over a field of characteristic 0, then formal groups are essentially the same as Lie algebras, and the formal group ring of a formal group is isomorphic to the universal enveloping algebra of the corresponding Lie algebra.

Example 2.6. Suppose G_a is the one dimensional formal group with formal group law $F(x,y) = x + y$. Then the formal group ring H over R is the R-algebra generated by elements $D^{(i)}$ for i a non-negative integer, with $D^{(0)} = 1$, $D^{(i)}D^{(j)} = \binom{i+j}{i} D^{(i+j)}$. If R contains the rational numbers this is just the ring of polynomials in one variable $D = D^{(1)}$, with $D^{(i)} = D^i/i!$. The dual H^* can be identified with the ring of Laurent power series $R[[x]]$, in such a way that the elements x^i are a dual "basis" to the elements $D^{(i)}$.

The category of (left) modules over a Hopf algebra H is defined to be the category of left modules over H considered as an R-algebra. (If A is a right module over R we can turn it into a left module by defining $ha = aS(h)$.) If A and B are modules over H then $A \otimes_R B$ and $\mathrm{Hom}_R(A, B)$ can be made into H-modules in a canonical way as follows.

$$h(a \otimes b) = \Delta(h)(a \otimes b)$$

$$(hf)(a) = \sum h_{(1)}(f(S(h_{(2)})(a)))$$

In particular the modules over a Hopf algebra form a tensor abelian category.

If P is any free module over R then the tensor algebra $T^*(P) = \oplus_{n\geq 0} \otimes^n (P)$ has a natural coalgebra structure, given by $\Delta(p_1 \otimes \cdots \otimes p_n) = \sum_{0\leq j\leq n}(p_1 \otimes \cdots \otimes p_j) \otimes (p_{j+1} \otimes \cdots \otimes p_n)$. We denote the largest cocommutative subcoalgebra of this by $B(P)$, so that $B(P) = \oplus_{n\geq 0}(\otimes^n P)^{S_n}$, where $(\otimes^n P)^{S_n}$ is the subspace of $(\otimes^n P)$ fixed by the natural action of the symmetric group S_n. Over a field of characteristic 0 the natural map from $B(P)$ to the symmetric algebra of P is an isomorphism of vector spaces, but this is not true in positive characteristic. Warning: the symmetric coalgebra $B(P)$ is not the universal cocommutative coalgebra cogenerated by P (although some books incorrectly state that it is, probably because of the analogy with the symmetric algebra). The relation between them is that $B(P)$ is an irreducible component of the universal cocommutative coalgebra cogenerated by P; see [A]. We can recover P from $B(P)$ as the space of primitive elements, i.e., the elements p with $\Delta(p) = p \otimes 1 + 1 \otimes p$. A Hopf algebra is the formal group ring of a formal group over R if and only if it is a free R module and its underlying coalgebra is isomorphic to $B(P)$ for some free R module P.

3. Vertex groups.

In this section we define the concept of a vertex group G, which can be thought of informally as a group with a space of functions with singularities on it. We will later define G vertex algebras for any vertex group G as analogues of algebras acted on by an ordinary group G. The definition of a vertex group has to be weak enough to include the examples later, and strong enough so that it is possible to define the concept of an associative algebra over a vertex group.

In the definition below it is helpful to think of H as the Hopf algebra given by the group ring of some group G, so that H^* can be thought of as something like the ring of functions on the underlying space of G, and the left and right actions of H on H^* correspond to the left and right actions of G on itself. The module K below should then be thought of as some sort of ring of rational or algebraic functions with singularities on G. The axioms for a vertex structure are based on obvious properties of the space of singular functions on a group. In particular the singular functions form an algebra over the regular functions on which we can define the operations of left and right translation by elements of the group and the operation induced by taking inverses of elements of the group.

Definition 3.1. *Suppose that H is a Hopf algebra over a commutative ring R. An* **elementary vertex structure** *on H consists of an R-module K with the following extra structures.*

(1) K has the structure of an associative algebra over H^*.

(2) K has the structure of a 2 sided H-module, and the natural map from H^* to K is a homomorphism of 2 sided H-modules. The product on K is invariant under the left

and right actions of H. This means that $h(ab) = \sum(h_{(1)}a)(h_{(2)}b)$ and similarly for the right action. This axiom can be thought of as saying that K is closed under "left and right translation".

(3) *There is an R linear map S (called the antipode) from K to K extending the antipode S on H^*, and $S(ab) = S(b)S(a)$ whenever each of a and b is in H or H^* or K. This axiom can be thought of as saying that K is closed under the inversion map of G.*

Definition 3.2. *An* **(elementary) vertex group** *G is a Hopf algebra H with an (elementary) vertex structure K, such that H is cocommutative, K is commutative, and $S^2 = 1$. We call H the group ring of the vertex group G, and we call K the ring of singular functions on G.*

If we replace the condition $S^2 = 1$ by the weaker condition that S is an isomorphism of R modules from K to K then we call G a **braided elementary vertex group**. There are several other obvious variations of these definitions (which we will not use in this paper): if we add the condition that H is the Hopf algebra of a formal group we get the definitions of vertex formal groups and braided vertex formal groups, and if we replace the condition that the vertex structure is cocommutative by the condition that the Hopf algebra H is "braided quasi commutative" or "quasi triangulated" we get the concept of a "braided vertex quantum group".

The definition of an elementary vertex group is not really general enough for some examples. The main problem is that the only singularities of functions of several variables we allow are essentially singularities of functions of two variables, and there are some examples where we want to allow more general singularities; see the end of section 8 for examples and a provisional definition of a non elementary vertex group. However the definition above is adequate for most of the examples in this paper (possibly because most of the examples are related to quantum field theories that are either free or in small dimensions). We will usually drop the adjective "elementary" from now on.

Example 3.3. If H is any cocommutative Hopf algebra, then taking $K = H$ gives H the structure of a vertex group, called the trivial vertex group structure. In particular vertex groups are generalizations of cocommutative Hopf algebras (and hence of groups and Lie algebras).

Example 3.4. The simplest nontrivial example of a vertex group G is given as follows. Suppose that H is the Hopf algebra of the 1-dimensional formal group G_a of example 2.6, so that H has a basis of elements $D^{(i)}$ for $i \geq 0$ and H^* can be identified with $R[[x]]$ as in example 2.6. We let K be the quotient field $R[[x]][x^{-1}]$ of H^* consisting of formal Laurent series over R, with S acting as $S(x^i) = (-1)^i x^i$ and H acting as derivations in the obvious way. (The right and left actions of H on K are identical.) If G is the vertex group given by H and K then we will see later that (classical) vertex algebras are exactly the same as the commutative G vertex algebras as defined in section 6.

Example 3.5. Suppose the ring R contains the inverse $1/N$ of some integer N and has an element ζ with $\zeta^N = -1$. Then there is a variation of example 3.4 where we take K to be the ring $R[[x^{1/N}]][x^{-1/N}]$ of all formal Laurent series in $x^{1/N}$, with S acting as $S(x^{1/N}) = \zeta x^{1/N}$. This gives examples of braided vertex groups which are not vertex groups (because the antipode S does not have period 2).

Example 3.6. Take H to be the formal group of the algebraic group SL_2 (over any commutative ring R). If we represent an element of $SL_2(R)$ as $\binom{ab}{cd}$ then H^* is the ring of formal power series in $a-1, b, c, d-1$ modulo the ideal $(ad-bc-1)$. We can define a vertex structure by putting $K = H^*[b^{-1}]$, and this defines a vertex formal group. Under the same conditions as example 3.5 we can define a braided vertex formal group by putting $K = H^*[b^{-1/N}]$.

If R is an integral domain we can often define K to be the full quotient field of H^*, or even the separable algebraic closure of this quotient field, but this definition of K usually seems either too large or too small: for most of the examples later in this paper it is only necessary to invert a few elements of K and inverting more makes theorem 9.1 much weaker, and on the other hand we also sometimes want to allow transcendental extensions as in example 3.9 below.

Example 3.7. Take H to be the formal group of SL_n, so that H^* is the ring of formal power series in the elements $a_{ij} - \delta_i^j$ $(1 \leq i, j \leq n)$ modulo the ideal generated by $\det(a_{ij}) - 1$, where the a_{ij}'s are the entries in an n by n matrix. If D_k is the determinant of the top right $k \times k$ submatrix, so that $D_n = 1$ and $D_1 = a_{1n}$, then we can define a vertex structure by letting K be H^* with some subset of the D_k's inverted. (So example 3.6 is the case with $n = 2$ where we invert D_1.)

Example 3.8. We can generalize example 3.7 as follows. We let G be a split simple algebraic group with a fixed choice of Cartan subgroup T and Borel subgroup B. We define $\chi_1, \ldots, \chi_n$ to be the fundamental characters of T in the Weyl chamber. These characters can be extended uniquely to functions on G which are right invariant under B and left invariant under $\bar{B}$, which we will again denote by $\chi_1, \ldots, \chi_n$. For example if G is SL_n and T the diagonal matrices and B the upper triangular matrices then χ_i is given by the determinant of the top left $i \times i$ submatrix. Now take ω to be an element of G representing the opposition involution of the Weyl group, so that ω takes B to $\bar{B}$, and define functions D_k by $D_k(g) = \chi_k(\omega g)$, so that these functions are invariant under left and right multiplication by elements of B. We define a vertex structure by inverting some subset of the elements D_k, considered as formal power series in H^*. (Notice that changing ω only results multiplying D_k by an invertible power series, so that this vertex structure does not depend on the choice of representative ω.) Notice that the product of the functions D_k vanishes exactly on the complement of the big cell of the Bruhat decomposition, so that elements of K can be thought of informally as singular functions defined near the identity of G whose poles are in the complement of the big cell.

Example 3.9. Suppose that R is an algebra over the rational numbers and H is the Hopf algebra of the formal group of G_a as in example 3.4. Let K be the ring $R[[x]][x^{-1}, \log(x)]$ where $\log(x)$ is a new indeterminate (written as $\log(x)$ rather than y for mnemonic reasons) with $D^{(i)}(\log(x)) = (-1)^{i-1}x^{-i}/i$ if $i > 0$ and $S(\log(x)) = \log(x)$. (We could also define $S(\log(x)) = \log(x) + c$ for any $c \in R$ if we did not mind that $S^2 \neq 1$.) This defines a vertex structure on H which gives a vertex group. Similarly in examples 3.6 to 3.8 we could introduce formal logs of the functions b or D_k that we inverted.

We can define homomorphisms of vertex groups as follows. The definition is the obvious one if one thinks of a vertex group as a group ring.

Definition 3.10. *Suppose G_1 and G_2 are vertex groups with group rings H_1 and H_2 and* *rings of singular functions K_1 and K_2. Then a homomorphism of vertex groups from G_1 to G_2 consists of a homomorphism of Hopf algebras (over R) from H_1 to H_2, together with a H_2^*-algebra homomorphism from K_2 to K_1 which commutes with the antipode and the left and right actions of H_1.*

Example 3.11. If G_1 and G_2 are "really" discrete groups, in other words if H_1 and H_2 are their group rings and $K_1 = H_1^*$, $K_2 = H_2^*$, then a homomorphism of vertex groups from G_1 to G_2 is the same as a homomorphism of groups from H_1 to H_2.

Example 3.12. Suppose G_1 is the trivial group (or rather the vertex group corresponding to it). If G_2 is also an honest group, then there is a unique homomorphism from G_1 to G_2. If G_2 is an arbitrary vertex group this is not usually true. For example if G_2 is the vertex group of example 3.4 with $K_2 = R[[x]][x^{-1}]$ then there is no homomorphism from G_1 to G_2, because the homomorphism from $H_2^* = R[[x]]$ to $H_1 = R$ cannot be extended to the field K_2. The significance of this is as follows. If G_1 and G_2 are groups then any homomorphism from G_1 to G_2 induces a forgetful functor from G_2 algebras to G_1 algebras in the obvious way. In particular if we take G_1 to be the trivial group this shows (in a rather roundabout way!) that any G_2 algebra has an underlying associative algebra structure. Similarly for G vertex algebras a homomorphism from G_1 to G_2 induces a natural forgetful functor from G_2 vertex algebras to G_1 vertex algebras. However we cannot use this to show that every G_2 vertex algebra has an underlying associative algebra structure, because the map from the trivial vertex group to G_2 need not exist.

Example 3.13. We can construct products of vertex groups with the usual universal properties. For example the product $G_1 \times G_2$ of two vertex groups G_1, G_2 is constructed as follows. The underlying Hopf algebra of $G_1 \times G_2$ is $H_1 \otimes H_2$, and the ring of singular functions is $(H_1 \otimes H_2)^* \otimes_{H_1^* \otimes H_2^*} K_1 \otimes K_2$.

Example 3.14. Suppose G_1 is the vertex group with $K_1 = R[[z]][z^{-1}]$ of example 3.4 and G_2 is the 2 dimensional vertex group $G_1 \times G_1$, so that $K_2 = R[[x, y]][x^{-1}, y^{-1}]$. For each $(a, b) \in R^2$ there is a homomorphism from H_1 to H_2 taking x to az and y to bz. This extends to a homomorphism of vertex groups if and only if $a \neq 0$ and $b \neq 0$.

4. Relaxed multilinear categories.

We can define multilinear maps between the representations of a vertex group, but the composition of multilinear maps is not in general multilinear. We deal with this problem by defining relaxed multilinear categories, where the composition of multilinear maps need not be multilinear, but can still be compared with multilinear maps. Symmetric relaxed multilinear categories have the following two properties: the representations of a vertex group form a symmetric relaxed multilinear category, and it is possible to define algebraic objects like commutative rings in a symmetric relaxed multilinear category. The reader willing to assume this can skip the rest of this section, which consists mainly of content-free category theory.

Soibelman has defined a similar notion in unpublished notes [So], except that his version is more general in several ways; for example, he allows braided rather than symmetric categories.

We will start by discussing what is needed in order to be able to define an associative algebra in some additive category. Obviously it is sufficient to assume the category is a tensor category, in other words for every two objects there is a tensor product given with suitable properties. If we are given good isomorphisms between $A \otimes B$ and $B \otimes A$ for all A and B then we get the notions of symmetric and braided multilinear categories. In any additive symmetric tensor category we can define most common algebraic structures, such as associative algebras, commutative algebras, Lie algebras (ignoring problems in characteristic 2), Hopf algebras, and so on. If $A_1, \ldots, A_n, B$ are objects of any additive tensor category we have a space of multilinear maps from $A_1, \ldots, A_n$ to B, defined as the maps from $A_1 \otimes \cdots \otimes A_n$ to B.

We can weaken the definition of tensor category by assuming that we are just given spaces of multilinear maps from $A_1, \ldots, A_n$ to B for objects A_i and B, with suitable properties, but are not given objects $A_1 \otimes \cdots \otimes A_n$ representing them. (For example, composition is defined under obvious conditions, and is associative, and the identity maps behave in the obvious way.) An additive category with such a structure is called a multilinear category, and if we add conditions about the action of the symmetric group on spaces of multilinear maps we get the concepts of symmetric and braided multilinear categories. In a multilinear category we can still define associative algebras, commutative algebras, Lie algebras, and so on, but we cannot define things like coalgebras and Hopf algebras because these cannot be defined just in terms of multilinear maps but require maps to tensor products. For a multilinear category to be a tensor category it is necessary that spaces of multilinear maps should be representable, but this is not sufficient because there is no reason in general why the objects $(A \otimes B) \otimes C$, $A \otimes B \otimes C$, and $A \otimes (B \otimes C)$ should be isomorphic. A multilinear category with only one object is an operad.

A multilinear category is sometimes called an additive pseudo tensor category but this seems a rather misleading name: "tensor" suggests that tensor products exist, and "pseudo" then says that in fact they do not.

We now want to define "relaxed multilinear categories", which are a generalization of multilinear categories for which the composition of multilinear maps is not always a multilinear map. We start by defining the sieve category $Sieve_n$ as follows. Its objects are sieves of depth $k \geq 0$ on $1, 2, \ldots, n$, by which we mean a sequence of equivalence relations $E_0, E_1, \ldots, E_k$ on $1, \ldots, n$ such that

1 E_k is the "indiscrete" equivalence relation where any two elements are equivalent, and E_0 is the "discrete" equivalence relation where any element is only equivalent to itself

2 Any equivalence class in any equivalence relation is an interval $\{i, i+1, \ldots, j-1, j\}$

3 Any equivalence class of E_{i+1} is a union of equivalence classes of E_i (so that the equivalence relations are increasingly fine).

A sieve can be thought of as a record of someone's attempt to multiply n non-commuting elements of a ring. For example suppose someone tries to calculate the product $a_1a_2a_3a_4a_5a_6$. At the end of the first day they might have calculated $a_1a_2a_3$ and a_5a_6. At the end of the second day they might have worked out $a_1a_2a_3$ and $a_4a_5a_6$, and by the third day they might have calculated $a_1a_2a_3a_4a_5a_6$. This would correspond to the sieve

$$\begin{aligned}
E_0 &= \{\{1\}\{2\}\{3\}\{4\}\{5\}\{6\}\} \\
E_1 &= \{\{1,2,3\}\{4\}\{5,6\}\} \\
E_2 &= \{\{1,2,3\}\{4,5,6\}\} \\
E_3 &= \{\{1,2,3,4,5,6\}\}
\end{aligned}$$

There are two other useful ways of representing sieves of width n and depth d. The second is as some parentheses, such as $(((()()()))((())(()())))$, with the property that there are n innermost pairs of parentheses, each of which is contained in exactly d other pairs of parentheses, and such that there is an outermost pair containing everything. The pairs of parentheses correspond to equivalence classes of the equivalence relation, in a way that should be obvious on comparing the expression above with the sieve in the example above. For clarity we represent all innermost pairs of parentheses $()$ by a blob $\bullet$, and miss out the outermost pair and any other pair which is uniquely determined by the condition that the depth should be d. For example we would abbreviate the example above to $(\bullet\bullet\bullet)(\bullet(\bullet\bullet))$.

The third way of representing sieves of width n and depth d is as rooted trees such that all branches are of length d, there are n "leaves", and all the edges going upwards from any given vertex are totally ordered. There is one vertex of the tree for every pair of parentheses (or to every equivalence class of every equivalence relation), and an edge between vertices if one of the vertices corresponds to a pair of parentheses which is a maximal pair inside the parentheses corresponding to the other vertex.

The number of sieves of width $n > 0$ and depth d is d^{n-1}, as it is easy to see by induction on n.

We say that a sieve p is a refinement of q if every equivalence relation in q is also in p, and we make the sieves into a category by saying that there is a unique morphism from any sieve to any refinement. The category of sieves has an initial element, consisting of the sieve with just two equivalence relations E_0 and E_1.

The idea of a relaxed multilinear category is that instead of one space of multilinear maps from $A_1, \ldots, A_n$ to B we should have several spaces, one for each sieve of size n (so that in some sense the spaces of multilinear maps depend on "the order in which we calculate the product of elements of the A_i's"). The space corresponding to a sieve should be related to the space of any refinement of that sieve.

More precisely, a relaxed multi category is given by the following data:

(1) A set of objects.

(2) For each collection of objects $A_1, \ldots A_n, B$ ($n \geq 1$) we are given a functor from $Sieve_n$ to sets. The value of this functor at $p \in Sieve_n$ is denoted by $Multi_p(A_1, \ldots, A_n; B)$ and is called the set of multi morphisms from $A_1, \ldots, A_n$ to B of type p. (It can be thought of as some sort of space of Taylor series expansions of multilinear maps.) If p is the initial object of $Sieve_n$ then we call $Multi_p(A_1, \ldots; B)$ the set of multi morphisms and sometimes miss out p from the notation.

(3) If $A_{11}, \ldots, A_{n,m_n}, B_1, \ldots, B_n, C$ are objects, $p_i \in Sieve_{m_i}$ are sieves of the same depth,

and $q \in Sieve_n$, then we are given a composition map

$$\begin{aligned}&Multi_{p_1}(A_{11}, \dots, A_{1m_1}; B_1) \times \cdots \times Multi_{p_2}(A_{n1}, \dots, A_{nm_n}; B_n)\\&\times Multi_q(B_1, \dots, B_n; C)\\\to &Multi_{q(p_1,\dots,p_n)}(A_{11}, \dots, A_{nm_n}; C)\end{aligned}$$

where $q(p_1, \dots, p_n)$ is the sieve given by first putting the sieves p_i "side by side" and then putting the sieve q "on top of them". In terms of parentheses, this means we replace the i'th innermost pair of parentheses of q by p_i, and in terms of trees we join the trees together by making the i'th leaf of q into the root of p_i.

These data should satisfy the following axioms, which we will state only vaguely as we do not use them later.

(1) Composition is associative.
(2) There is an identity morphism in $Multi(A; A)$ for all A, with the obvious properties.
(3) Composition is "compatible" with the morphisms in the categories $Sieve_n$.

If the spaces of multi morphisms are all abelian groups and composition is multilinear then we call the category a relaxed multilinear category.

If we are given isomorphisms from $Multi(A_1, \dots; B)$ to $Multi(A_{\sigma(1)}, \dots, B)$ for all σ in the symmetric group and these isomorphisms satisfy various obvious conditions we call the category a relaxed symmetric multilinear category. Similarly we can define relaxed braided multilinear categories.

Example 4.1. Any additive tensor category or multilinear category is a relaxed multilinear category, and in these cases the spaces of multilinear maps do not depend on the choice of sieve p. (In other words the functors from $Sieve_n$ take all morphisms of $Sieve_n$ to the identity morphism.)

We can define associative rings in any relaxed multilinear category as follows. An associative ring consists of an object A and maps $f_n \in Multi(A, A, \dots A; A)$, $f_1 =$ identity, for all $n \geq 1$ (with n copies of A mapping to A) such that any composition of these maps f_m is the image of some f_n under the map from multilinear maps to compositions of multilinear maps. Informally we can think of the associativity property in a relaxed multilinear category like this: the products $(ab)c$ and $a(bc)$ cannot be compared directly because they lie in different spaces, but they can both be compared with abc using the map from the trivial sieve to the sieves generated by the equivalence relations $\{\{a, b\}\{c\}\}$ and $\{\{a\}\{b, c\}\}$.

Note that more complicated multilinear maps in a ring in a relaxed multilinear category are no longer always uniquely determined by compositions of bilinear maps, so we cannot sensibly define associativity of a bilinear map in $Multi(A, A; A)$ but can only define associativity of a sequence of multilinear maps f_n as above.

Similarly we can define commutative rings, Lie algebras, and so on in any relaxed symmetric multilinear category. As before, we need to specify all possible products of bilinear maps (as well as just bilinear maps) as part of the definitions of these things.

There are several variations of the definition of a relaxed multi category. For example, instead of using sieves we could use collections of intervals on $1, 2, \dots, n$ such that any two intervals in the collection are either disjoint or one contains the other, and all 1 and n point sets are in the collection. (The union of all equivalence relations of a sieve is a collection with this property.) Then composition of multi maps is easier to axiomatize than if we use sieves, but it is harder to show that the representations of a vertex group satisfy the axioms.

The difference between an additive tensor category, a multilinear category, and a relaxed multilinear category can be illustrated by pretending that all multilinear maps are representable; we will of course denote the representing objects as tensor products. In a tensor category the object $A \otimes B \otimes C$ representing trilinear maps is isomorphic to $(A \otimes B) \otimes C$. In a multilinear category these need not be isomorphic even if both sides exist, but we would expect a canonical map from $A \otimes B \otimes C$ to $(A \otimes B) \otimes C$ corresponding to the composition of two bilinear maps being a trilinear map. In a relaxed multilinear category we would expect a map in the other direction from $(A \otimes B) \otimes C$ to $A \otimes B \otimes C$, corresponding to the fact that any trilinear map can be related to compositions of bilinear maps.

We can illustrate the differences by drawing the diagrams that have to commute for a bilinear map to be associative. For tensor categories, the following diagram has to commute:

$$
\begin{array}{ccccc}
(A \otimes A) \otimes A & & \cong & & A \otimes (A \otimes A) \\
\downarrow & & & & \downarrow \\
A \otimes A & \longrightarrow & A & \longleftarrow & A \otimes A
\end{array}
$$

For multilinear categories whose multilinear maps are representable the following diagram has to commute:

$$
\begin{array}{ccccc}
(A \otimes A) \otimes A & \longleftarrow & A \otimes A \otimes A & \longrightarrow & A \otimes (A \otimes A) \\
\downarrow & & & & \downarrow \\
A \otimes A & \longrightarrow & A & \longleftarrow & A \otimes A
\end{array}
$$

For relaxed multilinear categories whose multilinear maps are representable the following diagram has to commute. Notice that the arrows in the top row go in the opposite direction from the previous diagram. Moreover this is only the first of an infinite number of diagrams describing associativities of more copies of A that we need to define an associative product.

$$
\begin{array}{ccccc}
(A \otimes A) \otimes A & \longrightarrow & A \otimes A \otimes A & \longleftarrow & A \otimes (A \otimes A) \\
\downarrow & & \downarrow & & \downarrow \\
A \otimes A & \longrightarrow & A & \longleftarrow & A \otimes A
\end{array}
$$

A. Joyal pointed out to me the following similarities between Stasheff's A_∞ spaces and associative objects in a relaxed multilinear category. An A_∞ space is a sort of space with a product that is associative up to homotopy, and the homotopies are well defined up to higher homotopies, and so on. More precisely an A_∞ space A is a space A together with maps $K_n \times A^n \mapsto A$ for all $n \geq 2$, where K_n is a certain $n-2$-dimensional cell complex whose cells correspond to topological types of rooted trees with n leaves together with an ordering on the branches going upwards from each node. These are almost the same as sieves as defined above; the only difference is that in a sieve all the branches have the same height, and this seems to be a minor technical requirement which could probably be removed by some slight changes of definition. Moreover the boundary maps in the complexes K_n correspond to the maps between sieves given by refinement. By adjointness we can think of an A_∞ space as a space A with maps from A^n to A^{K_n} for all n (satisfying various conditions) which is very similar to the definition of an associative object in the relaxed multilinear category of representations of a vertex group G, except that we use A^{K_n} instead of the spaces of singular A valued functions on G^n parameterized by sieves.

5. The representations of a vertex group.

In this section we will describe how to make the representations of a vertex group G into a symmetric relaxed multilinear category, so that it is possible to define commutative ring objects in this category. We let G be a vertex group with group ring H and ring of singular functions K.

The category C of representations of G is defined to be the category of representations of the group ring H. (Of course the multilinear maps in these two categories will usually be different!) In particular the category C has tensor products and Hom functors, as it is just the representations of some cocommutative Hopf algebra. There is a functor Γ from C to R-modules taking any G module to its fixed point set. Note that $\mathrm{Hom}_R(x, y)$ is an object of C, while $\mathrm{Hom}_G(x, y)$ is the morphisms from x to y which is just an R module. The relation between them is $\mathrm{Hom}_G(x, y) = \Gamma(\mathrm{Hom}_R(x, y))$.

We will first define the multilinear maps from $A_1, \ldots, A_n$ to B in C. To motivate this we first describe the H-multilinear maps in a slightly unusual way. The R-multilinear maps are just the elements of $\mathrm{Hom}_R(A_1 \otimes \cdots \otimes A_n, B)$, and the H multilinear ones are just the H invariant elements of this space. Now consider the map taking $a_1 \in A_1, \ldots, a_n \in A_n$ to $a_1^{g_1} a_2^{g_2} \cdots a_n^{g_n} \in B$ for $g_i \in H$. For fixed a_i this can be thought of as a G invariant B valued function on G^n, by which we mean an element of $\mathrm{Hom}_H(H \otimes \cdots \otimes H, B)$. Moreover this map from $A_1 \otimes \cdots \otimes A_n$ to B valued functions on G^n is G^n-invariant. So to summarize, the multilinear maps can be thought of as G^n invariant maps from $A_1 \otimes \cdots \otimes A_n$ to the G-invariant B-valued functions on G^n.

We will define G multilinear maps in the same way, except that we allow the functions on G^n to have "singularities". Informally we want this space to be the space of functions on G^n which are allowed to have singularities "of type K" whenever two of the components of G^n are equal. The formal definition goes as follows. The space $\mathrm{Hom}(H^n, B)$ of functions from G^n to B is a module over the ring H^{n*} of functions on G^n. For each $1 \le i < j \le n$ there is a homomorphism from H^* to H^{n*} induced by the map from H^n to H taking $g_1 \otimes \cdots \otimes g_n$ to $g_i S(g_j)$ (= "$g_i g_j^{-1}$"). We define the localization of a module M over H^{n*} at (i, j) to be $M \otimes_{H^*} K$, where M is made into an H^* module using the homomorphism f_{ij} from H^* to H^{n*}.

Definition 5.1. *We define the space $Fun(G^n, B)$ of singular functions from G^n to B to be the localization at all $(i, j), 1 \le i < j \le n$ of the space of functions from G^n to B.*

The module $\mathrm{Hom}(H^n, B)$ also appears in the computation of the cohomology of the G-module B using the homogeneous standard resolution: the cohomology groups of B are just the cohomology groups of a complex whose terms are the G-invariant elements $\mathrm{Hom}(H^n, B)^G$. We will not use this connection in this paper.

Example 5.2. Suppose G is the vertex formal group of example 3.4, so that $H^* = R[[x]]$ and $K = R[[x]][x^{-1}]$. Then the functions from G^3 to B can be identified with the space of formal power series $B[[x_1, x_2, x_3]]$ in 3 variables with values in B. The localization of this at (i, j) for $1 \le i < j \le 3$ is just $B[[x_1, x_2, x_3]][(x_i - x_j)^{-1}]$, so the space of singular functions from G^3 to B is the space

$$B[[x_1, x_2, x_3]][(x_1 - x_2)^{-1}, (x_2 - x_3)^{-1}, (x_1 - x_3)^{-1}].$$

Definition 5.3. *The G-module $Multi_R^K(A_1, \ldots, A_n, B)$ of multilinear maps is defined to be the space of H^{n*} invariant maps from $A_1 \otimes \cdots \otimes A_n$ to the space of singular B-valued functions on G^n. The R-module $Multi_G^K(A_1, \ldots, A_n, B)$ of G invariant singular multilinear maps is defined to be the module of G-invariant elements of $Multi_R^K(A_1, \ldots, A_n, B)$.*

We have defined the 4 different spaces of multilinear maps $Multi_R(A_1, \ldots, A_n, B)$, $Multi_G(A_1, \ldots, A_n, B)$, $Multi_R^K(A_1, \ldots, A_n, B)$, and $Multi_G^K(A_1, \ldots, A_n, B)$. The first two are the "usual" spaces of multilinear and invariant multilinear maps over the Hopf algebra H, and the other two are the corresponding spaces of singular multilinear maps.

To make C into a relaxed multilinear category we not only have to define the multilinear maps, but also the spaces $Multi_p(A_1, \ldots, B)$ for nontrivial sieves p, and the maps between them for $p < q$, and the compositions of multilinear maps.

We first define the multilinear maps of type p, for p in the sieve category $Sieve_n$. These can be thought of informally as various sorts of Taylor series expansions of multilinear maps.

Definition 5.4. *The space $Fun_p(G^n, B)$ of singular B-valued functions of type p is defined recursively as follows. Represent p as a tree of depth d, consisting of n leaves attached to a tree q of depth $d-1$. Suppose that X is the space of singular B-valued functions of type q. Then $Fun_p(G^n, B)$ is obtained from X in two steps: first we construct the space Y of all equivariant G^n valued functions with values in X, then we localize Y at all pairs i, j for i and j leaves of p joined to the same leaf of q. Here "equivariant" means equivariant under the natural action of G^m, where m is the number of leaves of q.*

This space can be thought of as some space of singular functions "of type p" on G^n with values in B, where n is the number of nodes of the tree p. This space is also acted on by G^n.

Example 5.5. With G as in example 3.4, we have

$$\begin{aligned}
Fun_{(\bullet\bullet\bullet)}(G^3, B) &= B[[x_1, x_2, x_3]][(x_1 - x_2)^{-1}, (x_2 - x_3)^{-1}, (x_1 - x_3)^{-1}] \\
Fun_{((\bullet\bullet)\bullet)}(G^3, B) &= B[[x_1, x_2]][(x_1 - x_2)^{-1}][[x_3]][x_3^{-1}] \\
Fun_{(\bullet(\bullet\bullet))}(G^3, B) &= B[[x_2, x_3]][(x_2 - x_3)^{-1}][[x_1]][x_1^{-1}]
\end{aligned}$$

Definition 5.6. *The G-module $Multi_p^K(A_1, \ldots, A_n, B)$ of singular multilinear maps of type p is defined to be*

$$\mathrm{Hom}_{H^n}(A_1 \otimes \cdots \otimes A_n, Fun_p(G^n, B)).$$

The R-module $Multi^K_{Gp}(A_1,\dots,A_n,B)$ of G-invariant singular multilinear maps of type p is defined to be the G-invariant elements of $Multi^K_p(A_1,\dots,A_n,B)$.

The composition of multilinear maps of types $p_1,\dots,p_n$ and q is easily checked to be a multilinear map of type $q(p_1,\dots,p_n)$.

Finally we have to describe the map from multilinear maps of type p to multilinear maps of type q whenever $p<q$. The only possible problem in defining this map is showing that the localizations extend. The way to do this is best explained by an example. If we have an expression like $(x_1-x_2)^{-k}$ we can expand it as a series

$$\sum_{i,j\geq 0} C(i,j)(x_1-y_1)^i(y_1-y_2)^{-k-i-j}(y_2-x_2)^j$$

for some constants $C(i,j)=(-k)!/i!j!(-k-i-j)!$. In general there is an analogue of this power series expansion for any cocommutative Hopf algebra H.

6. G vertex algebras.

In this section we give the definition of a G vertex algebra.

We start with some motivation for the definition. Suppose that V is an associative algebra acted on by a group G, with the action of $g\in G$ on $v\in V$ written as v^g. For each fixed $v_1,\dots,v_n\in V$ we can form the products $v_1^{g_1}v_2^{g_2}\cdots v_n^{g_n}$, which we think of as a function from G^n to V. These function have the following properties:

1. Identity: $\cdots v_n^{g_n}1^g v_{n+1}^{g_{n+1}}\cdots = \cdots v_n^{g_n}v_{n+1}^{g_{n+1}}\cdots$
2. Associativity and G invariance:

$$(v_{11}^{g_{11}}v_{12}^{g_{12}}\cdots)^{g_1}(v_{21}^{g_{21}}v_{22}^{g_{22}}\cdots)^{g_2}\cdots = v_{11}^{g_{11}g_1}v_{12}^{g_{12}g_1}\cdots v_{21}^{g_{21}g_2}v_{22}^{g_{22}g_2}\cdots$$

Conversely if we have a module V over a ring R together with functions from G^n to V as above which are multilinear in the v_i's then we can define the structure of an associative G algebra on V by defining the product v_1v_2 to be $v_1^1v_2^1$ and defining the G action on V by v^g. It is easy to check that this means that modules V with a set of functions as above are equivalent to associative algebras with an action of G.

We now define a G vertex algebra where G is a vertex group. We defined a vertex group earlier as a cocommutative Hopf algebra over R with a vertex structure, but we can think of it informally as being a group together with some ring of "singular" functions on it. A G vertex algebra is informally defined to be a module over R together with multilinear singular maps denoted by $v_1^{g_1}v_2^{g_2}\cdots v_n^{g_n}$ from G^n to V for all $v_1,\dots,v_n\in V$, such that the axioms above are satisfied. In other words we just copy the strange definition of an associative G-ring in the paragraph above, and replace functions on G with singular functions on G. The formal definition, which makes precise what we mean by a singular function, is as follows.

Definition 6.1. *A G vertex algebra is an associative algebra in the relaxed multilinear category of G-modules.*

If we give G the trivial vertex structure such that the singular V-valued functions on G^n are defined to be the ordinary functions $\mathrm{Hom}(G^n,V)$ on G^n then we see from the previous remarks that a G vertex algebra is exactly the same as an associative algebra acted on by G (although defined in a rather odd way).

It is now easy to extend many concepts in ring theory to G vertex algebras as in the following examples.

Example 6.2. A G invariant left ideal of a G vertex algebra V is an R submodule I such that $v_1^{g_1} v_2^{g_2} \dots v_n^{g_n}$ is a singular map from G^n to I, whenever $v_1, \dots, v_{n-1} \in V$, $v_n \in I$. Similarly we can define right and two sided ideals.

Example 6.3. A left module over the G vertex algebra V is an R module M together with multilinear singular maps $v_1^{g_1} \cdots v_n^{g_n} m$ from G^n to M for all $v_i \in V, m \in M$, satisfying the obvious axioms about associativity and the identity element. This is the definition of a module without an action of G; we can define modules with an invariant action of G in the same way by using multilinear singular maps $v_1^{g_1} \cdots v_n^{g_n} m^g$ from G^{n+1} to M instead. We can define submodules, homomorphisms, and multilinear maps of modules in the obvious way. Unlike the case of rings there seems to be no reason in general why multilinear maps should be representable (though they often are in practice), but when they are representable we can use this to define tensor products of modules. (The problem of when this tensor product is associative seems rather hard.)

Example 6.4. We can define a commutative G vertex algebra to be one such that the multilinear singular functions are invariant under the obvious action of the symmetric group. We can define supercommutative G vertex algebras in a similar way.

Example 6.5. If V and W are two G vertex algebras then their tensor product is also a G vertex algebra, with the G vertex algebra structure defined by

$$(v_1 \otimes w_1)^{g_1} \cdots = (v_1^{g_1} \cdots) \otimes (w_1^{g_1} \cdots).$$

If V and W are commutative then so is $V \otimes W$. (Note that if V and W are supercommutative then there should be some extra signs $(-1)^{\deg(v_i)\deg(w_j)}$ for $i < j$ added to the definition of the product in $V \otimes W$.) More generally, if V and W are G_1 and G_2 vertex algebras, then $V \otimes W$ is a $G_1 \otimes G_2$ vertex algebra. If $G_1 = G_2 = G$ then there is a homomorphism from G to $G \otimes G$, which can be used to turn $V \otimes W$ into a G vertex algebra as in the first part of this example.

Example 6.6. If G is the vertex group of example 3.4 then it is easy to check that commutative G vertex algebras are the same as vertex algebras (as defined in [K, 1.3] for example). In fact if V is a vertex algebra and $v_1, \dots, v_n$ are any elements of it, and $v_i(z)$ is the vertex operator of v_i, then $v_1(z_1) \cdots v_n(z_n)1$ is an element of

$$V[[z_1, \dots, z_n]][\prod_{i<j}(z_i - z_j)^{-1}]$$

and hence defines a singular multilinear map from $V \times \cdots \times V$ to V, and it is not hard to check that the axioms for a vertex algebra in [K, 1.3] are equivalent to saying that this sequence of multilinear maps makes V into a commutative G vertex algebra. The field algebras of [K, 4.11], which are "non-commutative" vertex algebras, seem to be the same as associative G vertex algebras for G as in example 3.4.

Example 6.7. If G is the vertex group whose underlying Hopf algebra is the universal enveloping algebra of the Virasoro algebra, with the ring of singular functions given by inverting an element as in example 3.4, then G vertex algebras are more or less the same as vertex operator algebras, provided that we restrict to the positive energy representations of the Virasoro algebra.

Example 6.8. If G is the product of two copies of the vertex group of example 3.4, then G can be thought of as a vertex group corresponding to 2 dimensional quantum field theories. A typical G vertex algebra can be constructed by taking the tensor product of 2 vertex algebras; this corresponds to the usual decomposition of fields in 2 dimensional field theory into their left moving and right moving parts. We can do the same thing if G is the product of 2 copies of the vertex group in example 6.7; this gives the vertex group underlying much of conformal field theory and string theory, and as before we can construct many examples by taking a tensor product of 2 vertex operator algebras.

One reason why nontrivial examples of G vertex algebras are hard to find is that there are almost no examples of finite dimensional G vertex algebras, except for those that are really just associative algebras. It is not hard to see why this should be so: a nontrivial G vertex algebra should have some sort of singularity, say a pole of order 1, otherwise it would just be an associative algebra. However from a pole of order 1 we can usually construct poles of all orders by algebraic operations, so we get an infinite dimensional space of possible singularities. This is not possible unless the G vertex algebra we started with is infinite dimensional.

7. Vertex operators.

The most powerful and general way of constructing G vertex algebras is to write down a set of vertex operators satisfying the "normal ordering" condition below. This is closely related to the well known method of constructing vertex algebras by writing down a set of commuting vertex operators; see [K, 4.5]. For the rest of this section we suppose that G is a vertex group with group ring H. We now define several rings of "singular functions" and some modules over them.

If V is an R module we define $V[[x]]$ to be $Hom_R(H, V)$ (where x stands for $\dim(G)$ variables). If G is a formal group this is isomorphic as an R module to the space of power series in $\dim(G)$ variables with coefficients in V, and is an H module (using the trivial action of H on V). Similarly we define $V[[x_1, \ldots, x_n]]$ to be $Hom_R(H\otimes\cdots\otimes H, V)$, which is isomorphic to the space of power series in $n\dim(G)$ variables with coefficients in V and is an $H\otimes\cdots\otimes H$ module. It is also a module over the ring $R[[x_1, \ldots x_n]] = \mathrm{Hom}(H\otimes\cdots\otimes H, R)$. We define $V((x_1, \ldots, x_n))$ to be $V[[x_1, \ldots, x_n]]$ localized at all $1 \le i \le n$ and at all pairs $1 \le i < j \le n$ as in section 5. For example, if G is as in example 3.4 then

$$V((x_1, x_2)) = V[[x_1, x_2]][x_1^{-1}x_2^{-1}(x_1 - x_2)^{-1}].$$

Definition 7.1. *A vertex operator on an R module A is an R linear map from A to $B((x)) = Hom_R(H, B) \otimes_{H^*} K$. More generally a vertex operator from $A_1 \times \cdots \times A_n$ to B is an R multilinear map from $A_1 \times \cdots \times A_n$ to $B((x_1, \ldots, x_n))$.*

Example 7.2. With G as in example 3.4, a vertex operator from A to B is just a linear map from A to the space of formal Laurent series in B.

A vertex operator on A can be thought of informally as a singular function from G to operators from A to B.

It is not usually true that the composition of vertex operators is a vertex operator.

Definition 7.3. *We say that a sequence of vertex operators $v_1, \dots, v_n$ is* **compatible** *if their composition is the image of a unique singular multilinear map. More precisely suppose each v_i is a vertex operator from A_{i-1} to A_i, in other words an R linear map from A_{i-1} to $A_i((x_i))$. Then the composition is automatically a linear map from A_0 to $A_n((x_n))\cdots((x_1))$, and the condition that the composition is the image of a unique singular multilinear map means that it is in the image of a unique map from A_0 to $A_n((x_n,\dots,x_1))$. We say that a set S of vertex operators on A is compatible if any finite sequence of vertex operators in S is compatible.*

Definition 7.4. *We say that a vertex operator from A to B is a* **creation vertex operator** *if it is (induced by) a linear map from A to $B[[x]] = Hom(H,B)$.*

Definition 7.5. *We say that a vertex operator from A to B is an* **annihilation vertex operator** *if it factors through $B\otimes_R K$ via the natural map from $B\otimes_R K = B\otimes_R H^*\otimes_{H^*} K$ to $\mathrm{Hom}(H,B)\otimes_{H^*} K$.*

Roughly speaking, a creation vertex operator is a vertex operator that is nonsingular, and an annihilation vertex operator is one that is "rational" rather than "transcendental".

Definition 7.6. *We say that a product of vertex operators is* **normally ordered** *if for some $k \geq 0$ the first k operators in the product are creation vertex operators and the remainder are annihilation vertex operators.*

Definition 7.7. *We say that a set of vertex operators S on an R-module A satisfies the* **normal ordering condition** *if the composition of any two vertex operators in S can be written as a linear combination of normally ordered products of pairs of vertex operators in S.*

Lemma 7.8. *If a set of vertex operators satisfies the normal ordering condition then it is a compatible set of vertex operators.*

Proof. The main point is that any normally ordered product of annihilation and creation operators is a vertex operator. It follows immediately that if a set of vertex operators satisfies the normal ordering condition then any composition of vertex operators in S can be written as a linear combination of normally ordered compositions of vertex operators in S, by repeatedly rewriting product of operators so that creation vertex operators occur on the left, and hence is a vertex operator. So the set S is a compatible set of vertex operators. This proves lemma 7.8.

The point of the preceding definitions is the following theorem, which will be the main tool for finding compatible sets of vertex operators. It is an analogue of the trivial theorem that any set of operators on a module generates an algebra acting on the module.

Theorem 7.9. *Any compatible set of vertex operators on a module A generates a G vertex algebra acting on A.*

If A is a module over a commutative ring V which is generated as a V module by an element 1_A, then A obviously has the structure of a commutative ring, isomorphic to the kernel of V by the ideal of elements annihilating 1_A. The next theorem is a generalization of this to vertex algebras.

Theorem 7.10. *Suppose an R module A is acted on by a commutative G vertex algebra V, and suppose that A contains an element 1_A fixed by G and such that A is generated as a V-module by 1_A (which means that there is no proper G-submodule of A containing 1_A and acted on by V) and such that the vertex operator of any element of V applied to 1_A is nonsingular. Then the module A has a unique structure of a commutative vertex algebra such that the identity is 1_A and such that the action of any element of V on A is given by the action of some element of A.*

Proof. We define a map from V to A by mapping any v to $v^1(1_A)$ (which is well defined as we assumed $v^g(1_A)$ to be nonsingular in g). This is a homomorphism of V modules, and as V is commutative the kernel is a two sided ideal of V, so the image of V is a commutative vertex algebra. As this image is a V submodule containing 1_A, it must be the whole of A by assumption. Therefore A has the structure of a commutative G vertex algebra. This proves theorem 7.10.

The usual method of using the theorems in this section to construct G vertex algebras goes as follows. First we write down a lot of annihilation and creation operators satisfying the normal ordering condition. Then we try to find a commuting set of vertex operators in the vertex algebra they generate. There is often then an obvious element 1_A satisfying the conditions of theorem 7.10, so this makes A into a commutative G vertex algebra.

Example 7.11. We show how to construct the vertex algebra of a lattice using theorem 7.10. (See [K, 5.4] for a similar construction.) We assume for simplicity that all inner products in the lattice L are even. We let G be the vertex group of example 3.4 with underlying Hopf algebra H the ring spanned by the elements $D^{(i)}$. We let V be the universal commutative H-algebra generated by the group ring $R[L]$ of the lattice L. For each α in L we define a creation operator $e^{\alpha+}(z)$ to be multiplication by the element $\sum z^i D^{(i)}(e^\alpha)$. We define the annihilation operator $e^{\alpha-}(z)$ to be the homomorphism of rings with derivation from V to $V[z,z^{-1}]$ taking e^β to $z^{(\alpha,\beta)}e^\beta$. These annihilation and creation operators generate a (non commutative) G vertex algebra acting on V, by theorem 7.9. The operators $e^\alpha(z)=e^{\alpha+}(z)e^{\alpha-}(z)$ all commute with each other, and applying theorem 7.10 shows that they induce the structure of a vertex algebra on V. This is the usual vertex algebra of a lattice L.

8. Free field G vertex algebras.

In this section we construct some examples of G vertex algebras closely related to generalized free quantum fields.

To keep notation simple we will first describe a special case of the construction with just one (scalar) field. Afterwards we will list various ways to generalize it. We assume that the underlying Hopf algebra of G is the universal enveloping algebra of the Lie algebra of the group of translations of spacetime, so it is a polynomial algebra generated by the elements $\frac{\partial}{\partial x_i}$ for $1 \le i \le d$. The dual H^* is the ring of formal power series $R[[x_1,\dots,x_d]]$. We will take K to be $H^*[(-x_1^2+\cdots+x_n^2)^{-1}]$ (so we allow functions to have singularities along the light cone). We will say that an element f of H^* or K is even if $S(f)=f$ and odd if $S(f)=-f$, where S is the antipode with $S(x_i)=-x_i$. We fix an even element $\Delta(x)$ of K, which we call the propagator.

We let A be a free module over H generated by an element ϕ (which will be the free field). The underlying space of the G vertex algebra V we are constructing is the symmetric algebra $S^*(A)$. This is an associative commutative algebra acted on by G, and we will call the product of elements a and b in this algebra the normal ordered product of a and b and denote it by $:ab:$

We will construct an annihilation vertex operator $\phi^-(x)$ and a creation vertex operator $\phi^+(x)$, and we define $\phi(x)$ to be $\phi^-(x)+\phi^+(x)$. We define the annihilation operator $\phi^-(x)$ to be the derivation of $R[[x]]$ algebras from $V[[x]]$ to $V[[x]]$ which commutes with the action of G and such that

$$\phi^-(x)(\phi) = \Delta(x).$$

We define the creation operator from V to $V[[x]]$ by

$$\phi^+(x)(v) = \sum_i x^i D^{(i)}(\phi)v.$$

Lemma 8.1. *The annihilation and creation operators satisfy the following commutation relations.*

$$\begin{aligned}
[\phi^+(x), \phi^+(y)] &= 0\\
[\phi^-(x), \phi^+(y)] &= \Delta(x-y)\\
[\phi^-(x), \phi^-(y)] &= 0\\
[\phi(x), \phi(y)] &= 0
\end{aligned}$$

Proof. The first equality is trivial because the ring V is commutative, and the fourth follows from the first three and the fact that Δ is even. The third equality follows because $[\phi^-(x), \phi^-(y)]$ is a commutator of 2 derivations on $V[[x, y]]$ and is therefore also a derivation, and it commutes with G and vanishes on V so it is zero. For the second equality we calculate that

$$\begin{aligned}
\phi^-(x)(\sum_j y^j D^{(j)}(\phi)) &= \sum_j y^j D^{(j)}\Delta(x)\\
&= \Delta(x-y)
\end{aligned}$$

which implies the second equality because $\phi^-(x)$ is a derivation. This proves lemma 8.1.

Theorem 8.2. *There is a unique structure of a commutative G vertex algebra on V such that the vertex operator $\phi(x)$ is the vertex operator of some element.*

Proof. This follows easily from lemma 8.1 and theorem 7.10 and some routine checks.

The elements of the G vertex algebra V are just the usual Wick polynomials in ϕ, and the vertex algebra product of elements of V is just the usual expansion of products of Wick polynomials in terms of normal ordered products. See [S-W], section 3.2, for examples.

The construction above can easily be generalized in several ways as follows.

1. We can allow more than one field ϕ, in which case Δ should be changed to a function $\Delta_{\phi\psi}$ depending on the fields ϕ and ψ.
2. We can enlarge G to a semidirect product of spacetime translations with some other group, such as the Lorentz group. This other group will act on the space of fields in 1 above, so we can define spinor fields, vector fields, and so on.

3. If Δ satisfies some differential equation (such as the wave equation), in other words if it is annihilated by some element D of H, then $D(\phi)$ generates a proper ideal of the G vertex algebra V so we can quotient out by it.
4. Suppose that we have a function $\Delta_n(g_1, g_2, \ldots, g_n)$ of n variables $g_i \in G$ which are g invariant, with Δ_n nonsingular for $n > 2$. Then we can define a new G vertex algebra structure on $S^*(A)$ so that $\phi(x_1)\phi(x_2)\cdots\phi(x_n)$ to be $:\phi(x_1)\phi(x_2)\cdots\phi(x_n): +\Delta(x_1, \cdots, x_n)$ More generally we define products of the form $\phi^{n_1}(x_1)\phi^{n_2}(x_2)\cdots$ to be a sum of terms, each of which is formed by repeatedly pulling out n factors of the form $\phi(x_i)$ with not all i's the same, and replacing them with Δ_n. Derivatives of ϕ are handled by differentiating Δ. More generally we can assume that we are given functions Δ_n for all $n \geq 2$ as above. The functions Δ_n are the "irreducible n-point functions".
5. In example 4 we need to assume that the functions Δ_n are nonsingular for $n > 2$ in order to fit into the framework of G vertex algebras. We can allow the functions Δ_n to be singular (but we have to assume that if we identify some but not all of the variables of Δ with each other then the result is a well defined function) if we are willing to enlarge the notion of a vertex group. In particular we have to allow singular multilinear functions to have more general sorts of singularities, so the more general definition of a vertex group should specify not just the singular functions of one variable, but also singular functions of many variables. We will leave the precise definition as an exercise for the reader.

Example 5 above gives examples corresponding to nontrivial quantum field theories defined perturbatively, which corresponds to the fact that the G vertex algebras are only defined over the ring of formal power series in the coupling constants.

9. The main identity.

In this section we prove an identity for commutative G vertex algebras where G is a vertex formal group associated to some algebraic groups over $\mathbf{C}$. Roughly speaking, the identity says that if we have a vertex differential operator acting on a commutative vertex algebra, then integrating it over a cycle of dimension n increases its order as a differential operator by at most n.

In the case of classical vertex algebras (when G as in example 3.4 is associated to the one dimensional additive algebraic group) this identity we will prove is equivalent to the identities originally used to define vertex algebras [B] (see [K]). We will start off by explaining why, as motivation for the proof. The vertex algebra identity states that

$$\sum_{i\in\mathbf{Z}}\binom{m}{i}(u_{q+i}v)_{m+n-i}w = \sum_{i\in\mathbf{Z}}(-1)^i\binom{q}{i}(u_{m+q-i}(v_{n+i}w) - (-1)^q v_{n+q-i}(u_{m+i}w)).$$

For simplicity we will only discuss the case $m = n = q = 0$, when it becomes (one version of) the Jacobi identity $(u_0v)_0w = u_0(v_0w) - v_0(u_0w)$; the general case follows from a similar argument after including a suitable rational function of x, y and z. This Jacobi identity can be deduced from the fact that u_0 is a differential operator of degree 1 as follows. The fact that u_0 is a differential operator of degree 1 means that $(u_0)v(y) - v(y)u_0$ is a differential operator of degree 0 and must therefore be of the form $t(y)$ for some $t \in V$;

putting $y = 0$ then shows that $t = u_0 v$, so that $u_0(v(y)w) - v(y)u_0 w = (u_0 v)(y)w$, and now integrating y around 0 gives the Jacobi identity above. This justifies the statement that the main identity for a vertex algebra is equivalent to the fact that integrating $u(x)$ along a 1-cycle is a vertex differential operator of degree 1.

To prove that the integral u_0 of $u(x)$ along a 1-cycle is a vertex differential operator of degree 1 we have to show that the double commutator

$$[[u_0, v(y)]w(z)]$$

is 0, and each term in the double commutator can be written as the integral of $u(x)v(y)w(z)$ along a suitable cycle in the x plane with the points 0, y, and z removed. More precisely, if $C_{a,b,c\ldots}$ is a 1-cycle going once clockwise around the points $a, b, c\ldots$ and not containing other elements of the points $0, x, y$ then

$$C_{0,y,z} - C_{0,y} - C_{0,x} + C_0 = 0$$

in the first homology group, which gives us the relation

$$(u_0)v(y)w(z) - w(z)(u_0)v(y) - v(y)(u_0)w(z) + v(y)w(z)u_0 = 0.$$

In other words, the fact that u_0 is a vertex differential operator follows from linear relations between elements of a certain homology group.

In the higher dimensional case we proceed in a similar way, showing that certain operators are vertex differential operators of higher degree using relations between elements of homology groups, and this can be thought of as a higher dimensional generalization of the vertex algebra identity. At first sight there seems to be a serious problem in carrying out this program: the higher dimensional spaces whose homology groups we work with are not only rather complicated, but there are far too many of them to calculate all their homology groups. Rather surprisingly, we do not need to know the exact homology groups, and we do not even need to find any explicit relations. It turns out that the identities we want to prove follow just from the existence of "sufficiently many" relations. We will prove the existence of enough relations by bounding the dimensions of the homology group and using the fact that if we have a set of elements in the homology group with cardinality greater than its rank then there must be a nontrivial linear relation between them.

We let G be a finite dimensional real vertex formal group. The underlying formal group of G is the formal group of some connected real algebraic group, and we assume that the vertex structure on G is given by inverting some element of the coordinate ring of this algebraic group (so that K is of the form $RG^*[1/p] = R[[x_1, \ldots]][1/p]$ for some polynomial p in $x_1, \ldots$). We let U stand for a unipotent algebraic subgroup of G of some dimension n which is not contained in the divisor of p. As U is unipotent the exponential map from the Lie algebra of $U(\mathbf{C})$ to $U(\mathbf{C})$ is a diffeomorphism, so that $U(\mathbf{C})$ is a vector space over $\mathbf{C}$ with some group structure (which is of course different from the additive group structure if U is not abelian).

The function p restricts to a function on U with zero set given by some divisor D. We choose a compact n cycle $C \subset U(\mathbf{C})$ in the complement of the divisor D, which represents an element of $H_n(U(\mathbf{C}) - D)$ which we also denote by C.

We recall the definition of a differential operator on a ring. If S is a commutative algebra over a commutative ring R, then a differential operator D of order at most $n \in \mathbf{Z}$ is defined to be an operator which is zero if $n < 0$, and if $n \geq 0$ it is defined to be an operator such that $[D, s]$ is a differential operator of order at most $n - 1$ for any element $s \in S$. A differential operator D is called normalized if $D(1) = 0$. The differential operators of order at most 0 are just multiplications by elements in the center of S, and normalized differential operators of order at most 1 are the same as derivations of S, or in other words operators such that $D(ab) = aD(b) + D(a)b$ for all $a, b \in S$. Every differential operator can be written uniquely as the sum of a normalized differential operator and multiplication by a constant. If D and E are differential operators of orders m and n then DE is a differential operator of order at most $m + n$, and $[D, E]$ is a differential operator of order at most $m + n - 1$. In particular the normalized differential operators of order at most 1 form a Lie algebra acting on S, called the Lie algebra of derivations of S. A derivation of S can be thought of as an infinitesimal automorphism of S. We define vertex differential operators of commutative G vertex algebras in the same way.

Theorem 9.1. *If a is an element of a commutative G vertex algebra V with G as above, and n is the dimension of the subgroup U, then $\int_C a(z)d^n z$ is a vertex differential operator of order at most n. In other words if $a_0, \dots a_n \in V$ then*

$$[a_0(z_0), [a_1(z_1), [\dots, [a_n(z_n), \int_C a(z)d^n z] \cdots]]] = 0.$$

Note that $\int a(z)dz\, b(y) \neq b(y) \int a(z)dz$ even though $a(z)b(y) = b(y)a(z)$. The reason for this is that the two integrals are taken over different cycles in the subset where $a(z)b(y)$ is holomorphic, and these two cycles need not be homologous if $a(z)b(y)$ has singularities.

The proof of theorem 9.1 is in two steps. We first show that the number of linearly independent elements of the form

$$\left(\prod_{i \in S, 0 \leq i \leq N} a_i(z_i)\right) \int_C a(z)d^n z \left(\prod_{i \notin S, 0 \leq i \leq N} a_i(z_i)\right) \qquad 9.2$$

for the 2^{N+1} subsets S of $0, 1, 2, \dots, N$ is bounded by a polynomial in N of degree at most n (rather than the obvious bound 2^{N+1}) as N tends to infinity. This is because the number of linearly independent elements is bounded by the rank of certain homology groups, and we can bound these ranks by a polynomial in N. We then show that the possible relations between these elements are so restricted that this crude bound implies (and is even equivalent to) a single explicit relation, which is the one given in theorem 9.1.

Lemma 9.3. *Suppose $U(\mathbf{C})$ is a complex connected unipotent algebraic Lie group of dimension n and D is a closed algebraic subset. Then the rank of the homology group $H_n(U(\mathbf{C}) - g_1(D) \cup g_2(D) \cup \cdots \cup g_N(D))$ for elements g_i in general position is bounded by a polynomial of degree n in N (whose coefficients depend only on U and D).*

Proof. The one point compactification of $U(\mathbf{C})$ is a sphere of dimension $\dim(U(\mathbf{C})) = 2n$, so by Spanier-Whitehead duality it is sufficient to prove that the rank of the homology group $H_{n-1}(g_1(D) \cup g_2(D) \cup \cdots \cup g_N(D))$ is bounded by a polynomial of degree n in N. There is a spectral sequence converging to the homology of $g_1(D) \cup g_2(D) \cup \cdots \cup g_N(D)$ whose E_2 term is given by the homology groups of all finite intersections of the $g_i(D)$'s (see [G, theorem 5.4.1 and the remarks in section 5.6]). The rank of $H_{n-1}(g_1(D) \cup g_2(D) \cup \cdots \cup g_N(D))$ is bounded by the sum of the ranks of the terms $H_{n-1-(k-1)}(g_1(D) \cap g_2(D) \cap \cdots \cap g_k(D))$ of the E_2 term of the spectral sequence of total degree n, and the rank of any homology group of any intersection of the $g_i(D)$'s is bounded by some constant, so it is sufficient to show that the number of ways of choosing k of the g_i's so that their intersection has vanishing $H_{n-1-(k-1)}$ is bounded by a polynomial in N of degree at most n. As the g_i's are in general position, the intersection of k of them has dimension at most $2(n-k)$, so that the H_{n-k}'th homology of this intersection vanishes whenever $n-k > 2n-2k$, or $k > n$. Hence the number of ways of choosing k of the g_i's so that their intersection has non vanishing $H_{n-1-(k-1)}$ is bounded by the number of subsets of N elements of size at most n, which is a polynomial of degree n in N. This proves lemma 9.3.

Lemma 9.4. *The number of linearly independent elements of the form 9.2 is bounded by a polynomial in N of degree n.*

Proof. Each term of the form 9.2 is given by integrating

$$a(z)a_0(z_0)\cdots a_n(z_n)d^n z$$

over some n-cycle in $G - z_0(D) \cup \cdots \cup z_N(D)$. By lemma 9.3 the rank of the homology group generated by these n-cycles is bounded by a polynomial in N of degree n, which proves lemma 9.4.

In particular there must be some nonzero relation for sufficiently large N, because the number of expressions of the form 9.2 increases like 2^N, but the number of linearly independent ones is bounded by some polynomial in N.

Lemma 9.5. *Suppose that the coefficients c_S are the coefficients of a nonzero relation of smallest possible degree N as above. Then $c_S = (-1)^{|S|}c_\emptyset$ for some nonzero constant $c_\emptyset$.*

Proof. We are given that

$$\sum_S c_S \left(\prod_{i\in S, 0\le i\le N} a_i(z_i)\right) \int_C a(z)d^n z \left(\prod_{i\notin S, 0\le i\le N} a_i(z_i)\right) = 0$$

for all $a_0, \ldots, a_N \in V$. But if we put $a_i = 1$, $z_i = 0$ we find a relation of smaller degree. All coefficients of this relation must be identically 0 as the c_S's were by assumption the coefficients of a nonzero relation of smallest degree. But the coefficients of this smaller relation are of the form $\pm(c_S + c_{S\cup i})$ for $S \subset \{1, \ldots, i-1, i+1, \ldots, N\}$, so $c_S = -c_{S\cup i}$. This implies that $c_S = (-1)^{|S|}c_\emptyset$ for all S. This proves lemma 9.5.

In particular this shows that $\int_C a(z)d^n z$ is a differential operator of some order. We now pin down the order more precisely by looking more carefully at the set of all possible relations.

Lemma 9.6. *Suppose that n' is the degree of the unique nonzero relation of smallest degree, as in lemma 9.5. Then the maximum number of linearly independent elements of the form 9.2 is exactly equal to the number of subsets of $\{0, 1, \ldots, N\}$ of size at most n' and is therefore a polynomial in N of degree exactly n'.*

Proof. We show that the set of elements of the form

$$\left(\prod_{i\in S, 0\leq i\leq N} a_i(z_i)\right)\int_C a(z)d^nz\left(\prod_{i\notin S, 0\leq i\leq N} a_i(z_i)\right)$$

with $|S| \leq n'$ form a maximal linearly independent set of functions of the $a_i(z_i)$'s. In fact by using the relation of lemma 9.5 we see that any relation can be written as a sum of relations with at most n' factors to the left of the integral. To complete the proof of the lemma we have to show there are no nontrivial linear relations between these terms. Suppose there is a nontrivial relation with coefficients c_S, with $c_S = 0$ if $|S| > n'$. We can assume that we have chosen this relation so that the maximum value of $|S|$ with $c_S \neq 0$ is as small as possible. Now set $a_i = 1$, $z_i = 0$ for $i \notin S$. Then we get a relation of degree at most n' with a nonvanishing coefficient of $\left(\prod_{i\in S, 0\leq i\leq N} a_i(z_i)\right)\int_C a(z)d^nz$, which is impossible. Hence the maximal number of linearly independent elements is exactly equal to the number of subsets of size at most n' of a set of size N, which is a polynomial in N of degree exactly n'. This proves lemma 9.6.

We can now complete the proof of theorem 9.1. By lemma 9.4 the maximal number of linearly independent elements of the form 9.2 is bounded by a polynomial of degree at most n. In particular there must exist some relation of minimal degree n' by lemma 9.5. By lemma 9.6 we see that $n' \leq n$, so by lemma 9.5 the relation 9.1 holds, in other words $\int_C a(z)d^nz$ is a vertex differential operator of order at most $n' \leq n$. This proves theorem 9.1.

Theorem 9.1 can easily be generalized in several ways as follows.

1 The condition that C should lie inside some unipotent subgroup U can be removed; it is put in only because it simplifies the proof slightly and is satisfied in all the examples we use later.

2 The restriction that G is finite dimensional is also usually unnecessary; for example we could take G to be the formal group of the Virasoro algebra and take U to correspond to the 1 dimensional group generated by L_{-1}.

3 We can also look at vertex differential operators $a(x)$ and find that $\int_C a(z)d^nz$ is a vertex differential operator of order at most the order of a plus the dimension of U.

4 We can also look at vertex differential operators $a(x, y, z, \ldots)$ in several parameters, and find that $b(y, z, \ldots) = \int_C a(x, y, z, \ldots)d^nx$ is a vertex differential operator with order as above.

5 We can include a singular function f of several variables in the integrand without affecting the argument. For example if a is an element of a commutative G vertex algebra V with G as above, and n is the dimension of the subgroup U, then $\int_C a(z)f(z, z_0, \ldots, z_n)d^nz$ is a vertex differential operator of order at most n.

6 We do not need to restrict the vertex structure on G to be given by inverting a function p on an algebraic group, and we can allow things like the vertex structure generated by $f^{1/n}$ for some integer n.

7 Finally the restriction to vertex groups over the reals is unnecessary and theorem 9.1 can be generalized to vertex groups over any field R by using étale cohomology instead of singular cohomology.

10. G Vertex algebras and the Yang-Baxter equation.

In this section we show how to construct examples of G vertex algebras from solutions of the Yang-Baxter equation. The idea is to start with an G algebra, and deform it using a solution of the Yang-Baxter equation into a commutative G vertex algebra. (Note that in many examples we end up with a commutative G vertex algebra, even though the algebra we start with is not commutative!)

We first consider the case of associative algebras V. We say that a linear map R from $V \otimes V$ to $V \otimes V$ is an R matrix if it satisfies the following conditions:

1. $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ (Yang-Baxter equation.) Both sides act on $V \otimes V \otimes V$, and R_{ij} means R acting on the i'th and j'th factors of the tensor product.
2. $R(1 \otimes v) = 1 \otimes v$, $R(v \otimes 1) = v \otimes 1$.
3. $R_{12}m_{12} = m_{12}R_{23}R_{13}$, $R_{12}m_{23} = m_{23}R_{12}R_{13}$, where m_{ij} is the product map from $V \otimes V \otimes \cdots V$ to $V \otimes \cdots V$ given by multiplying the i'th and j'th factors.

Note that we have added extra conditions saying that the R matrix is "compatible" with the algebra structure.

Lemma 10.1. *Suppose V is a commutative algebra and R is an R matrix for V. Then the bilinear map $m_{12}R_{12}$ is another associative algebra structure on V (with the same identity element 1).*

Proof. The only nontrivial thing to check is associativity, which follows from

$$\begin{aligned} &m_{12}R_{12}m_{23}R_{23} \\ =&m_{12}m_{23}R_{12}R_{13}R_{23} \\ =&m_{12}m_{12}R_{23}R_{13}R_{12} \\ =&m_{12}R_{12}m_{12}R_{12}. \end{aligned}$$

Now we assume that V is acted on by a group G. The group $G \times G$ then acts on $V \otimes V$ and on $Hom(V \otimes V, V \otimes V)$. We denote the image of the matrix R under $g \times g'$ by $R^{g,g'}$. We assume that R satisfies the following axioms. The first three are the obvious analogues of the ones above, the fourth is just the definition of the action of $G \times G$ on R, and the fifth is a sufficient condition for the G-invariance of the new algebra structure on V.

1. $R_{12}^{g_1,g_2}R_{13}^{g_1,g_3}R_{23}^{g_2,g_3} = R_{23}^{g_2,g_3}R_{13}^{g_1,g_3}R_{12}^{g_1,g_2}$ (Yang-Baxter equation.)
2. $R^{g_1,g_2}(1 \otimes v) = 1 \otimes v$, $R^{g_1,g_2}(v \otimes 1) = v \otimes 1$.
3. $R_{12}^{g_1,g_2}m_{12} = m_{12}R_{23}^{g_1,g_2}R_{13}^{g_1,g_2}$, $R_{12}^{g_1,g_2}m_{23} = m_{23}R_{12}^{g_1,g_2}R_{13}^{g_1,g_2}$.
4. $(R^{g_1,g_2}(u \otimes v))^{g_1',g_2'} = R^{g_1g_1',g_2g_2'}((u \otimes v)^{g_1',g_2'})$
5. $R^{gg_1,gg_2} = R^{g_1,g_2}$

If we want we can use the fifth axiom to define $R^g = R^{1,g}$ so that $R^{g_1,g_2} = R^{g_1^{-1}g_2}$, so R really only depends on one element of G.

As before, we can use R^{g_1,g_2} to twist a G invariant algebra product on V to get a new G-invariant algebra product.

We can now try to construct G vertex algebras in the same way, except that the matrix R^{g_1,g_2} is a singular function of g_1, g_2; more precisely, R^g should be an element of $Hom(V \otimes V, Hom(H, V \otimes V) \otimes_{H^*} K)$. The quantum groups literature contains many examples of solutions of equation 1 above with singularities for g_1, g_2 complex numbers, or more generally elements of some Riemann surface. These do not usually satisfy equation 3 because the space V is usually taken to be a finite dimensional vector space rather than an algebra, but equation 3 shows that there is at most one extension of R to any algebra generated by this finite dimensional space, such as the tensor algebra. This gives a large number of G vertex algebras constructed using solutions of the Yang-Baxter equation.

Example 10.2. We take G as in example 3.4, and show how to construct the vertex algebra of a lattice L using the method above. For simplicity we assume all inner products in L are even; the general case can be done in a similar way using a twisted group ring of L. We take V to be the universal commutative H algebra generated by the group ring $\mathbf{Z}[L]$ of L. There is a singular solution to the Yang-Baxter equation on $\mathbf{Z}[L]$, given by

$$R^{x,y}(e^a \otimes e^b) = (x-y)^{(a,b)} e^a \otimes e^b.$$

This extends uniquely to an R matrix on V satisfying the conditions above, which can be used to make V into a vertex algebra. This is the usual vertex algebra of an even lattice.

11. Cohomology of G vertex algebras.

We have seen earlier that many G vertex algebras can be constructed by deforming the product on a G-algebra. In this section we will show how to describe the infinitesimal deformations of a G-algebra into a G vertex algebra using G-equivariant cohomology of associative algebras.

We start by recalling how to classify the infinitesimal deformations of an associative algebra V over a commutative ring R. This means that we want to define an associative algebra structure on $V[\epsilon]$ over $R[\epsilon]$, where $\epsilon^2 = 0$. For simplicity we will work with associative algebras without identity elements; the main difference below if we use algebras with identity elements is that we should use normalized cochains instead of cochains. If we write the new algebra product as $ab + \epsilon f(a,b)$ then associativity is equivalent to

$$af(b,c) - f(ab,c) + f(a,bc) - f(a,b)c = 0$$

so f is just a 2-cycle for the standard complex used to calculate the associative algebra cohomology $H^2(V,V)$. (Recall that the n-cochains of the standard resolution for calculating the Hochschild cohomology groups $H^n(V,M)$ for a 2 sided module M over the associative algebra V are the multilinear maps from $A, A, \ldots, A$ to M; see [C-E].) The "trivial" deformations are those that are induced by an infinitesimal automorphism $a \mapsto a + \epsilon g(a)$ of the underlying R-module, when the corresponding 2-cocycle f is given by $f(a,b) = ag(b) - g(ab) + g(a)b$, in other words f is just the coboundary of the 1-cochain g.

Therefore the infinitesimal deformations of the algebra V are classified by the cohomology group $H^2(V,V)$ of the associative algebra V with coefficients in the 2-sided module V.

Now suppose that V is acted on by a group (or cocommutative Hopf algebra) G. Then we can calculate the infinitesimal deformations of the G-algebra V by using G-equivariant cohomology groups $H^2_G(V,V)$. The G-equivariant cohomology groups are calculated (and defined) by replacing the (standard) cochains above by the G-invariant cochains.

Finally suppose that G is a vertex group. We define the cohomology groups $H^n_G(V,M)$ by using the G-invariant singular multilinear functions from $V, V, \dots V$ to M in place of the multilinear functions from $V, V \dots, V$ to M. (The boundary operator is defined by the same formula as for associative algebras.) We then find by the argument above that there is a map from $H^2_G(V,V)$ for an associative G-algebra V to G vertex algebras over $R[\epsilon]$. The group $H^2_G(V,V)$ can be thought of a roughly the tangent space at A of the moduli space of G vertex algebra structure on A. Some examples of nontrivial elements of $H^2_G(V,V)$ can easily be obtained from the G vertex algebras of generalized free fields, because we can just take a 2-point function Δ of the form $\epsilon/(x-y)^2$ for example.

We can also define other sorts of cohomology groups involving vertex groups G, such as vertex group cohomology. We first note that most definitions in group cohomology (in particular the homogeneous and inhomogeneous standard complexes) can easily be extended to the case of arbitrary cocommutative Hopf algebras. We can now define the vertex group cohomology groups $H^n(G,M)$ for a vertex group G and a G-module M by replacing the spaces of multilinear maps $Hom(H, H \dots, H, M)$ in the homogeneous standard complex (where H is the underlying cocommutative Hopf algebra) by the corresponding spaces of singular multilinear maps.

12. Relation to quantum field theory.

There are several mathematical structures closely related to quantum field theory; for example, Wightman's axioms [S-W] and some closely related variations [G-J], Segal's axioms for a topological field theory [Se], and G vertex algebras as described above. We give an informal description of the relation between G vertex algebras and the other theories above. The four theories above are closely related but not exactly the same.

The relation with Wightman's axioms is easiest to describe. A quantum field theory satisfying Wightman's axioms is determined by its correlation functions [S-W], and if these correlation functions have "good" operator product expansions then they are the correlation functions of some G vertex algebra, with G the vertex group whose underlying Hopf algebra is the universal enveloping algebra of the Poincaré algebra, and where we allow some sort of singularities on the light cone. Unfortunately it is difficult to decide when a G vertex algebra comes from a quantum field theory satisfying the Wightman axioms. The main problem is in reconstructing the Hilbert space. It is usually not too hard to construct a real vector space with a symmetric bilinear form on it, but deciding when this form is positive (semi) definite is usually rather hard. (For example a special case of this is the problem of deciding which highest weight representations of the Virasoro algebra are unitary.)

Note that expressions like $\phi_1(x_1)\phi_2(x_2)$ are interpreted in quite different ways in quantum field theories and G vertex algebras: in quantum field theories we think of the ϕ_i's as distribution valued operators defined on a manifold containing points x_i, and the product is a product of operators. In G vertex algebras the elements x_i are thought of as elements of some group and $\phi(x)$ should be thought of as the transform of the element ϕ under the action of the group element x. The product does not always make sense for fixed x_1 and x_2, but is only defined as some sort of singular function of x_1 and x_2.

The relation with quantum field theories that occur in physics is similar: provided that good operator product expansions exist, there is often a G vertex algebra with the same correlation functions. The main problem is that most realistic quantum field theories are only defined at the level of perturbation theory, in other words, the correlation functions are formal power series in the coupling constants that (probably) do not converge for any nonzero values. We can get round this by the following trick: we define the G vertex algebra over the ring of formal power series in the coupling constants. So many quantum field theories are now well defined mathematical objects: they are G vertex algebras over formal power series rings. Notice that this does not solve the important problem of making sense of these theories non perturbatively; all we have done is change the question of existence of quantum field theories into an equally hard question about properties of G vertex algebras over formal power series rings. More precisely, we would really like to construct some sort of moduli space of G vertex algebras, such that at points of its compactification the G vertex algebras somehow degenerate into the formal power series G vertex algebras above.

The relation of G vertex algebras with Segal's topological field theories is harder to describe; in fact, there seems to be no particularly easy way to go between them. The reason for this seems to be that G vertex algebras like to work with groups G and work best when spacetime can be regarded as a group (for example, if spacetime is flat). On the other hand, Segal's axioms work with arbitrary manifolds, most of which have nothing to do with groups. A good example is given by 2 dimensional conformal field theories. It is well known that vertex operator algebras (which are more or less vertex $Virasoro$-algebras) are good at describing the genus 0 part of a conformal field theory, but are bad at describing the higher genus part (although the genus 1 case has been pushed through by Zhu [Z]). This is because a genus 0 Riemann surface is more or less the additive group $\mathbf{C}$ (at least if a point is missed out), while Riemann surfaces of genus greater than 1 are not directly related to groups. The additive group $\mathbf{C}$ is of course more or less the "group" that acts on vertex algebras.

To summarize, quantum field theories satisfying the Wightman axioms, topological field theories, and G vertex algebras are related but different mathematical structures, each of which captures part but not all of the notion of a quantum field theory. Algebraic quantum field theories emphasize Lorentz invariance, locality, and unitarity, but have the disadvantage that it is extraordinarily difficult to construct nontrivial examples of them. G vertex algebras emphasize locality, Poincare invariance, and the operator product expansion, and can cope with the Feynman path integral (regarded as a trace on the G vertex algebra), but are not very good at dealing with unitarity or curved spacetimes. Segal's axioms emphasize the Feynman path integral and are particularly good at dealing with curved spacetimes.

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