The Cellular Structure of the Leech Lattice.

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(This is chapter 25 of “Sphere packings, lattices, and groups”, edited by Conway and Sloane.)

We complete the classification of the holes in the Leech lattice, and of the associated Delaunay cells, by showing that there are precisely 284 types of shallow hole.

1. Introduction.

In Chapter 23 of [C-S] it was shown that there are 23 types of deep hole in the Leech lattice Λ_{24}, and that these holes are in one-to-one correspondence with the Niemeier lattices. The existence of this correspondence, and the recently discovered correspondence between the conjugacy classes in the Monster group and certain modular functions [C-N] suggested that it might be worth enumerating the shallow holes in the Leech lattice and completing the classification of its Delaunay cells, in case any “deep structure” became apparent. Although this has not yet happened, the complete list of deep and shallow holes has already found several uses, and it seems worth while to put it on record. The main result is the following.

Theorem 1. *There are 307 types of hole in the Leech lattice, consisting of 23 types of deep hole and 284 types of shallow hole. They are listed in Table 1.*

The neighborhood graph of the deep holes may be seen in Chapter 17 of [C-S].

2. Names for the holes.

We use the notation of Chapter 23 of [C-S], and describe sets of Leech lattice points by graphs, with a node for each lattice point, and where two nodes x and y are

$$\text{not joined if } N(x-y)=4,$$
$$\text{joined by a single edge if } N(x-y)=6,$$
$$\text{joined by two edges if } N(x-y)=8.$$

Larger numbers of joins will not arise here.

It was shown in Chapter 23 of [C-S] that the vertices of a deep hole in Λ_{24} are described by a graph that is a disjoint union of extended Coxeter-Dynkin diagrams having total subscript (or dimension) 24 and constant Coxeter number h. (The Coxeter numbers are shown in Table 2 below.) There are just 23 possible combinations, which can be seen in the first 23 lines of Table 1. Using the same graphical notation we prove the following result.

Lemma 2. *The vertices of a shallow hole in the Leech lattice are sets of 25 points of* Λ_{24} *for which the corresponding graph is a union of spherical Coxeter-Dynkin diagrams.*

Proof. Theorems 5 and 6 of Chapter 23 of [C-S] show that the graph is a union of spherical Coxeter-Dynkin diagrams, and by dimension considerations the hole must contain at least 25 vertices. But the fundamental roots corresponding to such a diagram (whether or not it is connected) are linearly independent in the vector space they lie in, and so are affinely independent. This shows that the graph cannot contain more than 25 nodes. It is just as easy to verify that any such set of 25 points is the vertex set of a shallow hole. The argument is similar to Theorem 7 of Chapter 23 of [C-S], which is the corresponding result for deep holes. Thus all the shallow holes in the Leech lattice are simplices.

3. The volume formula.

Let $P_1, P_2, \ldots, P_N$ be a system of representatives for all the holes in Λ_{24} under the full automorphism group $\cdot\infty$ of Λ_{24}. Let vol(P_i) denote the volume of P_i and $g(P_i)$ the order of its automorphism group (i.e. the subgroup of $\cdot\infty$ fixing P_i). Then we have the *volume formula*:

$$\text{volume of a fundamental domain of } \Lambda_{24} = \sum_{i=1}^{N} vol(P_i) \times \text{ no. of images of } P_i \text{ under } \cdot 0$$
$$= \sum_{i=1}^{N} \frac{|\cdot 0|}{g(P_i)} vol(P_i) \ ,$$

where $|\cdot 0|$ denotes the order of $\cdot 0$.

The volume of a hole P can be expressed in terms of familiar concepts in the Lie theory (cf. [B]). For a deep hole,

$$vol(P) = \frac{1}{24!} h\sqrt{d} \ ,$$

and for a shallow hole

$$vol(P) = \frac{1}{24!} \sqrt{sd} \ ,$$

where h is the Coxeter number, d is the determinant of the Cartan matrix of the spherical Coxeter-Dynkin diagram, and s is the norm of the Weyl vector. For a connected component the values of these quantities are shown in Table 2 (note that $s = h(h+1)n/12$). For a disconnected graph, h and d are the products of the values of the components, while s is the sum. One can also show that the radius of a shallow hole is $\sqrt{2 - \frac{1}{s}}$.

Table 2

	$a_n (or\ A_n)$	$d_n (or\ D_n)$	$e_6 (or\ E_6)$	$e_7 (or\ E_7)$	$e_8 (or\ E_8)$
$h:$	$n+1$	$2n-2$	12	18	30
$d:$	$n+1$	4	3	2	1
$s:$	$\frac{n(n+1)(n+2)}{12}$	$\frac{(n-1)n(2n-1)}{6}$	78	$\frac{399}{2}$	620

4. The enumeration of the shallow holes.

We shall only give a brief description of how the shallow holes were enumerated and Theorem 1 proved. Two methods of classification were used.

Method 1. This method was used to find all the shallow holes that contain a particular spherical Coxeter-Dynkin diagram X as a component, by finding all occurrences of X as a set of points in the Leech lattice. This method was only used when X has at least seven points, for otherwise the classification becomes too complicated. We drew the graph of all points in Λ_{24} not joined to any point of X, and then deleted enough points from this graph to make the union of X with the remaining graph be a collection of spherical diagrams with a total of 25 points. The group orders of such holes, found by inspection, are usually 1 or 2.

As an example we consider the diagram $X = a_{15}$. We find from Figure 23.9 of [C-S] that there are just nine types of ordered a_{15} diagrams in the Leech lattice. When we identify reversals these reduce to five distinct types, which are drawn in Figures 1(a)-(e) as heavy lines. (The background of these figures is taken from Figure 27.3 of [C-S], and shows all edges between pairs of the 35 points nearest to the center of a deep hole of type A_{24}. This is a convenient portion of the Leech lattice to work with, and in particular contains all the points of Λ_{24} not joined to the a_{15} diagrams.)

In four of figures (a)-(e) we find that in the subgraph not joined to the a_{15} diagram (indicated by shaded nodes and broken lines) it is impossible to find a union of disjoint spherical Coxeter-Dynkin diagrams with a total of $10(= 25-15)$ nodes, and so the corresponding type of a_{15} diagram cannot be part of a shallow hole. However, for the fifth type (figure (e)), the disjoined subgraph contains 11 points, and by omitting each point in turn we can break it into Coxeter-Dynkin diagrams, as shown in the figure. Three of the eleven possibilities fail (those marked with an x), since they leave a graph containing an *extended* diagram. The remaining eight succeed.

Thus there are eight types of shallow hole with a component of type a_{15} in their diagram: $a_{15}e_8a_1^2$, $a_{15}e_7a_3$, $a_{15}e_6d_4$, $a_{15}d_5^2$, $a_{15}d_6a_4$, $a_{15}d_7a_2a_1$, $a_{15}d_9a_1$, and $a_{15}d_{10}$.

Method 2. We take a given deep hole and find all shallow holes having a face in common with it. The environs of a deep hole are described in sufficient detail in chapter 24 of [C-S] to make this quite easy. The group orders of these holes can be found by considering the subgroup of those automorphisms of the corresponding deep hole that fix the face in question and then making due allowance for the fact that the shallow hole may touch several deep holes in the same manner.

The complete enumeration was carried out by combining the two methods and using the volume formula as a check. A second check was supplied by the fact that the quantity sd for each hole must be a perfect square (since the volume of a hole is a rational number). Most of the enumeration was performed twice: first by L.Q. and J.H.C., then independently by R.E.B who completed the list. The group orders were computed by both teams.

Remarks. (1) It turns out that (in contrast to the deep holes) the diagram of a shallow hole does not determine it uniquely. There are two holes of each of the types a_17a_8, $a_{17}d_7a_1$, $a_{11}d_7a_3a_2^2$ and $a_9^2a_4a_3$.

(2) There is a unique hole of type $a_1^{24}a_1$. This has the shape of a regular simplex with 25 vertices, although its group is not transitive on the vertices. The group is the Mathieu group M_{24} acting with orbits of size 24 and 1. This hole has the smallest radius, namely $\sqrt{2-\frac{2}{25}}$, while the shallow hole of largest radius is d_{25}, with radius $\sqrt{2-\frac{1}{4900}}$. The deep holes have radius $\sqrt{2}$.

(3) If the closed polytopes corresponding to the deep holes of type D_4^6 are deleted from $\mathbf{R^{24}}$, the resulting space is disconnected.

Acknowledgment. We should like to thank R. A. Parker for several helpful discus-

sions.

Table 1.

A list of all 307 holes in the Leech lattice. The first 23 entries are the deep holes. The entries give the name of a hole P_i, the order $g(P_i)$ of its automorphism group, its scaled volume

$$svol(P_i) \; = \; vol(P_i) \; \cdot \; \frac{24!}{|\cdot 0|} \; ,$$

the norm $s(P_i)$ of its Weyl vector, and the determinant $d(P_i)$ of the Cartan matrix. The volume formula then becomes

$$\sum_i \frac{svol(P_i)}{g(P_i)} \; = \; \frac{24!}{|\cdot 0|} \; = \; 74613 \; .$$

The name of a hole indicates the orbits of its automorphism group on the components of the diagram. Thus the hole $a_7^2 a_3^2 a_3 a_1^2$ has two equivalent components of type a_7, two of type a_1, and three of type a_3, only two of which are equivalent under the automorphism group.

Table 1(a).

name	g	svol	s	d
D_{24}	2	92		4
A_{24}	10	125		25
A_17E_7	12	1944		36
$D_{16}E_8$	2	1800		4
$A_{15}D_9$	16	2048		64
D_{12}^2	8	1936		16
A_{12}^2	52	2197		169
$A_{11}D_7E_6$	24	20736		144
$D_{10}E_7^2$	8	23328		16
$A_9^2D_6$	80	20000		400
E_8^3	6	27000		1
D_8^3	48	21952		64
A_8^3	324	19683		729
$A_7^2D_5^2$	256	131072		1024
E_6^4	432	186624		81
D_6^4	384	160000		256
A_6^4	1176	117649		2401
$A_5^4D_4$	3456	559872		5184
D_4^6	138240	2985984		4096
A_4^6	30000	1953125		15625
A_3^8	688128	16777216		4^8
A_2^{12}	138568320	387420489		3^{12}
A_1^{24}	1002795171840	68719476736		2^{24}
d_{25}	1	140	9800/2	4

a_{25}	1	195	2925/2	26
$d_{24}a_1$	1	186	8649/2	8
$a_{24}a_1$	2	255	2601/2	50
$a_{23}a_2$	2	288	2304/2	72
$d_{22}a_2a_1$	1	282	6627/2	24
$d_{21}a_4$	1	240	5760/2	20
$a_{21}a_3a_1$	1	396	1782/2	176
$d_{20}d_5$	1	200	5000/2	16
$a_{20}a_5$	1	315	1575/2	126
$d_{19}e_6$	1	162	4374/2	12
$a_{19}d_6$	1	240	1440/2	80
$a_{19}a_4a_1^2$	2	520	1352/2	400
$d_{18}e_7$	1	126	3969/2	8
$a_{18}e_7$	1	171	1539/2	38
$a_{18}a_6a_1$	1	399	1197/2	266
$d_{17}e_8$	1	92	4232/2	4
$a_{17}e_8$	2	141	2209/2	18
$a_{17}a_8$	2	297	1089/2	162
$a_{17}a_8$	2	297	1089/2	162
$a_{17}e_7a_1$	2	222	1369/2	72
$a_{17}d_7a_1$	1	288	1152/2	144
$a_{17}d_7a_1$	2	288	1152/2	144
$a_{17}d_6a_2$	2	342	1083/2	216
$a_{17}a_6a_2$	2	441	1029/2	378
$a_{17}a_5a_3$	2	468	1014/2	432
$a_{17}a_4a_3a_1$	2	600	1000/2	720
$d_{16}d_9$	1	152	2888/2	16
$d_{16}a_9$	1	230	2645/2	40
$d_{16}e_8a_1$	1	122	3721/2	8
$d_{16}a_8a_1$	1	306	2601/2	72
$d_{16}e_7a_2$	1	186	2883/2	24
$d_{16}e_6a_3$	1	252	2646/2	48
$d_{16}a_6a_2a_1$	1	462	2541/2	168
$d_{16}d_5a_4$	1	320	2560/2	80
$d_{16}a_5a_4$	1	390	2535/2	120
$a_{16}d_9$	1	204	1224/2	68
$a_{16}e_8a_1$	1	187	2057/2	34
$a_{15}d_{10}$	2	200	1250/2	64
$a_{15}d_9a_1$	2	264	1089/2	128
$a_{15}e_8a_1^2$	2	248	1922/2	64
$a_{15}e_7a_3$	2	264	1089/2	128
$a_{15}d_7a_2a_1$	2	408	867/2	384
$a_{15}e_6d_4$	2	288	864/2	192
$a_{15}d_6a_4$	2	360	810/2	320

$a_{15}d_5d_5$	2	320	800/2	256
$d_{14}d_{10}a_1$	1	188	2209/2	32
$d_{14}a_{10}a_1$	1	286	1859/2	88
$d_{14}a_9a_1a_1$	1	380	1805/2	160
$d_{14}e_8a_2a_1$	1	186	2883/2	24
$d_{14}e_7a_3a_1$	1	256	2048/2	64
$d_{14}a_7a_2a_1a_1$	1	576	1728/2	384
$d_{14}e_6a_4a_1$	1	330	1815/2	120
$d_{14}a_6a_4a_1$	1	490	1715/2	280
$d_{14}d_5a_5a_1$	1	408	1734/2	192
$a_{14}a_9a_2$	2	405	729/2	450
$a_{14}e_8a_2a_1$	1	285	1805/2	90
$a_{14}d_7a_2a_2$	1	450	750/2	540
$a_{14}a_6a_3a_2$	1	630	630/2	1260
$a_{14}a_4^2a_2a_1$	2	825	605/2	2250
$d_{13}d_{12}$	1	136	2312/2	16
$d_{13}a_{12}$	1	208	1664/2	52
$d_{13}a_{11}a_1$	1	276	1587/2	96
$d_{13}a_9a_2a_1$	1	420	1470/2	240
$d_{13}e_8a_4$	1	160	2560/2	20
$d_{13}a_8a_4$	1	360	1440/2	180
$d_{13}e_7a_5$	1	204	1734/2	48
$d_{13}a_7d_5$	1	304	1444/2	128
$d_{13}e_6a_6$	1	252	1512/2	84
$a_{13}a_{12}$	1	273	819/2	182
$a_{13}d_{10}a_2$	1	294	1029/2	168
$a_{13}a_9a_3$	1	420	630/2	560
$a_{13}e_8a_4$	1	245	1715/2	70
$a_{13}e_8a_2a_1a_1$	1	378	1701/2	168
$a_{13}e_7a_4a_1$	1	350	875/2	280
$a_{13}d_7a_5$	1	336	672/2	336
$a_{13}a_7a_4a_1$	1	560	560/2	1120
$a_{13}e_6d_5a_1$	1	336	672/2	336
$a_{13}a_6a_6$	1	441	567/2	686
$d_{12}^2a_1$	2	180	2025/2	32
$d_{12}d_{10}a_2a_1$	1	276	1587/2	96
$d_{12}d_9a_4$	1	240	1440/2	80
$d_{12}e_8d_5$	1	136	2312/2	16
$d_{12}d_8d_5$	1	208	1352/2	64
$d_{12}e_7d_6$	1	156	1521/2	32
$d_{12}d_7e_6$	1	180	1350/2	48
$a_{12}^2a_1$	4	351	729/2	338
$a_{12}e_8d_5$	1	208	1664/2	52
$a_{12}e_8a_4a_1$	1	325	1625/2	130

$a_{12}e_7a_6$	1	273	819/2	182
$a_{12}d_7e_6$	1	234	702/2	156
$d_{11}e_8e_6$	1	114	2166/2	12
$d_{11}e_7^2$	2	112	1568/2	16
$a_{11}d_{10}a_3a_1$	1	408	867/2	384
$a_{11}d_{10}a_2^2$	2	432	864/2	432
$a_{11}d_9a_5$	2	324	729/2	288
$a_{11}a_9d_4a_1$	1	480	480/2	960
$a_{11}e_8e_6$	1	174	1682/2	36
$a_{11}e_8d_5a_1$	1	276	1587/2	96
$a_{11}e_8a_2^2a_1^2$	2	576	1536/2	432
$a_{11}d_8e_6$	2	228	722/2	144
$a_{11}e_7d_7$	1	204	867/2	96
$a_{11}e_7a_5a_1^2$	2	456	722/2	576
$a_{11}d_7e_6a_1$	2	300	625/2	288
$a_{11}d_7a_6a_1$	1	420	525/2	672
$a_{11}d_7a_5a_2$	2	468	507/2	864
$a_{11}d_7a_3a_2^2$	2	648	486/2	1728
$a_{11}d_7a_3a_2^2$	2	648	486/2	1728
$a_{11}a_7d_5a_1a_1$	1	576	432/2	1536
$a_{11}e_6e_6a_1^2$	2	360	600/2	432
$a_{11}e_6d_5a_3$	2	384	512/2	576
$a_{11}e_6d_5a_2a_1$	2	468	507/2	864
$a_{11}e_6d_4a_4$	2	420	490/2	720
$a_{11}d_6a_5a_3$	2	504	441/2	1152
$a_{11}a_6d_4a_2^2$	2	756	378/2	3024
$a_{11}d_5a_5a_3a_1$	2	672	392/2	2304
$a_{11}d_5a_4a_2^2a_1$	2	900	375/2	4320
$a_{11}a_5a_5d_4$	2	576	384/2	1728
$d_{10}^2a_3a_1^2$	2	384	1152/2	256
$d_{10}a_{10}a_5$	1	330	825/2	264
$d_{10}d_9a_5a_1$	1	312	1014/2	192
$d_{10}a_9d_6$	1	260	845/2	160
$a_{10}a_9a_3a_2a_1$	1	600	750/2	960
$d_{10}e_8e_7$	1	94	2209/2	8
$d_{10}d_8d_6a_1$	1	248	961/2	128
$d_{10}a_8e_7$	1	198	1089/2	72
$d_{10}a_8a_5a_2$	1	486	729/2	648
$d_{10}e_7^2a_1$	2	148	1369/2	32
$d_{10}e_7d_7a_1$	1	192	1152/2	64
$d_{10}e_7d_6a_2$	1	228	1083/2	96
$d_{10}e_7a_6a_2$	1	294	1029/2	168
$d_{10}e_7a_5a_3$	1	312	1014/2	192
$d_{10}e_7a_4a_3a_1$	1	400	1000/2	320

$d_{10}a_7d_6a_2$	1	384	768/2	384
$d_{10}a_7a_3^2a_1^2$	2	832	676/2	2048
$d_{10}d_6^2a_3$	2	320	800/2	256
$d_{10}d_6a_5a_4$	1	420	735/2	480
$d_{10}d_6a_5a_3a_1$	1	528	726/2	768
$d_{10}a_6a_5a_3a_1$	1	672	672/2	1344
$d_{10}a_5^3$	2	540	675/2	864
$a_{10}a_9d_6$	1	330	495/2	440
$a_{10}e_8e_7$	1	143	1859/2	22
$a_{10}e_8e_6a_1$	1	231	1617/2	66
$a_{10}e_8a_4a_2a_1$	1	495	1485/2	330
$a_{10}e_7a_7a_1$	1	352	704/2	352
$a_{10}d_7a_6a_2$	1	462	462/2	924
$d_9^2a_7$	2	240	900/2	128
$d_9a_9a_6a_1$	1	420	630/2	560
$d_9e_8^2$	2	76	2888/2	4
$d_9d_8^2$	2	176	968/2	64
$d_9a_8^2$	2	324	648/2	324
$d_9a_7^2a_1^2$	4	544	578/2	1024
$a_9a_9e_7$	2	270	729/2	200
$a_9^2d_7$	4	320	512/2	400
$a_9^2d_6a_1$	4	420	441/2	800
$a_9^2d_5a_1^2$	4	560	392/2	1600
$a_9^2d_4a_2a_1$	4	660	363/2	2400
$a_9^2a_4a_3$	4	600	360/2	2000
$a_9^2a_4a_3$	2	600	360/2	2000
$a_9e_8^2$	2	115	2645/2	10
$a_9e_8e_7a_1$	1	190	1805/2	40
$a_9e_8d_5a_2a_1$	1	420	1470/2	240
$a_9e_8a_4^2$	2	425	1445/2	250
$a_9d_8e_7a_1$	1	260	845/2	160
$a_9a_8^2$	2	405	405/2	810
$a_9e_7d_7a_2$	1	300	750/2	240
$a_9e_7a_6a_3$	1	420	630/2	560
$a_9e_7a_4^2a_1$	2	550	605/2	1000
$a_9a_7d_5a_3a_1$	1	640	320/2	2560
$a_9d_5a_4^2a_2a_1$	2	900	270/2	6000
$a_9a_4^2a_4^2$	4	875	245/2	6250
$e_8^3a_1$	6	61	3721/2	2
$e_8^2a_8a_1$	2	153	2601/2	18
$e_8^2e_7a_2$	2	93	2883/2	6
$e_8^2e_6a_3$	2	126	2646/2	12
$e_8^2a_6a_2a_1$	2	231	2541/2	42
$e_8^2d_5a_4$	2	160	2560/2	20

$e_8^2a_5a_4$	2	195	2535/2	30
$e_8a_8d_5a_4$	1	360	1440/2	180
$e_8a_8e_6a_2a_1$	1	351	1521/2	162
$e_8e_7^2a_3$	2	128	2048/2	16
$e_8e_7a_7a_2a_1$	1	288	1728/2	96
$e_8e_7e_6a_4$	1	165	1815/2	30
$e_8e_7a_6a_4$	1	245	1715/2	70
$e_8e_7d_5a_5$	1	204	1734/2	48
$e_8a_7e_6a_4$	1	300	1500/2	120
$e_8a_7d_5^2$	2	304	1444/2	128
$e_8e_6^2a_5$	2	207	1587/2	54
$e_8e_6a_6d_5$	1	252	1512/2	84
$d_8^3a_1$	6	232	841/2	128
$d_8^2e_7a_1a_1$	2	248	961/2	128
$d_8^2e_6a_3$	2	264	726/2	192
$d_8^2d_6a_2a_1$	2	360	675/2	384
$d_8^2d_5d_4$	2	288	648/2	256
$d_8^2d_5a_4$	2	320	640/2	320
$d_8e_7^2a_2a_1$	2	228	1083/2	96
$d_8e_7d_6a_3a_1$	1	320	800/2	256
$d_8e_7a_7a_2a_1$	1	384	768/2	384
$d_8e_7a_5a_4a_1$	1	420	735/2	480
$d_8e_7a_5a_3a_1a_1$	1	528	726/2	768
$d_8a_7e_6a_4$	1	360	540/2	480
$d_8a_7d_5^2$	2	352	484/2	512
$d_8e_6a_5^2a_1$	2	468	507/2	864
$d_8d_6^2d_4a_1$	2	368	529/2	512
$d_8d_6^2a_3a_1^2$	2	512	512/2	1024
$d_8d_6d_5a_5a_1$	1	432	486/2	768
$a_8^3a_1$	12	513	361/2	1458
$a_8a_7d_5^2$	2	432	324/2	1152
$a_8e_6^2a_2^2a_1$	2	567	441/2	1458
$a_8e_6d_5a_4a_2$	1	540	360/2	1620
$e_7^3d_4$	6	140	1225/2	32
$e_7^2d_7a_4$	2	200	1000/2	80
$e_7^2e_6d_5$	2	156	1014/2	48
$e_7^2d_6d_5$	2	176	968/2	64
$e_7d_7a_7a_3a_1$	1	416	676/2	512
$e_7d_7d_6a_5$	1	264	726/2	192
$e_7d_7a_6a_5$	1	336	672/2	336
$e_7a_7^2a_3a_1$	2	544	578/2	1024
$e_7a_7a_6a_4a_1$	1	560	560/2	1120
$e_7a_7a_5a_4a_1a_1$	1	720	540/2	1920
$e_7e_6^3$	6	153	867/2	54

$e_7d_6^3$	3	216	729/2	128
$e_7a_6^3$	3	441	567/2	686
$e_7a_5^3a_1^3$	6	936	507/2	3456
$d_7a_7^2a_3a_1$	4	608	361/2	2048
$d_7a_7a_6a_3a_2$	1	672	336/2	2688
$d_7a_7a_3^2a_3a_2$	2	960	300/2	6144
$d_7d_6^3$	6	256	512/2	256
$d_7a_6^3$	3	490	350/2	1372
$d_7a_3^6$	24	1408	242/2	16384
$a_7^3a_1^4$	24	1024	256/2	8192
$a_7^2e_6d_5$	2	384	384/2	768
$a_7^2d_6d_5$	4	416	338/2	1024
$a_7^2d_5^2a_1$	8	544	289/2	2048
$a_7^2d_5a_4a_1^2$	4	800	250/2	5120
$a_7^2d_5a_3a_2a_1$	4	864	243/2	6144
$a_7^2a_5a_3a_1^2a_1$	4	1152	216/2	12288
$a_7^2a_3^2a_3a_1^2$	8	1280	200/2	16384
$a_7e_6^2a_5a_1$	2	432	432/2	864
$a_7d_6^2a_3^2$	4	576	324/2	2048
$a_7a_6^3$	3	588	252/2	2744
$a_7a_5^3a_1^3$	6	1152	192/2	13824
$a_7a_3^6$	24	1536	144/2	32768
$e_6^4a_1$	48	225	625/2	162
$e_6^3a_6a_1$	6	315	525/2	378
$e_6^3a_5a_2$	12	351	507/2	486
$e_6^3a_3a_2^2$	12	486	486/2	972
$e_6^2a_5^2a_3$	8	504	392/2	1296
$e_6^2a_5a_4a_2^2$	4	675	375/2	2430
$e_6^2a_5a_2^4$	8	891	363/2	4374
$e_6d_5a_5^2a_2^2$	4	756	294/2	3888
$e_6a_5^3d_4$	12	612	289/2	2592
$d_6^4a_1$	24	336	441/2	512
$d_6^3d_5a_1^2$	6	448	392/2	1024
$d_6^3d_4a_2a_1$	6	528	363/2	1536
$d_6^3a_4a_3$	6	480	360/2	1280
$d_6^3d_4a_1^3$	6	608	361/2	2048
$d_6^2a_5d_4a_3a_1$	2	672	294/2	3072
$d_6^2d_4^2a_3a_1^2$	4	768	288/2	4096
$d_6a_5^3d_4$	6	648	243/2	3456
$d_6d_4^4a_1^3$	24	960	225/2	8192
$a_6^4a_1$	24	735	225/2	4802
$a_6a_5^3d_4$	6	756	189/2	6048
$d_5a_5^4$	16	720	200/2	5184
$d_5d_4^5$	120	640	200/2	4096

$d_5a_4^5$	20	1000	160/2	12500
$a_5^4d_4a_1$	48	936	169/2	10368
$a_5^4a_2a_1^3$	48	1512	147/2	31104
$a_5a_4^5$	20	1125	135/2	18750
$a_5a_2^{10}$	720	3645	75/2	354294
$d_4^6a_1$	2160	832	169/2	8192
$d_4^5a_2a_1^3$	360	1344	147/2	24576
$d_4^4a_3a_1^6$	144	2048	128/2	65536
$d_4^4a_1^9$	432	2816	121/2	131072
$d_4a_3^7$	336	1792	98/2	65536
$d_4a_1^{21}$	120960	14336	49/2	8388608
$a_4^6a_1$	240	1375	121/2	31250
$a_4a_3^7$	168	1920	90/2	81920
$a_3^8a_1$	2688	2304	81/2	131072
$a_3a_2^{11}$	7920	4374	54/2	708588
$a_3a_1^{22}$	887040	16384	32/2	16777216
$a_2^{12}a_1$	190080	5103	49/2	1062882
$a_2a_1^{23}$	10200960	18432	27/2	25165824
$a_1^{24}a_1$	244823040	20480	25/2	33554432

Figure 1. The enumeration of shallow holes whose diagram contains a component a_{15}. For types (a)-(d) the set of points disjoined from a_{15} has no 10-point subgraph of mutually disjoined spherical diagrams, so only case (e) leads to holes.

(This figure is not yet included as I can't be bothered to draw it in TEX.)

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