Quantum vertex algebras. *7 March 1999.*

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Contents.

1. Introduction.

The purpose of this paper is to make the theory of vertex algebras trivial. We do this by setting up some categorical machinery so that vertex algebras are just "singular commutative rings" in a certain category. This makes it easy to construct many examples of vertex algebras, in particular by using an analogue of the construction of a twisted group ring from a bicharacter of a group. We also define quantum vertex algebras as singular braided rings in the same category and construct some examples of them. The constructions work just as well for higher dimensional analogues of vertex algebras, which have the same relation to higher dimensional quantum field theories that vertex algebras have to one dimensional quantum field theories.

One way of thinking about vertex algebras is to regard them as commutative rings with some sort of singularities in their multiplication. In algebraic geometry there are two sorts of morphisms: regular maps that are defined everywhere, and rational maps that are not defined everywhere. It is useful to think of a commutative ring R as having a regular multiplication map from $R \times R$ to R, while vertex algebras only have some sort of rational or singular multiplication map from $R \times R$ to R which is not defined everywhere. One of the aims of this paper is to make sense of this, by defining a category whose bilinear maps can be thought of as some sort of maps with singularities.

The main idea for constructing examples of vertex algebras in this paper is a generalization of the following well-known method for constructing twisted group rings from bicharacters of groups. Suppose that L is a discrete group (or monoid) and R is a commutative ring. Recall that an R-valued bicharacter of L is a map $r : L \times L \mapsto R$ such that

$$r(1,a) = r(a,1) = 1$$
$$r(ab,c) = r(a,c)r(b,c)$$
$$r(a,bc) = r(a,b)r(a,c).$$

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If r is any R-valued bicharacter of L then we define a new associative multiplication $\circ$ on the group ring $R[L]$ by putting $a \circ b = abr(a,b)$. We call $R[L]$ with this new multiplication the twisted group ring of L. The point is that this rather trivial construction can be generalized from group rings to bialgebras in additive symmetric tensor categories. We will construct vertex algebras by applying this construction to "singular bicharacters" of bialgebras in a suitable additive symmetric tensor category.

Section 2 describes how to generalize the twisted group ring construction to bialgebras, and constructs several examples of singular bicharacters that we will use later. Much of section 2 uses an extra structure on the spaces underlying many common vertex algebras that is often overlooked. It is well known that these spaces often have natural ring structures, but what is less well known is that this can usually be extended to a cocommutative bialgebra structure. The comultiplication turns out to be very useful for keeping track of the behavior of vertex operators; this is not so important for vertex algebras, but is very useful for quantum vertex algebras. It also allows us to interpret these spaces as the coordinate rings of gauge groups.

Section 3 contains most of the hard work of this paper. We have to construct a category in which the commutative rings are more or less the same as vertex algebras. The motivation for the construction of this category comes from classical and quantum field theory (though it is not necessary to know any field theory to follow the construction). The idea is to construct categories which capture all the formal operations one can do with fields. For examples, fields can be added, multiplied, differentiated, multiplied by functions on spacetime, and we can change variables and restrict fields. All of these operations are trivial but there are so many of them that it takes some effort to write down all the compatibility conditions between them. The categories constructed in section 3 are really just a way of writing down all these compatibility conditions explicitly. The main point of doing this is the definition at the end of section 3, where we define (A, H, S) vertex algebras to be the commutative rings in these categories. Here A is a suitable additive category (for example the category of modules over a commutative ring), H is a suitable bialgebra in A (and can be thought of as a sort of group ring of the group of automorphisms of spacetime), and S is something that controls the sort of singularities we allow.

One of the main differences between the (A, H, S) vertex algebras defined in section 3 and previous definitions is as follows. Vertex algebras as usually defined consist of a space $V(1)$ with some extra operations, whose elements can be thought of a fields depending one one spacetime variable. On the other hand (A, H, S) vertex algebras include spaces $V(1, 2, \dots, n)$ which can be thought of as fields depending on n spacetime variables for all n. The lack of these fields in several variables seems to be one reason why classical vertex algebras are so hard to handle: it is necessary to reconstruct these fields, and there seems to be no canonical way to do this. However if these fields are given in advance then a lot of these technical problems just disappear.

Section 4 puts everything together to construct many examples of vertex algebras. The main theorem of this paper is theorem 4.2, which shows how to construct a vertex algebra from a singular bicharacter of a commutative and cocommutative bialgebra. As examples, we show that the usual vertex algebra of an even lattice can be constructed like this from the Hopf algebra of a multiplicative algebraic group, and the vertex algebra of a (generalized) free quantum field theory can be constructed in the same way from the Hopf algebra of an additive algebraic group. (This shows that the vertex algebra of a lattice is in some sense very close to a free quantum field theory: they have the same relation as multiplicative and additive algebraic groups.)

The vertex algebras we construct in this paper do not at first sight look much like classical vertex algebras: they seem to be missing all the structure such as vertex operators, formal power series, contour integration, operator product expansions, and so on. We show that all this extra structure can be reconstructed from the more elementary operations we provide for vertex algebras. For example, the usual locality property of vertex operators follows from the fact that we define vertex algebras as *commutative* rings in some category.

All the machinery in sections 2 and 3 has been set up so that it generalizes trivially to quantum vertex algebras and higher dimensional analogues of vertex algebras. For example, we define quantum vertex algebras to be braided (rather than commutative) rings in a certain category, and we can instantly construct many examples of them from non-symmetric bicharacters of bialgebras. By changing a certain bialgebra H in the construction, we immediately get the "vertex G algebras" of [B98], which have the same relation to higher dimensional quantum field theories that vertex algebras have to one dimensional quantum field theories.

Finally in section 5 we list some open problems and topics for further research.

Some related papers are [F-R] and [E-K], which give alternative definitions of quantum vertex algebras. These definitions are not equivalent to the ones in this paper, but define concepts that are closely related (at least in the case of 1 dimensional spacetime) in the sense that the interesting examples for all definitions should correspond. Soibelman has introduced other foundations for quantum vertex algebras, which seem to be related to this paper. There is also a preprint [B-D] which defines vertex algebras as commutative rings or Lie algebras in suitable multilinear categories. (Soibelman pointed out to me that multi categories seem to have been first introduced by Lambek in [L].) It might be an interesting question to study the relationship of this paper to [B-D]. One major difference is that the paper [B-D] extends the genus 0 Riemann surfaces that appear in vertex algebra theory to higher genus Riemann surfaces, while in this paper we extend them instead to higher dimensional groups.

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Notation.

A	An additive symmetric tensor category.
C	A symmetric tensor category, with tensor product $\cup$; usually Fin or $Fin^{\not\equiv}$.
Δ	The coproduct of a bialgebra, or a propagator.
$D^{(i)}$	An element of the formal group ring of the one dimensional additive formal group.
η	The counit of a bialgebra.
Fin	The category of finite sets.
$Fin^{\not\equiv}$	The category of finite sets with an inequivalence relation.
Fun	A functor category.
H	A cocommutative bialgebra in A.
I, J	Finite sets.

L An integral lattice.
M A commutative ring in A, or a commutative cocommutative bialgebra.
r A bicharacter.
R A commutative ring or an R-matrix.
R-mod The category of R-modules.
S A commutative ring in some category, especially $Fun(C, A, T^*(H))$.
S^* A symmetric algebra.
T A cocommutative Hopf algebra in $Fun(C^{op}, A)$.
T_*, T^* $T_*(M)(I) = \otimes_{i\in I} M$, $T^*(H)(I) = \otimes_{i\in I} H$. See definition 3.3.
U, V Objects of $Fun(C, A, T^*(H), S)$.

2. Twisted group rings.

We let R be any commutative ring. Recall that a bialgebra is an algebra with a compatible coalgebra structure, with the coproduct and counit denoted by Δ and η, and a Hopf algebra is a bialgebra with an antipode. If a is an element of a coalgebra then we put $\Delta(a) = \sum a' \otimes a''$.

Recall from the introduction that any bicharacter r of a group L can be used to define a twisted group ring. We now extend this idea from group rings $R[L]$ to cocommutative bialgebras.

Definition 2.1. *Suppose that M and N are bialgebras over R and S is a commutative R-algebra. Then we define a bimultiplicative map from $M \otimes N$ to S to be a linear map $r : M \otimes N \mapsto S$ such that*

$$r(1 \otimes a) = \eta(a), \qquad r(a \times 1) = \eta(a)$$
$$r(ab \otimes c) = \sum r(a \otimes c')r(b \otimes c'')$$
$$r(a \otimes bc) = \sum r(a' \otimes b)r(a'' \otimes c)$$

where $\Delta(a) = \sum a' \otimes a''$, $\Delta(c) = \sum c' \otimes c''$, and η is the counit of M or N. We define an S-valued bicharacter of M to be a bimultiplicative map from $M \otimes M$ to S. We say the bicharacter r is symmetric if $r(a \otimes b) = r(b \otimes a)$ for all $a, b \in M$.

The S-valued bicharacters form a monoid, which is commutative if M is cocommutative. The identity bicharacter is defined by $r(a \otimes b) = \eta(a) \otimes \eta(b)$, and the product rs of two bicharacters r and s is given by

$$rs(a \otimes b) = \sum r(a' \otimes b')s(a'' \otimes b'').$$

If M is a Hopf algebra with antipode s then any S-valued bicharacter r has an inverse r^{-1} defined by

$$r^{-1}(a \otimes b) = r(s(a) \otimes b)$$

so the S-valued bicharacters form a group.

Example 2.2. Suppose that $M = R[L]$ is the group ring of a group L, considered as a bialgebra in the usual way (with $\Delta(a) = a \otimes a$ for $a \in L$). Any R^*-valued bicharacter of L

can be extended to a linear function from $M\otimes M$ to R, and this is an R-valued bicharacter of M. This identifies the bicharacters of the group L with the bicharacters of its group ring M.

In order to define quantum vertex algebras we need a generalization of commutative rings, called braided rings. The idea is that we should be able to write $ab = \sum b_i a_i$ for suitable a_i and b_i related in some way to a and b. For example, for a commutative ring we would have $a_i = a$, $b_i = b$. The definition of the elements a_i and b_i is given in terms of an R-matrix with $R(a\otimes b) = \sum a_i \otimes b_i$, where an R-matrix is defined as follows.

Definition 2.3. *An R-matrix for a ring M with multiplication map $m : M\otimes M \mapsto M$ in an additive symmetric tensor category consists of a map $R : M\otimes M \mapsto M\otimes M$ satisfying the following conditions.*

1. *(R is compatible with 1.) $R(1\otimes a) = 1\otimes a$, $R(a\otimes 1) = a\otimes 1$.*
2. *(R is compatible with multiplication.) $m_{23}R_{12}R_{13} = R_{12}m_{23} : M\otimes M\otimes M \mapsto M\otimes M$ and $m_{12}R_{23}R_{13} = R_{13}m_{12} : M\otimes M\otimes M \mapsto M\otimes M$.*
3. *(Yang-Baxter equation.) $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$.*

Here R_{13} is R restricted to the first and third factors of of $M\otimes M\otimes M$, and so on.

Definition 2.4. *A braided ring M in an additive symmetric tensor category is a ring M with an R-matrix R such that*

$$mR = m\tau : M\otimes M \mapsto M$$

(where $\tau : a\otimes b \mapsto b\otimes a$ is the twist map and $m : a\otimes b \mapsto ab$ is the product).

Example 2.5. Suppose that M is $Z/2Z$ graded as $M = M_0 \oplus M_1$, and define R by $R(a\otimes b) = (-1)^{\deg(a)\deg(b)} a\otimes b$. Then M is a braided ring with R-matrix R if and only if M is a super commutative ring.

Lemma/Definition 2.6. *Suppose that r is an R-valued bicharacter of a commutative cocommutative bialgebra M. Define a new multiplication $\circ$ on M by*

$$a\circ b = \sum a'b'r(a''\otimes b'')$$

(where $\Delta(a) = \sum a'\otimes a''$, $\Delta(b) = \sum b'\otimes b''$). Then this makes M into a ring, called the twisting of M by r. If r is symmetric then the twisting of M by r is commutative. If r is invertible (which is true whenever M is a Hopf algebra) then the twisting of M by r is a braided ring.

Proof. The element 1 is an identity for twisting of M by r because R is compatible with 1. The twisting is an associative ring because R satisfies the Yang-Baxter equation and is compatible with multiplication. It is easy to check that the twisting is commutative if r is symmetric and M is commutative. Finally we have to check that M has an R matrix if r is invertible. Define a bicharacter r' by

$$r'(a\otimes b) = \sum r(a'\otimes b')r^{-1}(b''\otimes a'').$$

We define the R matrix by

$$R(a\otimes b)=a'\otimes b'r'(b''\otimes a'')$$

where $\Delta(a)=\sum a'\otimes a''$, and $\Delta(b)=\sum b'\otimes b''$. It is easy to check that this satisfies the conditions for an R matrix for the twisting of M by r. This proves lemma 2.6.

Example 2.7. Suppose that L is a free abelian group or free abelian monoid with a basis $\alpha_1,\ldots,\alpha_n$, and suppose that we are given elements $r(\alpha_i,\alpha_j)\in S^*$ for some commutative R-algebra S. We write e^α for the element of the group ring of L corresponding to the element $\alpha\in L$. Then we can extend r to a unique S-valued bicharacter of the ring $M=R[L]$ by putting

$$r\big(\prod_i e^{m_i\alpha_i}\otimes\prod_j e^{n_j\alpha_j}\big)=\prod_{1\le i,j\le n} r(\alpha_i,\alpha_j)^{(m_i,n_j)}.$$

Example 2.8. Suppose that S is a commutative R algebra and that Φ is a free R-module, considered as an abelian Lie algebra. We let M be the universal enveloping algebra of Φ (in other words the symmetric algebra of Φ), so M is a commutative cocommutative Hopf algebra. Suppose that r is any linear map from from $\Phi\otimes\Phi$ to S. Then we can extend r an S-valued bicharacter of M by putting

$$r(\phi_1\cdots\phi_m\otimes\phi_1'\cdots\phi_n')=\sum_{\sigma\in S_m}\prod_{i=1}^m r(\phi_i\otimes\phi'_{\sigma(i)})$$

if $m=n$ and 0 otherwise for $\phi_i,\phi_i'\in\Phi$ (where S_m is the symmetric group of permutations of $1,2,\ldots,m$).

Example 2.9. Suppose that r is any bicharacter of a cocommutative bialgebra M. We can define an R-matrix for M by putting

$$R(a\otimes b)=\sum a'\otimes b'r(a''\otimes b'').$$

If R is any R-matrix for a ring M we can define a new associative multiplication on M by putting

$$a\circ b=mR(a\otimes b):M\otimes M\mapsto M$$

where $m:M\otimes M\mapsto M$ is the old multiplication. The composition of these two operations is just the twisting of M by r.

In the rest of this section we describe the construction of universal rings acted on by bialgebras, which we will need for the construction of vertex algebras. These universal rings can be thought of as something like the coordinate rings of function spaces or gauge groups.

Lemma/Definition 2.10. *Suppose that M is a commutative algebra over some ring and H is a cocommutative coalgebra. Then there is a universal commutative algebra $H(M)$ such that there is a map $h\otimes m\mapsto h(m)$ from $H\otimes M$ to $H(M)$ with*

$$h(mn)=\sum h'(m)h''(n),\qquad h(1)=\eta(h).$$

If H is a bialgebra then H acts on the commutative ring $H(M)$. If M is a commutative and cocommutative bialgebra (or Hopf algebra) then so is $H(M)$.

Proof. The existence of $H(M)$ is trivial; for example, we can construct it by writing down generators and relations. Equivalently we can construct it as the quotient of the symmetric algebra $S(H \otimes M)$ by the ideal generated by the images of H and $H \otimes M \otimes M$ under the maps describing the relations. If H is a cocommutative bialgebra then it acts on $H(M)$ by $h_1(h_2(m)) = (h_1 h_2)m$. If M has a coproduct $M \mapsto M \otimes M$ then this induces a map $M \mapsto H(M) \otimes H(M)$. As $H(M) \otimes H(M)$ is a commutative algebra acted on by H, this map extends to a map from $H(M)$ to $H(M) \otimes H(M)$ by the universal property of $H(M)$. It is easy to check that this coproduct makes $H(M)$ into a bialgebra. This proves lemma 2.10.

The ring $H(M)$ has the following geometric interpretation. Pretend that H^* is the coordinate ring of a variety G. Then $Spec(H(M))$ can be thought of as a sort of function space of all maps from G to $Spec(M)$. If H is a bialgebra then we can pretend that it is the group ring of a group G, and the action of H on $H(M)$ then corresponds to the natural action of G on this function space induced by the action of G on itself by left multiplication. If in addition M is a cocommutative Hopf algebra, then $Spec(M)$ is an affine algebraic group. The space $Spec(H(M))$ is also an affine algebraic group, and can be thought of as the gauge group of all maps from G to $Spec(M)$.

Example 2.11. Suppose that H is the supercommutative bialgebra with a basis 1, d, with $d^2 = 0$, $\Delta(d) = d \otimes 1 + 1 \otimes d$, such that d has odd degree. If M is any supercommutative ring then $H(M)$ is the ring of differential forms over M (where of course we replace "commutative" by "supercommutative" in lemma 2.10).

Example 2.12. Suppose that M is a polynomial algebra $R[\phi_1, \ldots, \phi_n]$. Let H be the commutative cocommutative Hopf algebra over R with basis $D^{(i)}$ for $i \geq 0$, where $D^{(i)}D^{(j)} = \binom{i+j}{i} D^{(i+j)}$ and $\Delta(D^{(i)}) = \sum_j D^{(j)} \otimes D^{(i-j)}$. (We can think of H as the formal group ring of the one dimensional additive formal group. If R contains the rational numbers then $D^{(i)} = D^i/i!$ (where $D = D^{(1)}$) and H is just the universal enveloping algebra $R[D]$ of a one dimensional Lie algebra.) Then $H(M)$ is the ring of polynomials in the variables $D^{(i)}(\phi_j)$ for $i \geq 0$, $1 \leq j \leq n$. More generally, if we take M to be a symmetric algebra $S^*(\Phi)$ for an R-module Φ, then $H(S^*(M)) = S^*(H \otimes \Phi)$.

Example 2.13. Suppose that L is a lattice and $R[L]$ its group ring and suppose that H is the formal group ring of the one dimensional additive group, as in example 2.12. Then $H(R[L])$ is the module underlying the vertex algebra of the lattice L. If instead we take H to be the polynomial ring $R[D]$ (with $\Delta(D) = D \otimes 1 + 1 \otimes D$) then $H(R[L])$ is isomorphic to the tensor product $R(L) \otimes S^*(L(1) \oplus L(2) \oplus \cdots)$ of the group ring $R[L]$ and the symmetric algebra of the sum of an infinite number of copies $L(n)$ of $L \otimes R$. This tensor product is also commonly used to construct the vertex algebra of a lattice. If R contains the rational numbers then it is equivalent to the first construction because $R[D]$ is then the same as the H defined in 2.12. However in non-zero characteristics it does not work quite so well; for example, we cannot define formal contour integrals as in example 4.7, because this requires divided powers of D.

We now show that bicharacters of M are more or less the same as $H \otimes H$-invariant bicharacters of $H(M)$.

Lemma 2.14. *Suppose that M and N are bialgebras and r is a bimultiplicative map from $M\otimes N$ to S, where S is a commutative algebra acted on by the bialgebra H. Then r extends uniquely to a H invariant bimultiplicative map from $H(M)\otimes N$ to S.*

Proof. By adjointness we get an algebra homomorphism from M to the algebra $Hom(N,S)$ of linear maps from the coalgebra N to the algebra S. By the universality property of $H(M)$ this extends uniquely to a H invariant homomorphism from $H(M)$ to $Hom(N,S)$, which by adjointness gives a map from $H(M)\otimes N$ to S such that $r(m_1m_2\otimes n)=\sum r(m_1\otimes n')r(m_2\otimes n'')$ (where $\Delta(n)=\sum n'\otimes n''$). To finish the proof we have to check that $r(m\otimes n_1n_2)=\sum r(m'\otimes n_1)r(m''\otimes n_2)$. The set of m with this property contains M because by assumption r is bimultiplicative on $M\otimes N$. It is also easy to check that it is closed under multiplication and under the action of H. Therefore it contains the smallest H-invariant subalgebra of $H(M)$ containing M, which is the whole of $H(M)$. This proves lemma 2.14.

Lemma 2.15. *Suppose that H is a cocommutative bialgebra and S is a commutative algebra acted on by $H\otimes H$. Suppose that M is a commutative and cocommutative bialgebra with an S-valued bicharacter r. Then r extends uniquely to a $H\otimes H$-invariant S-valued bicharacter $r: H(M)\otimes H(M)\mapsto S$ of $H(M)$.*

Proof. We apply lemma 2.14 to get a bimultiplicative H-invariant map from $H(M)\otimes M$ to S. Then we apply lemma 2.14 again to get a bimultiplicative $H\otimes H$-invariant map from $H(M)\otimes H(M)$ to S. This proves lemma 2.15.

3. Construction of some categories.

In this section we define a category $Fun(Fin^{\not\equiv},A,H,S)$ in which we can carry out the "twisted group ring" construction in order to produce vertex algebras. The definition of this category is strongly motivated by classical and quantum field theory, and commutative rings in this category are formally quite similar to quantum field theories.

In the rest of this paper we fix an additive tensor category A that is cocomplete and such that colimits commute with tensor products. (In fact we do not need all colimits in A; it would be sufficient for most applications to assume that A has countable colimits.) For example, A could be the category R-mod of modules over a commutative ring. Note that most of the constructions and definitions of section 2 work for any category A with the properties above.

Definition 3.1. *We define Fin to be the category of all finite sets, with morphisms given by functions. We define $Fin^{\not\equiv}$ to be the category whose objects are finite sets with an equivalence relation $\equiv$, and whose morphisms are the functions f preserving inequivalence; in other words, if $f(a)\equiv f(b)$ then $a\equiv b$. We define $\cup$ on Fin and $Fin^{\not\equiv}$ to be the disjoint union (where in $Fin^{\not\equiv}$, elements of I and J in the disjoint union $I\cup J$ are inequivalent). This makes Fin and $Fin^{\not\equiv}$ into (non-additive) symmetric tensor categories.*

We will write objects of $Fin^{\not\equiv}$ by using colons to separate the equivalence classes.

We could replace Fin and $Fin^{\not\equiv}$ by smaller equivalent categories; for example we could restrict the objects of Fin to be the finite sets of the form $\{1,2,\ldots,n\}$.

Note that $\cup$ is a coproduct in Fin but is not a coproduct in $Fin^{\not\equiv}$; in fact, $Fin^{\not\equiv}$ does not have coproducts. For example the coproduct of a one point set and a two point set with two equivalence classes does not exist.

Definition 3.2. *If C is a category we define the category $Fun(C, A)$ to be the category of functors V from C to A. The category $Fun(C, A)$ is additive and has a symmetric tensor product given by the pointwise tensor product $(U \otimes V)(I) = U(I) \otimes V(I)$.*

In applications the category C will be one of Fin, $Fin^{\not\equiv}$, or their opposite categories Fin^{op}, $Fin^{\not\equiv op}$.

Definition 3.3. *Suppose that M is any commutative ring in A. We define $T_*(M)$ in $Fun(Fin, A)$ by $T_*(M)(I) = \otimes_{i\in I} M$, where the action of $T_*(M)$ on morphisms of Fin is induced in the obvious way by the product and unit of M. (For example, if f is the morphism from $\{1, 2\}$ to itself with $f(1) = f(2) = 2$, then $T_*(M)(f)$ takes $x_1 \otimes x_2$ to $1 \otimes x_1x_2$.) If H is a cocommutative coalgebra in A then we define $T^*(H)$ in $Fun(Fin^{op}, A)$ in a similar way, using the coproduct and counit of H to define the action of $T^*(H)$ on morphisms.*

Example 3.4. If M is a commutative ring in A then $T_*(M)$ is a commutative ring in $Fun(Fin, A)$. If in addition M is a commutative cocommutative bialgebra then so is $T_*(M)$.

Example 3.5. If V is a commutative ring in $Fun(Fin^{\not\equiv}, A)$ then we can think of $V(I)$ as the space of (nonsingular) quantum fields $\phi(x_1, x_2, \ldots)$ depending on $|I|$ spacetime variables.

The space of fields in one spacetime variable is acted on by the group of automorphisms G of spacetime, and similarly the space of fields of $|I|$ spacetime variables is acted on by $|I|$ commuting copies of G. We now add a similar structure to the objects of $Fun(Fin, A)$. It is convenient to use a cocommutative bialgebra H instead of a group G; we can think of this bialgebra H as analogous to the group ring of the automorphisms of spacetime (or maybe to the universal enveloping algebra of the Lie algebra of infinitesimal automorphisms of spacetime).

Definition 3.6. *Suppose that T is a cocommutative bialgebra in $Fun(C^{op}, A)$. (In applications, T will be of the form $T^*(H)$ for a cocommutative bialgebra of A.) We define a T module in $Fun(C, A)$ to be an object V of $Fun(C, A)$ such that $V(I)$ is a module over $T(I)$ for all I and such that $f_*(f^*(g)(v)) = g(f_*(v))$ for $v \in V(I)$, $g \in T(J)$, $f : I \mapsto J$. The action of T on the tensor product of two T modules is defined in the usual way using the coalgebra structure of the $T(I)$'s. We define $Fun(C, A, T)$ to be the additive symmetric tensor category of T modules in $Fun(C, A)$.*

Example 3.7. Suppose that V is any commutative ring in A acted on by the cocommutative bialgebra H. Then $T_*(V)$ is a commutative ring in $Fun(Fin, A, T^*(H))$.

Recall that we can define the category of modules over any commutative ring in any additive symmetric tensor category, and it is again an additive symmetric tensor category.

Definition 3.8. *Suppose that T is a cocommutative bialgebra in $Fun(C^{op}, A)$ and suppose that S is a commutative ring in $Fun(C, A, T)$. We define $Fun(C, A, T, S)$ to be the additive symmetric tensor category of modules over S.*

Example 3.9. Suppose that we define S by letting $S(I)$ be the smooth functions depending on $|I|$ variables in spacetime. Then we would expect a field theory to be a module over S because we should be able to multiply a field by a smooth function to get a new field.

Commutative rings in $Fun(Fin, A, T^*(H), S)$ as defined above behave rather like classical field theories, or at least they have most of their formal properties. However quantum field theories do not fit into this framework. The problem is that in quantum field theory it is no longer true that the product of two nonsingular fields is a nonsingular field. For example, a typical formula in free quantum field theory is

$$\phi(x_1)\phi(x_2) =: \phi(x_1)\phi(x_2) : +\Delta(x_1 - x_2)$$

where the propagator $\Delta(x)$ usually has a singularity at $x = 0$. In particular if we take $x_1 = x_2$ we find that the product of two fields depending on x_1 is not defined. Instead, we can take the product of two fields depending on different variables x_1 and x_2, and it lies in the space $V(1 : 2)$ of fields that are defined whenever x_1 and x_2 are "apart" in some sense.

The category $Fun(C, A, T^*(H), S)$ has a natural tensor product $\otimes$ which can be used to define multilinear maps. We will now define a new tensor product in $Fun(C, A, T^*(H), S)$ by defining a new concept of multilinear maps, called singular multilinear maps. We assume that C is a symmetric tensor category (not necessarily additive) with the tensor product denoted by $\cup$. (As the notation suggests, this will often be some sort of disjoint union.)

Definition 3.10. *We let T be a cocommutative bialgebra in $Fun(C^{op}, A)$, and we let S be a commutative ring in $Fun(C, A, T)$. Suppose that $U_1, U_2, \ldots$ and V are objects of $Fun(C, A, T, S)$. We define a singular multilinear map from $U_1, U_2, \ldots$ to V to be a set of maps from $U_1(I_1) \otimes_A U_2(I_2) \cdots$ to $V(I_1 \cup I_2 \cdots)$ for all $I_1, I_2, \ldots \in C$, satisfying the following conditions.*

1 *The maps commute with the action of T.*
2 *The maps commute with the actions of $S(I_1)$, $S(I_2)$, ...*
3 *If we are given any morphisms from I_1 to I_1', I_2 to I_2', ..., then the following diagram commutes:*

$$\begin{array}{ccccc} U_1(I_1) & \otimes & U_2(I_2)\cdots & \longrightarrow & V(I_1 \cup I_2 \cdots) \\ \downarrow & & & & \downarrow \\ U_1(I_1') & \otimes & U_2(I_2')\cdots & \longrightarrow & V(I_1' \cup I_2' \cdots) \end{array}$$

As A is co-complete and co-limits commute with taking tensor products the singular multilinear maps are representable, so we define the "singular tensor products" $U_1 \odot U_2 \cdots$ to be the objects representing the singular multilinear maps. It is possible to write down an explicit formula for these singular tensor products as follows.

$$(U_1 \odot U_2 \cdots)(I) = \varinjlim_{I_1 \cup I_2 \cdots \mapsto I} (U_1(I_1) \otimes_A U_2(I_2) \cdots) \otimes_{S(I_1)\otimes S(I_2)\cdots} S(I)$$

where the limit is a direct limit taken over the following category. The objects $I_1 \cup I_2 \cdots \mapsto I$ of the category consist of objects $I_1, I_2, \ldots$ of C together with a morphism from $I_1 \cup I_2 \cdots$

to I. A morphism from $I_1 \cup I_2 \cdots \mapsto I$ to $I'_1 \cup I'_2 \cdots \mapsto I$ consists of morphisms from I_1 to I'_1, I_2 to I'_2, ..., making the following diagram commute:

$$\begin{array}{cccc} I_1 \cup I_2 & \cdots & \longrightarrow & I \\ \downarrow & & & \| \\ I'_1 \cup I'_2 & \cdots & \longrightarrow & I. \end{array}$$

J. M. E. Hyland told me that the product $\odot$ is similar to the "Day product" in category theory. The construction of $\odot$ can be extended to the case when C is a "symmetric multi-category" rather than a symmetric tensor category. Soibelman remarked that the conditions for V to be an algebra for $\odot$ are similar to the conditions for the functor V from C to A to be a functor of tensor categories.

Example 3.11. Suppose that $\cup$ is a coproduct in C; for example, we could take C to be Fin and $\cup$ to be disjoint union. Then singular tensor products are the same as pointwise tensor products. In later examples we will take C to be $Fin^{\not\equiv}$ and $\cup$ to be disjoint union, which is not a coproduct in $Fin^{\not\equiv}$.

The two tensor products $\odot$ and $\otimes$ are related in several ways, as follows. There is a canonical morphism from $U \odot V$ to $U \otimes V$, so that any ring is automatically a singular ring. Also there is a canonical "interchange" morphism

$$(U \otimes V) \odot (W \otimes X) \to (U \odot W) \otimes (V \odot X).$$

(Unlike the case of the interchange map for natural transformations, this interchange map is not usually an isomorphism.) The interchange map can be used to show that if U and V are singular rings then so is $U \otimes V$.

We define singular rings, singular Lie algebras, and so on, in $Fun(C, A, T, S)$ to be rings, Lie algebras, and so on using the singular tensor product. We define singular bialgebras a little bit differently: the product uses the singular tensor product, but the coproduct uses the pointwise tensor product $\otimes$. Note that for this to make sense we need to know that the pointwise tensor product of two singular algebras is a singular algebra; see the paragraph above. In general, we should use the pointwise tensor product $\otimes$ for "coalgebra" structures, and the singular tensor product $\odot$ for "algebra" structures.

If S is a commutative ring in $Fun(Fin^{\not\equiv}, A, T^*(H))$ then by restriction it is also a commutative ring in $Fun(Fin, A, T^*(H))$ (using the functor which gives any finite set the equivalence relation where all elements are equivalent.) We can embed the category $Fun(Fin, A, T^*(H), S)$ into $Fun(Fin^{\not\equiv}, A, T^*(H), S)$ by defining $V(I_1 : I_2 \cdots) = V(I_1 \cup I_2 \cdots) \otimes_{S(I_1) \otimes S(I_2) \cdots} S(I_1 : I_2 \cdots)$ for $I_1, I_2, \ldots \in Fin$. In particular singular multilinear maps are defined in $Fun(Fin, A, T^*(H), S)$. (Note that singular tensor products representing singular multilinear maps do not usually exist in $Fun(Fin, A, T^*(H), S)$, though they do exist in the larger category $Fun(Fin^{\not\equiv}, A, T^*(H), S)$.)

The main point of all this category theory is the following definition:

Definition 3.12. *Suppose that A is an additive symmetric tensor category, H is a cocommutative bialgebra in A, and S is a commutative ring in $Fun(Fin^{\not\equiv}, A, T^*(H))$. We define an (A, H, S) vertex algebra to be a singular commutative ring in $Fun(Fin, A, T^*(H), S)$.*

We define a quantum (A, H, S) vertex algebra to be a singular braided ring in $Fun(Fin, A, T^*(H), S)$.

Soibelman remarked that all the examples of quantum (A, H, S) vertex algebras in this paper have the extra property that the R matrix satisfies $R_{12}R_{21} = 1$, so perhaps this condition should be added to the definition of a quantum (A, H, S) vertex algebra.

Note that the vertex algebra is in $Fun(Fin, A, T^*(H), S)$ rather than $Fun(Fin^{\not\equiv}, A, T^*(H), S)$, although we can of course embed the former category in the latter if we wish. The reason for using $Fun(Fin, A, T^*(H), S)$ rather than $Fun(Fin^{\not\equiv}, A, T^*(H), S)$ is that we wish to have control over the connection between (say) $V(1, 2)$ and $V(1 : 2)$.

4. Examples of vertex algebras.

In this section we construct some examples of (A, H, S) vertex algebras by applying the twisted group ring construction of section 2 to the categories constructed in section 3. We also show how these are related to classical vertex algebras.

Lemma 4.1. *Suppose that r is an $H \otimes H$-invariant $S(1 : 2)$-valued bicharacter of a commutative cocommutative bialgebra $H(M)$ in A. Then H can be extended to a singular bicharacter of $T_*(H(M))$, which we also denote by r.*

Proof. We define r by

$$r(\bigotimes_{i\in I} a_i \otimes \bigotimes_{j\in J} b_j) = \sum \prod_{i\in I} \prod_{j\in J} r(a_i^{(j)} \otimes b_j^{(i)})$$

where $\Delta^{|J|-1}(a_i) = \sum \bigotimes_{j\in J} a_i^{(j)}$, $\Delta^{|I|-1}(b_j) = \sum \bigotimes_{i\in I} b_j^{(i)}$, and $r(a_i^{(j)} \otimes b_j^{(i)})$ is considered as an element of $S(I \cup J)$ using the obvious map from $S(i : j)$ to $S(I \cup J)$. Some routine checking then proves lemma 4.1.

The following theorem is the main theorem of this paper. It shows how to construct many examples of (A, H, S) vertex algebras, by giving a sort of generalization of the construction of the vertex algebra of a lattice.

Theorem 4.2. *Suppose that H is a cocommutative bialgebra in A and S is a commutative ring in $Fun(Fin^{\not\equiv}, A, T^*(H))$. Assume that we are given an $S(1 : 2)$-valued bicharacter r of a commutative and cocommutative bialgebra M in A. The bicharacter r of M extends to a bicharacter of $T_*(H(M))$ as in lemmas 2.15 and 4.1, which we also denote by r. Then the twisting of $T_*(H(M))$ by r is a quantum (A, H, S) vertex algebra if r is invertible, and is an (A, H, S) vertex algebra if r is symmetric.*

Proof. By lemma 2.10 and example 3.4, $T_*(H(M))$ is a commutative cocommutative bialgebra in $Fun(Fin, A, T^*(H), S)$. By lemmas 2.15 and 4.1 the bicharacter r extends to a singular bicharacter of $T_*(H(M))$ with values in S. By lemma 2.6 (extended to additive tensor categories) the twisting of $T_*(H(M))$ by r is a braided ring if r is invertible, and is a commutative ring if r is symmetric. Theorem 4.2 now follows from the definition 3.12 of (quantum) (A, H, S) vertex algebras.

The following theorem describes the relation between the (A, H, S) vertex algebras of this paper, and ordinary vertex algebras.

Theorem 4.3. *Suppose we take H to be the formal group ring of the one dimensional additive formal group, as in example 2.12. Define S by $S(I) =$ the R-algebra generated by $(x_i - x_j)^{\pm 1}$ for i and j not equivalent (so $S = R$ if all elements of I are equivalent). If V is a $(R\text{-mod}, H, S)$ vertex algebra, then $V(1)$ is an ordinary vertex algebra over the ring R.*

Proof. For every element u_1 of $V(1)$ we have to construct a vertex operator $u_1(x_1)$ taking $V(1)$ to $V(1)[[x_1]][x_1^{-1}]$. We do this as follows. If $u_2 \in V(2)$ then $u_1u_2 \in V(1:2) = V(1,2) \otimes S(1:2) = V(1,2)[(x_1 - x_2)^{\pm 1}]$. There is a map from $V(1,2)$ to $V(1)[[x_1, x_2]]$ taking w to the "Taylor series expansion" $\sum_{i,j} f_{12\to 1}(D_1^{(i)} D_2^{(j)} w) x_1^i x_2^j$. (Here $f_{12\to 1}$ is the map from $V(1,2)$ to $V(1)$ induced by the morphism of finite sets taking both 1 and 2 to 1, and D_1 and D_2 indicate the two different actions of H on $V(1,2)$.) This induces a map from $V(1,2)[(x_1 - x_2)^{\pm 1}]$ to $V(1)[[x_1, x_2]][(x_1 - x_2)^{-1}]$, and we denote the image of u_1u_2 under this map by $u_1(x_1)u_2(x_2)$. Then we define the vertex operator $u_1(x_1)$ by $u_1(x_1)u_2 = u_1(x_1)u_2(0) \in V(1)[[x_1]][x_1^{-1}]$.

This defines the vertex operators of elements of $V(1)$; now we have to check that they formally commute. We can define expressions like $u_1(x_1)u_2(x_2)u_3(x_3)\cdots \in V(1)[[x_1, \ldots]][\prod(x_i - x_j)^{-1}]$ in the same way as above. The fact that V is commutative implies that $u_1(x_1)u_2(x_2)u_3(0) = u_2(x_2)u_1(x_1)u_3(0)$. This in turn implies that the vertex operators $u_1(x_1)$ and $u_2(x_2)$ commute in the sense that

$$(x_1 - x_2)^N (u_1(x_1)u_2(x_2) - u_2(x_2)u_1(x_1))u_3 = 0$$

for N a sufficiently large integer, depending on u_1 and u_2. So we have constructed commuting vertex operators for all elements of $V(1)$, and this can easily be used to show that $V(1)$ is a vertex algebra. This proves theorem 4.3.

Example 4.4. Take L to be an even integral lattice. Choose a bicharacter c such that $c(\alpha, \beta) = (-1)^{(\alpha,\beta)} c(\beta, \alpha)$. (There are many ways to do this. For example we can choose a basis $\alpha_1, \alpha_2, \ldots$ and define $c(\alpha_i, \alpha_j)$ to be 1 if $i \geq j$ and $(-1)^{(\alpha_i,\alpha_j)}$ if $i < j$.) Define a symmetric $R[(x_1 - x_2)^{\pm 1}]$-valued bicharacter r of L by

$$r(\alpha, \beta)(x_1, x_2) = c(\alpha, \beta)(x_1 - x_2)^{(\alpha,\beta)}.$$

If V is the $(R\text{-mod}, H, S)$ vertex algebra constructed in theorem 4.2 with underlying object $T_*(H(R[L]))$ then $V(1)$ is just the usual vertex algebra of the even integral lattice L. If L is any integral lattice (not necessarily even) then we can do a similar construction with the following changes. We choose c so that $c(\alpha, \beta) = (-1)^{(\alpha,\beta)+(\alpha,\alpha)(\beta,\beta)} c(\beta, \alpha)$. The bicharacter r is no longer symmetric but is supersymmetric, so we end up with a vertex superalgebra rather than a vertex algebra.

Example 4.5. Now we write down some quantum deformations of example 4.4. Let L be an even lattice as in example 4.4, let q be an invertible element of the commutative ring R, and let A be the category of R modules. We define S by $S(I) = R$ if I has only one equivalence class, and $S(I) =$ the R-algebra generated by $(x_i - q^n x_j)$ for i and j not equivalent, n an integer, if I has more than 1 equivalence class. Choose a basis $\alpha_1 \ldots, \alpha_n$ for L and define r using lemma 2.7 by putting

$$r(\alpha_i, \alpha_j) = c(\alpha, \beta) \prod_{k=1}^{(\alpha_i,\alpha_j)} (x_1 - q^{(\alpha_i,\alpha_j)-2k} x_2)$$

where c is the bicharacter of example 4.4. By applying theorem 4.2 we get a $(R\text{-mod}, H, S)$ quantum vertex algebra. We see that

$$(x_1 - q^{(\alpha_i,\alpha_j)}x_2)e^{\alpha_1}(x_1)e^{\alpha_2}(x_2) = (q^{(\alpha_i,\alpha_j)}x_1 - x_2)e^{\alpha_2}(x_2)e^{\alpha_1}(x_1).$$

This is similar to many of the formulas of statistical mechanics in the book [J-M].

Example 4.6. We show how to construct (A, H, S) vertex algebras corresponding to generalized free quantum field theories. Suppose that Φ is a module over a commutative ring in A and H is a commutative cocommutative bialgebra in A. Then any linear map Δ from $\Phi \otimes \Phi$ to $S(1:2)$ gives a quantum (A, H, S) vertex algebra as follows. Use example 2.8 to extend r to a $S(1:2)$-valued bicharacter of the symmetric algebra M of Φ. Then use theorem 4.2 to make $T_*(H(M))$ into a quantum (A, H, S) vertex algebra. If r is symmetric then this is a (A, H, S) vertex algebra, and is closely related to generalized free quantum field theories, at least when H is finite dimensional abelian. (To obtain analogues of free quantum field theories in odd dimensions or dimension 2 we should allow slightly more general sorts of singularities, such as half integral powers or logarithms of $(x_1 - x_2)^2$ rather than just poles.) The function r gives the propagator of free fields, and the Greens functions $\langle|\phi_1(x_1)\cdots\phi_n(x_n)|\rangle$ can be recovered as $\eta(\phi_1(x_1)\cdots\phi_n(x_n))$ where η is the counit of $H(M)$ and $\phi_1, \ldots, \phi_n$ are elements of Φ.

Take H to be the additive formal group of dimension d for some positive even integer d. If we take Φ to be a one dimensional free module over R spanned by an element ϕ and put $r(\phi\otimes\phi) = (\sum(x_{1,i} - x_{2,i})^2)^{1-d/2}$ then V is the "H vertex algebra of a free scalar field" constructed in [B98]. It is obvious that we can just write down many quantum deformation of this H vertex algebra just by varying r; for example, we could take $r(\phi\otimes\phi) = (\sum(x_{1,i} - qx_{2,i})^2)^{1-d/2}$.

Example 4.7. In the theory of vertex algebras we often get contour integrals such as

$$\int_{x_1} a_1(x_1)a_2(x_2)a_3(x_3)dx_1.$$

We will show how to define such contour integrals for (A, H, S) vertex algebras, where H and S are as in theorem 4.3. Take $a_i \in V(i)$, where $V(i)$ can be identified with $V(1)$. We know that $a_1a_2a_3 \in V(1:2:3)$ using the multiplication of V. We also know that $V(1:2:3) = V(1,2,3)[(x_1 - x_2)^{\pm1}, (x_2 - x_3)^{\pm1}, (x_1 - x_3)^{\pm1}]$, so we can write $a_1a_2a_3$ as a finite sum of terms of the form

$$a_{123}(x_1 - x_2)^i(x_1 - x_3)^j(x_2 - x_3)^k.$$

Next we expand $(x_1 - x_3)^j$ as a possibly infinite series

$$(x_1 - x_3)^j = \sum_{n\geq 0}\binom{j}{n}(x_1 - x_2)^n(x_2 - x_3)^{j-n}.$$

Finally we replace each term $a_{123}(x_1 - x_2)^i(x_2 - x_3)^k$ by

$$f_*(D_1^{(i)}(a_{123}))(x_2 - x_3)^k \in V(2:3)$$

where f is the function from $\{1,2,3\}$ to $\{2,3\}$ with $f(1) = f(2) = 2$, $f(3) = 3$. This algebraically defined contour integral has most of the properties one would expect. For example we have the identity

$$\begin{aligned}&\int a_1(x_1)dx_1 \int a_2(x_2)dx_2a_3 - \int a_2(x_2)dx_2 \int a_1(x_1)dx_1a_3\\ =&\int \left(\int a_1(x_1)dx_1a_2\right)(x_2)dx_2a_3\end{aligned}$$

which can be used to prove the usual vertex algebra identities. Of course this identity depends on the simple choice of H and S we made; for more complicated choices of H and S we will usually get more complicated identities. In particular contour integrals can be defined in terms of the more elementary operations of a (A, H, S) vertex algebra. One reason for using the bialgebra H with divided powers (see example 2.12) rather then the universal enveloping algebra $R[D]$ is that the divided powers are needed to define the contour integrals.

Example 4.8. Take H as in example 2.12, and let $S(I)$ be R if I has at most one equivalence class, and the ring generated by the elements $(x_i - q^n x_j)^{\pm 1}$ for $i \not\equiv j$ and I having more than 1 equivalence class. Then if V is a quantum (A, H, S) vertex algebra, we can think of $V(1)$ as being some sort of "quantum vertex algebra". We will not give a definition of quantum vertex algebras here, because the philosophy of this paper is that (quantum) vertex algebras should be replaced by (quantum) (A, H, S) vertex algebras. Several sets of axioms for quantum vertex algebras have been proposed by various authors in [E-K], [F-R].

Example 4.9. The (ordinary) tensor product of two (ordinary) vertex algebras is a vertex algebra. The analogue of this for (A, H, S) vertex algebras is trivial to prove: the pointwise tensor product of any two singular commutative rings in $Fun(Fin^{\not\equiv}, A, H, S)$ is a singular commutative ring, and the pointwise tensor product of two objects of $Fun(Fin, A, T^*(H), S)$ is still in $Fun(Fin, A, T^*(H), S)$, so the pointwise tensor product of two (A, H, S) vertex algebras is an (A, H, S) vertex algebra. Note that the singular tensor product of two (A, H, S) vertex algebras is a singular commutative ring in $Fun(Fin^{\not\equiv}, A, T^*(H), S)$, but need not be in $Fun(Fin, A, T^*(H), S)$, so the singular tensor product of two (A, H, S) vertex algebras need not be an (A, H, S) vertex algebra.

Example 4.10. We can obtain many variations of vertex algebras by changing H and S. For example we could take H to be the universal enveloping algebra of the Virasoro algebra to get things similar to "vertex operator algebras". If we take H to be the tensor product of two copies of the Virasoro algebra then (A, H, S) vertex algebras are closely related to conformal field theory and string theory. If we let H be the universal enveloping algebra of various superalgebras then we get (A, H, S) vertex algebras related to supersymmetry.

5. Open problems.

In this section we list some suggestions for further research.

Problem 5.1. Are there natural quantum deformations of other well known vertex algebras, such as the monster vertex algebra [B86], [F-L-M], the vertex algebra of the lattice $II_{25,1}$ [B86], [K97], and the vertex algebras of highest weight representations of affine Lie algebras and the Virasoro algebra [F-Z], [K97]? Etingof and Kazhdan [E-K] construct "quantum vertex operator algebras" corresponding to the vertex algebras of affine Lie algebras, and it seems likely that their construction could be extended to give examples satisfying the definitions in this paper. Frenkel and Jing [F-J] previously constructed vertex operators related to of quantum affine Lie algebras.

Problem 5.2. Ordinary vertex algebras can be used to construct many examples of generalized Kac-Moody algebras. Is there a relation between quantum vertex algebras and some sort of quantized generalized Kac-Moody algebras, possibly those defined in [K95]?

Problem 5.3. The similarity of the formulas in solvable lattice models in [J-M] and quantum vertex algebras suggests that there may be some relation between these subjects.

Problem 5.4. We have constructed vertex algebras from bicharacters of bialgebras that are both commutative and cocommutative. If a bialgebra is cocommutative but not commutative then the bicharacters are usually not all that interesting (for the much same reason that one dimensional characters of a non-abelian group are not usually interesting). However there are nontrivial examples of bicharacters of bialgebras that are neither commutative or cocommutative. Can these be used to construct some sort of vertex algebras?

Problem 5.5. Construct $(R\text{-mod}, H, S)$ vertex algebras corresponding to the other standard examples of vertex algebras, such as the vertex algebras of affine and Virasoro algebras ([F-Z]), or the monster vertex algebra ([F-L-M]) or the vertex algebra of differential operators on a circle ([K97]).

Problem 5.6. Many of the constructions and definitions in section 3 do not use the fact that the category A is additive. Is there any use for these constructions in the non-additive case?

Problem 5.7. Do these constructions for braided rather than symmetric tensor categories. In particular it should be possible to allow nonintegral powers of $x_i - x_j$, which often arise from non-integral lattices or from conformal field theory.

Problem 5.8 A cobraided Hopf algebra (as defined in in [K, definition VIII.5.1]) is a Hopf algebra with a bicharacter r with the extra property that $\mu^{op} = r * \mu * \bar{r}$. This suggests that it might be possible to replace commutative, cocommutative bialgebras by something more general, maybe cobraided bialgebras. In particular theorem 4.2 should be extended to the case when M is cobraided rather than cocommutative.

Problem 5.9 Instead of twisting a group ring by a bicharacter, we can also twist it by a 2-cocycle (preferably normalized). We can define "multiplicative 2-cocycles" of arbitrary cocommutative bialgebras with values in any algebra S acted on by the bialgebra, and use these to construct more general twistings. We can also define multiplicative n-cochains, cocycles, and coboundaries, and use these to define multiplicative analogues $H^n(M, S^*)$ of cohomology groups. Note that the usual (additive) cohomology $H^n(M, S)$ of bialgebras depends only on the underlying associative algebra and the counit of M and on the module structure of S, and should not be confused with these multiplicative cohomology groups $H^n(M, S^*)$ that also depend on the coproduct of M and the algebra structure of S. Find some examples of vertex algebras constructed using singular 2-cocycles rather than singular bicharacters. There are many examples that can be constructed like this in a formal (and not very interesting) way from a perturbative quantum field theory.

Problem 5.10 It is possible to construct singular 2-cocycles which look formally similar to the Greens functions of perturbative quantum field theories. At the moment this just seems to be little more than a formal triviality, but may be worth investigating further.

Problem 5.11 I. Grojnowski and S. Bloch independently suggested replacing the Hopf algebra H of example 4.4 by the formal group ring of the formal group of an elliptic curve. Over the rationals this makes no difference, but over finite fields or the integers we seem to get something different. The underlying space of the vertex algebra we get can be thought of as the coordinate ring of the gauge group of maps from a (formal) elliptic curve to an algebraic torus. The problem is to find a use for this construction!

Problem 5.12 Develop the theory of categories with two symmetric tensor products satisfying the conditions suggested in section 3 (and maybe some others), and find more examples of them. Soibelman pointed out that Beilinson and Drinfeld [B-D] have some categories which have both a tensor product and a separate multilinear structure.

Problem 5.13 The study of orbifolds of vertex algebras (in other words, fixed subalgebras under finite automorphism groups) is notoriously hard (see [D-M] for example), though this ought to be an easy and natural operation. The difficulties appear to be caused partly by the fact that vertex algebras seem to have something missing from their structure. Does the theory of orbifolds for (A, H, S) vertex algebras (with their extra structure of fields of several spacetime variables) become any easier?

Problem 5.14 Soibelman suggested that the examples of associative algebras of automorphic forms in the meromorphic tensor category of [So, Theorem 8] might be some sort of (A, H, S) vertex algebras. These may be related to the algebras in [K96].

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