Chapter 17

The 24-dimensional odd unimodular lattices.

R. E. Borcherds.

This chapter completes the classification of the 24-dimensional unimodular lattices by enumerating the odd lattices. These are (essentially) in one-to-one correspondence with neighboring pairs of Niemeier lattices.

The even unimodular lattices in 24 dimensions were classified by Niemeier [Nie2] and the results are given in the previous chapter, together with the enumeration of the even and odd unimodular lattices in dimensions less than 24. There are twenty-four Niemeier lattices, and in the present chapter they will be referred to by their components D_{24}, $D_{16}E_8$, with the Leech lattice being denoted by Λ_{24}, and also by the Greek letters $\alpha, \beta, \ldots$ (see Table 16.1).

The **odd** unimodular lattices in 24 and 25 dimensions were classified in [Bor1]. In this chapter we list the odd 24-dimensional lattices. Only those with minimal norm at least 2 are given, i.e., those that are strictly 24-dimensional, since the others can easily be obtained from lower dimensional lattices (see the summary in Table 2.2 of chapter 2).

There is a table of all the 665 25-dimensional unimodular lattices and the 121 even 25-dimensional lattices of determinant 2 on my home page

(currently http://www.dpmms.cam.ac.uk/home/emu/reb/.my-home-page.html).

Two lattices are called **neighbors** if their intersection has index 2 in each of them ([Kne4], [Ven2]).

We now give a brief description of the algorithm used in [Bor1] to enumerate the 25-dimensional unimodular lattices.

The first step is to observe that there is a one-to-one correspondence between 25 dimensional unimodular lattices (up to isomorphism) and orbits of norm -4 vectors in the even Lorentzian lattice $II_{25,1}$ given as follows: the lattice Λ corresponds to the norm -4 vector v if and only if the sublattice of even vectors of Λ is isomorphic to the lattice $v^\perp$. So we can classify 25 dimensional unimodular lattices if we can classify negative norm vectors in $II_{25,1}$.

We can classify orbits of vectors of norm $-2n \leq 0$ in $II_{25,1}$ by induction on n as follows. First of all the primitive norm 0 vectors correspond to the Niemeier lattices as in section 1 of chapter 26. So there are exactly 24 orbits of primitive norm 0 vectors, and any norm 0 vector can be obtained from a primitive one by multiplying by some constant.

Suppose we have classified all orbits of vectors of norms $-2m$ with $0 \geq -2m > -2n$, and that we have a vector v of norm $-2n$. We fix a fundamental Weyl chamber for the reflection group of $II_{25,1}$ as in chapter 26. We look at the root system of the lattice $v^\perp$, and find that one of the following 3 things can happen:

1. There is a norm 0 vector z with $(z, v) = -1$. It turns out to be trivial to classify such norm $-2n$ vectors v: there is one orbit corresponding to each orbit of norm 0 vectors. They correspond to lattices $v^\perp$ which are the sum of a Niemeier lattice and a one dimensional lattice generated by a vector of norm $2n$.
2. There is no norm 0 vector z with $(z, v) = 1$ and the root system of $v^\perp$ is nonempty. In this case we choose a component of the root system of $v^\perp$ and let r be its highest root. Then the vector $u = v + r$ has norm $-2(n-1)$, and the assumption about no

norm 0 vectors z with $(z, v) = 1$ easily implies that u is still in the Weyl chamber of $II_{25,1}$. Hence we have reduced v to some known vector u of norm $-2(n-1)$, and with a little effort it is possible to reverse this process and construct v from u.

3. Finally suppose that there are no roots in $v^\perp$. As v is in the Weyl chamber this implies that $(v, r) \leq -1$ for all simple roots r. By theorem 1 of chapter 27 there is a norm 0 (Weyl) vector w_{25} with the property that $(w_{25}, r) = -1$ for all simple roots r. Therefore the vector $u = v - w_{25}$ has the property $(u, r) \leq 0$ for all simple roots r. So u is in the Weyl chamber, and has norm $-2n - (u, w_{25})$ which is larger than $-2n$ unless v is a multiple of w_{25}. So we can reconstruct v from the known vector u as $v = u + w_{25}$.

In every case we can reconstruct v from known vectors, so we get an algorithm for classifying the norm $-2n$ vectors in $II_{25,1}$. (This algorithm breaks down in higher dimensional Lorentzian lattices for two reasons: it is too difficult to classify the norm 0 vectors, and there is usually no analogue of the Weyl vector w_{25}.)

We now apply the algorithm above to find the 121 orbits of norm -2 vectors from the (known) norm 0 vectors, and then apply it again to find the 665 orbits of norm -4 vectors from the vectors of norm 0 and -2.

The neighbors of a strictly 24 dimensional odd unimodular lattice can be found as follows. If a norm -4 vector $v \in II_{25,1}$ corresponds to the sum of a strictly 24 dimensional odd unimodular lattice Λ and a 1-dimensional lattice, then there are exactly two norm-0 vectors of $II_{25,1}$ having inner product -2 with v, and these norm 0 vectors correspond to the two even neighbors of Λ.

The enumeration of the odd 24-dimensional lattices. Figure 17.1 shows the neighborhood graph for the Niemeier lattices, which has a node for each lattice. If A and B are neighboring Niemeier lattices, there are three integral lattices containing $A \cap B$, namely A, B, and an odd unimodular lattice C (cf. [Kne4]). An edge is drawn between nodes A and B in Fig. 17.1 for each strictly 24-dimensional unimodular lattice arising in this way. Thus there is a one-to-one correspondence between the strictly 24-dimensional odd unimodular lattices and the edges of our neighborhood graph. The 156 lattices are shown in Table 17.1. Figure 17.1 also shows the corresponding graphs for dimensions 8 and 16.

For each lattice Λ in the table we give its components (in the notation of the previous chapter) and its even neighbors (represented by 2 Greek letters as in Table 16.1). The final column gives the orders $g_1.g_2$ of the groups $G_1(\Lambda)$, $G_2(\Lambda)$ defined as follows. We may write $Aut(\Lambda) = G_0(\Lambda).G_1(\Lambda).G_2(\Lambda)$ where G_0 is the reflection group. The group G_1 is the subgroup of $Aut(\Lambda)$ of elements fixing a fundamental chamber of the Weyl group and not interchanging the 2 neighbors. The group $G_2(\Lambda)$ has order 1 or 2 and interchanges the two neighbors of Λ if it has order 2. (It turns out that $G_2(\Lambda)$ has order 2 if and only if the two components of Λ are isomorphic.) The components are written as a union of orbits under $G_1(\Lambda)$, with parentheses around two orbits if they fuse under $G_2(\Lambda)$.

The first lattice in the table is the odd Leech lattice O_{24}, which is the only one with no norm 2 vectors. The number of norm 2 vectors is given by the formula

$$8h(A) + 8h(B) - 16$$

where $h(A)$ and $h(B)$ are the Coxeter numbers of the even neighbors of the lattice. These Coxeter numbers satisfy the inequality $h(B) \leq 2h(A)-2$ and the lattices for which equality holds are indicated by a thick line in figure 17.1. The Weyl vector $\rho(\Lambda)$ of the lattice Λ has norm given by the formula $\rho(\Lambda)^2 = h(A)h(B)$.