Lattices like the Leech lattice.

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Introduction.

The Leech lattice has many strange properties, discovered by Conway, Parker, and Sloane. For example, it has covering radius $\sqrt{2}$, and the orbits of points at distance at least $\sqrt{2}$ from all lattice points correspond to the Niemeier lattices other than the Leech lattice. (See Conway and Sloane [6, Chaps. 22-28].) Most of the properties of the Leech lattice follow from the fact that it is the Dynkin diagram of the Lorentzian lattice $II_{25,1}$, as in Conway [4]. In this paper we show that several other well-known lattices, in particular the Barnes-Wall lattice and the Coxeter-Todd lattice (see [6, Chap. 4]) are related to Dynkin diagrams of reflection groups of Lorentzian lattices; all these lattices have properties similar to (but more complicated than) those of the Leech lattice. Conway and Norton [5] showed that there was a strange correspondence between some automorphisms of the Leech lattice, some elements of the monster, some sublattices of the Leech lattice and some of the sporadic simple groups. Many of these things also correspond to some Lorentzian lattices behaving like $II_{25,1}$ and to some infinite dimensional Kac-Moody algebras.

Most of the notation and terminology is standard. For proofs of the facts about the Leech lattice that we use, see the original papers of Conway, Parker and Sloane in chapters 23, 26, and 27 of Conway and Sloane [6], or see Borcherds [1]. Lattices are always integral, and usually positive definite or Lorentzian, although they are occasionally singular. A root of a lattice means a vector r of positive norm such that reflection in the hyperplane of r is an automorphism of the lattice and such that r is primitive, i.e., r is not a nontrivial multiple of some other lattice vector; by a strong root we mean a root r such that (r, r) divides (r, v) for all v in the lattice. (For example, any vector of norm 1 is a strong root.) The symbols $a_n, b_n, \ldots, e_8$ stand for the spherical Dynkin diagrams, and their corresponding affine Dynkin diagrams are denoted by $A_n, B_n, \ldots, E_8$. In some of the examples we give later E_8 also stand for the E_8 lattice. The symbols $I_{n,1}$ and $II_{n,1}$ stand for odd and even unimodular Lorentzian lattices of dimension $n+1$, which are unique up to isomorphism. The automorphism group $\mathrm{Aut}(R)$ of a Lorentzian lattice R means the group of automorphisms that fix each of the two cones of negative norm vectors (so $\mathrm{Aut}(R)$ has index 2 in the "full" automorphism group of R).

The reflection group of a Lorentzian lattice is the group generated by the reflections of its roots. For any Lorentzian lattice, one of the two components of the norm -1 vectors can be identified with hyperbolic space, and all automorphisms of the lattice act as isometries on this space. In particular, a reflection of the lattice can be thought of as a reflection in hyperbolic space, so the reflection group of a lattice is a hyperbolic reflection group.

Section 1 contains several results useful for practical calculation of Dynkin diagrams of Lorentzian lattices, Section 2 contains some results about the reflection group of a sublattice fixed by some group, and Section 3 applies the results of Sections 1 and 2 to the Leech lattice to produce several lattices whose reflection groups either have finite index in the automorphism group of behave like the reflection group of the Lorentzian lattice $II_{25,1}$.

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1. Reflection groups of Lorentzian lattices.

In this section we give some theorems which help in calculating the Dynkin diagram of a reflection group of a Lorentzian lattice. Vinberg [7] described an algorithm for finding the Dynkin diagram of any hyperbolic reflection group. In the special case of hyperbolic reflection groups of Lorentzian lattices, his conditions for a root to be simple can be sharpened slightly, which reduces the amount of work needed for practical calculations for some of the lattices in Section 3. Most of the results of this section are not used later in this paper, but they are useful in checking the examples in Section 3. We let R be any Lorentzian lattice.

In a finite group acting on a lattice every positive root can be written uniquely as a sum of simple roots. This is not usually true for hyperbolic reflection groups as the simple roots are not always linearly independent, but in the case of Lorentzian lattices there is still a "canonical" way to write a root as a sum of simple roots as follows.

Theorem 1.1. *If r is a positive root of some hyperbolic reflection group W acting on the lattice R then r can be written uniquely as $r = \sum n_i s_i$ where the s's are a finite number of distinct simple roots of W, the n's are positive integers, and the following condition holds:*

> *Let T_r be the (possibly singular) lattice generated by linearly independent elements t_i with $(t_i, t_j) = (s_i, s_j)$. Then $t = \sum n_i t_i$ is conjugate to some t_i under the reflection group of T_r generated by the reflections of the roots t_i. (T_r is singular if the s's are linearly independent, but the quotient of T_r by the kernel of its quadratic form is Lorentzian or positive definite.)*

Proof. Let T be the (possibly infinite dimensional and singular) lattice generated by linearly independent elements t_i for every simple root s_i of W, with the inner product on T defined by $(t_i, t_j) = (s_i, s_j)$, and let W' be the reflection group of T generated by the reflections of the roots t_i. (T can also be described as the root lattice of the Kac-Moody algebra of the Dynkin diagram of W.) We let c be a vector of R which has negative inner product with all s_i, and define the height on R or T by $ht(x) = -(x, c)$. As W' is a Weyl group with linearly independent roots, every positive root of W' can be written uniquely as a sum of simple roots. (A root of W' is a vector of T conjugate to some t_i under W', and is called positive if its height is positive.)

We now check that every root of W is the image of a unique root of W' under the map from T to R taking t_i to s_i. It is sufficient to check this for simple roots of W, as any root of W is conjugate to a simple root, and it will follow for simple roots if we show that no simple root s can be written as a nontrivial sum $\sum m_i s_i$ with m_i positive integers. If it could be, then one of the s_i's, say s_0, must be s because $(s, \sum m_i s_i) = (s, s) > 0$ and all simple roots except s have inner product at most 0 with s. Now the fact that $ht(s) = \sum m_i ht(s_i) \geq ht(s_0) = ht(s)$ implies that $m_0 = 1$ and there are no other m's, because all simple roots have positive height. Hence every positive root r of W can be written uniquely in the form $r = \sum n_i s_i$ such that $\sum n_i t_i$ is a root of W'. $t = \sum n_i t_i$ is a root of W' if and only if t is conjugate to one of the t_i's under W', or equivalently under the subgroup of W' generated by the reflections of the t_i's appearing in the sum for t. Q.E.D.

We use this to prove a strengthened form of Vinberg's condition (Vinberg [7]) for a root to be simple. This corollary is not used later in the paper, but is sometimes useful for practical calculations.

Corollary 1.2. *Let c be an element of R which has inner product at most 0 with all simple roots of some fundamental domain of some hyperbolic reflection group W acting on R and call $-(c, r)$ the height of r. Let r be a root of W of positive height. Then the following are equivalent:*

(1) r is simple.

(2) r has inner product at most 0 with all simple roots s such that $ht(s) \leq ht(r)\min(1, |s|/(|r|\sqrt{2}))$.

(3) r has inner product at most 0 with all simple roots s such that there is an integer n satisfying the two conditions

(a) $ht(s)/ht(r) \leq 1/n \leq s^2/r^2$

(b) if $n = 1$ then $r^2 \leq s^2/2$.

Proof. It is obvious that (1) implies (2) and an easy argument shows that any root s satisfying the condition in (3) also satisfies the condition in (2), so that (2) implies (3). Hence we have to show that if r is not simple there is a simple root s satisfying the condition in (3) and having positive inner product with r.

Assume r is not simple. We can write $r = \sum n_i s_i$ as in 1.1 with n_i positive and s_i simple. Let s be a simple root of smallest possible height having positive inner product with r. If s has height 0 then it satisfies the conditions of (3), so we can assume that s has positive height. The simple root s must be equal to some s_i, so $ht(s) \leq ht(r)$, so we can define a positive integer n by $n \leq ht(r)/ht(s) < n+1$. The reflection of r in the hyperplane of s is a positive root equal to $r - 2s(s,r)/(s,s)$, so $ht(r) \geq 2ht(s)(s,r)/(s,s)$. Also $2(s,r) \geq (r,r)$ as r is a root and (s,r) is positive, so $ht(r)/ht(s) \geq n \geq 2(r,s)/(s,s) \geq (r,r)/(s,s)$ (as $2(r,s)/(s,s)$ is an integer). It remains to check that if $n = 1$ then $s^2 \geq 2r^2$. In this case $ht(s) > ht(r)/2$, so $r - 2s(s,r)/(s,s) = r - s$ as it must have height at least 0 (so it cannot be $r - ms$ for $m > 1$), so $2(r,s) = (s,s)$. Also $(r, r-s) \leq 0$ because $r - s$ is a sum of simple roots of height less than that of s (as $ht(s) > ht(r)/2$), so by the choice of s they all have inner product at most 0 with r. Hence $(r,r) \leq (r,s) = (s,s)/2$. Q.E.D.

Remarks. The condition in (3) is in some sense the best possible. Vinberg's condition was that r is simple if it has inner product at most 0 with all simple roots s such that $ht(s)/|s| < ht(r)/|r|$, which is weaker than the condition in (2) so that the corollary above slightly reduces the amount of calculation needed to prove that a root is simple (by a factor of about 1.5 for some typical examples). However, Vinberg's condition remains true for all hyperbolic reflection groups, while the corollary above is only true for those whose simple roots are primitive vectors of some Lorentzian lattice.

The following corollary is sometimes a quick way to show that a root is simple and will be used in Section 3.

Corollary 1.3. *(Notation is as in 1.2.) Suppose that r is a root of positive height not conjugate under W to any positive roots of smaller height. Then r is simple if and only* *if it has inner product at most 0 with all simple roots of height 0. ("Height" means the same as in 1.2.)*

Proof. We can write $r = \sum n_i s_i$ as in 1.1 with s_1 a simple root conjugate to r under W, so $ht(s_1) = ht(r)$ and all the other s_i's have height 0 as the height of r was minimal. Hence r has inner product at most 0 with all simple roots of positive height other than s_i, so by 1.2 it is simple if it has inner product at most 0 with all simple roots of height 0. Q.E.D.

Remark. Similarly, if x is a vector of norm at most 0 not conjugate to any vector of smaller height under W, then x is in the fundamental domain of the reflection group if and only if it has inner product at most 0 with all simple roots. The proof of this is trivial, as any root of positive height must have inner product at most 0 with r.

We have several ways to tell whether a root is simple. We also need to be able to tell when some set of simple roots is the complete set of simple roots of some reflection group. Vinberg gave a sufficient condition for this: if every critical subdiagram of a finite set of simple roots is affine, and every affine subdiagram is contained in an affine subdiagram of rank $\dim(R) - 2$ then the set of simple roots is complete. Here a critical diagram is a minimal non positive-definite diagram. Unfortunately many of the diagrams that occur in the examples in Section 3 have critical subdiagrams that are not affine; in these cases we can use the following theorem.

Theorem 1.4. *Let S be some finite subset of the simple roots of the hyperbolic reflection group W acting on a d-dimensional real Lorentzian space R, and let D be the set of all points of R having inner product at most 0 with all points of S. Suppose that the roots of S span R. Then the following two conditions are equivalent:*

(1) S is the set of all simple roots of W and any vector of D has norm at most 0.

(2) S contains a spherical Dynkin diagram T of rank $d-2$ contained in a Dynkin diagram of S which is spherical of rank $d-1$ or affine of rank $d-2$, and any such subset T of S is contained in another Dynkin diagram of S which is either spherical of rank $d-1$ or affine of rank $d-2$. (These diagrams may be disconnected. Recall that the rank of an affine Dynkin diagram is the number of points minus the number of components.)

(Vinberg showed that (1) is equivalent to D having finite volume; unfortunately it is often difficult to check directly whether D has finite volume when the roots are linearly dependent—there may be a large number of them.)

Proof. Let N be the $(d-1)$-dimensional sphere of rays of R leading from the origin, so there is a projection from nonzero points of R to N. Let H be the projection of one of the cones of negative norm vectors into N, and let P be the projection of D into H. If every vector of D has norm at most 0, then S is a complete set of simple roots of W, as any simple root of W not in S is in D and has positive norm. Hence (1) is equivalent to saying that P is in the closure $\bar{H}$ of H, or equivalently that all vertices of P are in $\bar{H}$.

A subset of the hyperplanes of S meet in a point of H if and only if they form a spherical Dynkin diagram. Hence condition (2) states that there is an edge of P containing a point of H, and any such edge contains two "good" vertices of P (where we call a vertex of P good if it lies in $\bar{H}$). Therefore (1) implies (2).

Conversely assume (2). There is at least one good vertex of P, so to prove (1) it is sufficient to show that any vertex of P joined by some edge to a good vertex of P is also good, as this will show all vertices are good. Any edge joined to a good vertex must contain a point in H, so by the assumption of (2) both the vertices on this edge are good. Q.E.D.

Similarly, (1) is equivalent to the condition that S contain at least one Dynkin diagram that is spherical of rank $d-1$ or affine of rank $d-2$, and every spherical diagram of rank $d-2$ contained in such a diagram is contained in a second such diagram. This is sometimes slightly easier to check than (2).

2. Sublattices fixed by a group.

In this section we investigate the relation between the automorphism group of a lattice R and the automorphism group of the sublattice R' fixed by some finite subgroup G of the group of "diagram automorphisms" of R. The case we are most interested in is when R is the lattice $II_{25,1}$ and G is a cyclic group. In this case R' often turns out to have many of the same properties as $II_{25,1}$; for example its Dynkin diagram is often related to some positive definite lattice in much the same way that the Dynkin diagram of $II_{25,1}$ is related to the Leech lattice. The theorems here describe the reflection group of R' in terms of the reflection group of R.

We fix some notation for the rest of this section: R is a Lorentzian lattice, D is a Weyl chamber for a group W which is some normal subgroup of $\mathrm{Aut}(R)$ generated by reflections whose roots span the vector space of R, G is a finite subgroup of $\mathrm{Aut}(R)$ fixing D (i.e., a group of diagram automorphisms), R' is the sublattice of R fixed by G, W' is the subgroup of elements of W commuting with G, and if r is any vector of R then r' is its projection into the real vector space of R'.

Lemma 2.1. *The sublattice R' of R fixed by G is a Lorentzian lattice and if r is a simple root of R the following conditions are equivalent:*

(1) The orbit of r under G is the Dynkin diagram a_1^n or a_2^n for some positive integer n.

(2) The space generated by the conjugates of r under G is positive definite.

(3) The projection r' of R into the vector space of R' has positive norm.

Any of these imply that some multiple of r' is a root of R' such that the reflection of r' is the restriction of some element W' of R'. If r is a strong root of R then the corresponding root of R' is also strong.

Proof. If s is any vector of R of negative norm then the sum of the conjugates of S under G is a vector in R' of negative norm, so R' is Lorentzian because it is a sublattice of a Lorentzian lattice and has a vector of negative norm.

It is obvious that (1) implies (2) and (2) implies (3). We assume (3) and deduce (1). Let r be a simple root of W such that r' has positive norm, and suppose that r has n conjugates $r_1, \ldots, r_n$ under G. Then

$$nr' = r_1 + \cdots + r_n,$$

so

$$(r, r_1) + \cdots + (r, r_n) = n(r, r') > 0.$$

If $r_i \neq r$ then $(r_i, r) \leq 0$ and is a multiple of $(r,r)/2$, so at most one of the terms (r, r_i) for $r \neq r_i$ is nonzero, and such a nonzero term must be $-(r,r)/2$. G acts transitively on the set of r_i's, so either they are all perpendicular in which case they form a Dynkin diagram a_1^n, or each has nonzero inner product with exactly one other r_i and this inner product is $-(r,r)/2$, in which case they form a Dynkin diagram $a_2^{n/2}$. This proves (1).

Finally if σ is the element of the Weyl group of a_1^n or a_2^n mapping ρ to $-\rho$ (where ρ is the Weyl vector of a_1^n or a_2^n) then the restriction of σ to R' maps r' to $-r'$ and fixes the orthogonal complement of r' in R'. Hence the reflection of r' is an automorphism of R' lifting to the element σ of W, and this implies that the smallest positive (real) multiple of r' that is in R' is a root of R'. (If r is a strong root then the conjugates of r form a Dynkin diagram a_1^n, and the corresponding root of R' is the sum of these conjugates and therefore also a strong root.) Q.E.D.

If r is strong or the conjugates of r form a Dynkin diagram of type a_2^n then the root of R' that is a multiple of r' is the sum of the conjugates of r; otherwise it may be half the sum of the conjugates of r.

Now we show that several ways of construction a reflection group of R' from R, W, and G all give the same group.

Theorem 2.2. *The following groups of automorphisms of the sublattice R' of R are the same.*

(1) The elements of W commuting with G.
(2) The elements of W fixing the subspace R'.
(3) The reflection group W' generated by the reflections of the vectors r' as r runs through the simple roots of W' whose projections r' have positive norm.
(4) Same as (3), with "simple roots" replaced by "roots".

Moreover the subgroups of W in (1) and (2) act faithfully on R'.

Proof. It is obvious that the subgroup (1) of W is contained in the subgroup (2), and that the groups (3) and (4) are the same. We will complete the proof by constructing an injective map from the subgroup (2) of W to the group (4), and then checking that the restriction of this map from the group (1) to the group (4) is onto.

The nonzero intersections of hyperplanes of W not containing R' with R' are hyperplanes of R', and by 2.1 the reflections of these hyperplanes are restrictions of elements of W to R'. Hence the intersection D' of D with R' is a Weyl chamber for the reflection group W'. If w is any element of the subgroup (2) of W then there is an element w' of W' such that ww' fixes the Weyl chamber D' of W', and by Lemma 2.1 w' can be lifted to an element of W', which we also denote by w'. Then ww' is an element of W fixing D and is therefore 1. This implies that the restriction map from the subgroup (2) of W to the group of automorphisms of R' is injective and maps into the group (4).

Finally we have to check that the composed map from (1) to (4) is surjective; to do this it is sufficient to check that any reflection of W' is the restriction of some element of (1), but this follows from 2.1. Q.E.D.

We now consider the special case where there is a nonzero vector c having bounded inner product with all simple roots of W (e.g., if W has a finite number of roots). This is a very strong restriction on W. The existence of such vectors of negative norm is equivalent to W having only a finite number of simple roots, and if c has norm 0 then the simple roots of W look a bit like the union of several cosets of some lattice together with a finite number of extra roots. (See Section 3.) Two examples of this case are $II_{25,1}$ or $I_{9,1}$ with W the group generated by reflections of vectors of norms 2 or 1, respectively. The simple roots can then be identified with the points of the Leech lattice (as in Conway [4]) or with the points of the E_8 lattice and in general there is a similar description of the Dynkin diagram of the lattice, as in Section 3. (Note that in the second case we are not using the full reflection group, which has only a finite number (10) of simple roots.) These facts are closely related to the facts that the covering radii of the Leech lattice and E_8 are $\sqrt{2}$ and 1. We now show that if R has such a vector c then we can say a lot more about R', and in particular it also has such a vector.

Lemma 2.3. *Let c be a nonzero vector in the fundamental domain D having bounded inner products with all simple roots of the group W, and assume that these simple roots span the vector space of R. If z is a norm 0 vector of D not proportional to c then the simple roots of W perpendicular to z form an affine Dynkin diagram of rank $\dim(R)-2$. (Recall that the rank of an affine Dynkin diagram is the number of points minus the number of components.)*

Proof. For any vector v of R perpendicular to z, the function mapping r in R to $r+(r,z)v-((r,v)+(v,v)(r,z)/2)z$ is easily checked to be an automorphism of R fixing all vectors perpendicular to z and v, and in particular fixing z. If the simple roots of W perpendicular to z do not have rank $\dim(R)-2$ we can find a vector v having inner product 0 with z and all the simple roots, and which is not a multiple of z. The automorphism of nv fixes z and all the simple roots perpendicular to z, and therefore fixes the Weyl chamber D and hence acts on the set of simple roots. We let r be any simple root of W not perpendicular to z and consider its images under the automorphism of nv for large n. The inner product of such an image with c is $-n^2(v,v)(r,z)(z,c)+$ terms in n^1 and n^0, and as the inner product of simple roots with c is bounded we must have $(v,v)(r,z)(z,c)=0$. However, (v,v) is nonzero because $(v,z)=0$ and v is not a multiple of z, (r,z) is nonzero by assumption on r, so $(z,c)=0$ and therefore z is a multiple of c as $(z,z)=0$ and z and c are both in D. Q.E.D.

Theorem 2.4. *Let c be a nonzero vector of the fundamental domain D of W having bounded inner product with all simple roots of W. Then: either*

(1) The smallest normal subgroup of $\mathrm{Aut}(R')$ containing W' is a reflection subgroup of finite index in $\mathrm{Aut}(R')$, or

(2) c is in R' and has norm 0, W' is a reflection subgroup of $\mathrm{Aut}(R')$ of infinite index and all simple roots of W' have bounded inner product with c. Any two conjugates of c under $\mathrm{Aut}(R')$ are conjugate under W', and the subgroup of W' fixing c is an affine reflection group that has a simple root for every orbit (under G) of simple roots of W perpendicular to c. (There may be no such roots, in which case W' is simply transitive on the conjugates of c under $\mathrm{Aut}(R')$.)
Also, if there are no roots of W' perpendicular to c then W' is normal in $\mathrm{Aut}(R')$.

Proof. First note that the subgroup of elements of $\mathrm{Aut}(R')$ that can be lifted to $\mathrm{Aut}(R)$ has finite index in $\mathrm{Aut}(R')$. If c has nonzero norm then W has a finite number of simple roots and hence has finite index in $\mathrm{Aut}(R)$ so W' has finite index in $\mathrm{Aut}(R')$ and we are in case (1), so we can assume that c has zero norm. If c is not fixed by G then the sum of two conjugates of c is a vector of nonzero norm with the same properties as c, so we can assume that c is fixed by G and hence is in R'.

The group of automorphisms of R' that can be lifted to $\mathrm{Aut}(R)$ has finite index in $\mathrm{Aut}(R')$, so there are only a finite number of conjugates of c in D', because any two conjugates of c under $\mathrm{Aut}(R)$ are conjugate under W. Suppose first that there is more than one conjugate of c under $\mathrm{Aut}(R')$ in D'. By Lemma 2.3 any conjugate of c in D' other than c has simple roots of W' perpendicular to it forming an affine Dynkin diagram of rank $\dim(R)-2$, so the same is true for c. Hence if W'' is any reflection group of R' containing W' with Weyl chamber D'', the index of W'' in its normalizer is the number of conjugates of c in D'' times the order of the group of automorphisms of R' fixing c and D'', which is finite. In particular if W'' is the smallest normal subgroup of $\mathrm{Aut}(R')$ containing W' then W'' is a normal reflection subgroup of finite index in $\mathrm{Aut}(R')$, so we are in case (1) again.

Hence we can assume that there is only one conjugate of c in D' under $\mathrm{Aut}(R')$. This implies that any two conjugates of c under $\mathrm{Aut}(R')$ are conjugate under W', and if there are no roots of W' perpendicular to c then W' is simply transitive on the conjugates of c, because W' is simply transitive on the conjugates of D'.

Finally assume that there are no roots of W perpendicular to c, and let W'' be the smallest normal subgroup of $\mathrm{Aut}(R')$ containing W'. If r is any root of W'', then (r'', c) is equal to (r', c) for some root r' of W' because W' is transitive on the conjugates of c under $\mathrm{Aut}(R')$, and in particular (r'', c) is nonzero. Hence W'' is simply transitive on the conjugates of c under W'', and as the same is true for W', W'' is the same as W', so W' is normal in $\mathrm{Aut}(R')$. Q.E.D.

Remark. Suppose that c is in R' and (r, c) divides (v, c) for every simple root r of W and every vector v of R (for example R could be $II_{25,1}$). Then the same is true for every root r of W' and every vector v of R'.

3. Examples.

The lattice $II_{25,1}$ has a nonzero norm 0 vector which has bounded inner product with all simple roots, and by applying the construction of the last section we can find many other Lorentzian lattices with the same property. These lattices have properties similar to those of $II_{25,1}$. For example, the Dynkin diagram of $II_{25,1}$ can be identified with the Leech lattice which is closely related to the fact that balls of radius $\sqrt{2}$ just cover the vector space of the Leech lattice, and similarly for other Lorentzian lattices we can describe their Dynkin diagrams in terms of some positive lattice, and can cover some vector space with balls and half planes.

We now describe the geometry of the simple roots when there is a non-zero norm 0 vector c of R having bounded inner product with all the simple roots of W, where W is a normal reflection subgroup of $\mathrm{Aut}(R)$. We let T_0 (respectively T_1) be the set of points of the real vector space of R having inner product 0 (respectively 1) with c, and we write V_0 and V_1 for the quotients of T_0 and T_1, where we identify two points of T_0 or T_1 if their difference is a multiple of c. V_1 is an affine space over the $\dim(R)-2$ dimensional vector space V_0, and V_0 inherits a positive definite inner product from R. For each simple root r

of W we let S_r be the subset of points of V_1 represented by norm 0 vectors of T_1 that have inner product at least 0 with r. (Note that every point of V_1 is represented by a unique norm 0 vector of T_1.) We write R_0 and R_1 for the lattice vectors in V_0 and V_1.

The fact that the Leech lattice is the Dynkin diagram of $II_{25,1}$ implies that the vector space of the Leech lattice is covered by balls of radius $\sqrt{2}$ about each lattice point, each ball corresponding to a simple root. This can be generalized by replacing $II_{25,1}$ with the lattice R above as follows.

Theorem 3.1. *The balls and half-spaces S_r of the affine space V_1 corresponding to the simple roots r have the following properties:*

(1) If r has height 0 S_r is a closed half-space; otherwise it is a closed ball with center $r/(r,c)$ and radius $|r/(r,c)|$. (Warning–S_r does not contain 0, because 0 is not in V_1!)

(2) The sets S_r cover V_1 and there are only a finite number of them intersecting any bounded subset of V_1.

(3) If two of these balls have radii r_1 and r_2 and the distance between their centers is d, then $d^2 \geq r_1^2 + r_2^2$. The center of any ball is not contained in any other set S_r. In particular, if any of the sets S_r are removed, the remaining sets do not cover V_1.

(4) The points z not in the interiors of any of the sets S_r are in natural 1:1 correspondence with the primitive norm 0 vectors of R in the fundamental domain D of W that are not multiples of c. The roots r such that z lies on the surface of S_r form an affine Dynkin diagram of rank $\dim(R) - 2$. (Warning–it is possible for any affine Dynkin diagram to occur; even "twisted" ones.)

(5) Let R_0 be the lattice of points of V_0 represented by points of R, and let L be the sublattice of R_0 of vectors perpendicular to all roots of W of height 0. Then L acts by translation on the set of S_r's and has only a finite number of orbits on this set.

The proof is routine and will be omitted.

The first example of this behavior was found by Conway [4] for the lattice $II_{25,1}$. R_0 and L are both isomorphic to the Leech lattice, and R_1 is the affine Leech lattice. The sets S_r are all balls of radius $\sqrt{2}$ with centers the points of the affine Leech lattice, and the points z not in the interiors of any of these balls are the so-called "deep holes" of the Leech lattice. The lattice points nearest to a deep hole form the affine Dynkin diagram of the Niemeier lattice of the norm 0 vector corresponding to a deep hole. A similar example is when R is $I_{9,1}$ and W is generated by the reflections of norm 1 vectors, when R_1 is the E_8 lattice and V_1 is covered by balls of radius 1 about each lattice point. Theorem 3.1 states that the general case is rather like this, except that the balls do not necessarily have the same radius (and may degenerate into half-spaces), their centers may form more than 1 orbit under the lattice L, and affine Dynkin diagrams with roots of different lengths can occur. (In the case of $II_{25,1}$, there is a natural correspondence between the orbits of primitive norm 0 vectors and the Niemeier lattices. For arbitrary Lorentzian lattices there is also a correspondence between the orbits of primitive norm 0 vectors and a finite number of positive definite lattices, but these lattices do not necessarily have the same determinant.) Roughly speaking the simple roots of W correspond to a finite number of cosets of the lattice L, with a finite number of simple roots left over if L has dimension less than that of R_0.

If we know the Dynkin diagram of the reflection group of some lattice, we can find all normal reflection subgroups whose simple roots have bounded inner product with some nonzero vector using the following lemma.

Lemma 3.2. *Let W be a hyperbolic reflection group with fundamental domain D, and let W' be a normal reflection subgroup with fundamental domain D' containing D. Then the group W is a spilt extension of W' by the reflection group H whose simple roots are the simple roots of W that are not roots of W'.*

Proof. W is a split extension of W' by the group G of elements of W fixing D'. This group certainly contains the reflections of any simple root of W that is not a root of W', and hence contains the group H generated by these reflections. If E is the union of all conjugates of D under H, then all faces of E are conjugates of faces of D' under W and hence are hyperplanes of W' because W' is a normal subgroup of W. Hence E is a union of fundamental domains of W' and in particular contains D', so H contains G and is therefore equal to G. Q.E.D.

This means that we can sometimes find interesting normal reflection subgroups of $\mathrm{Aut}(R)$ by finding subdiagrams of the Dynkin diagram of R which are affine or spherical and such that any simple root conjugate under $\mathrm{Aut}(R)$ to some root of this subdiagram is already in the subdiagram. For example, suppose R is $I_{9,1}$. Then the Dynkin diagram of R has 9 roots of norm 2 forming an extended E_8 Dynkin diagram and one root of norm 1. Hence the reflection group generated by the norm 2 vectors of R has 11 simple roots and index 2 in W, while the quotient of W by the subgroup generated by the norm 1 vectors is isomorphic to the affine E_8 Weyl group. In fact the simple roots of the reflection group generated by the norm 1 vectors are isometric to the E_8 lattice in the same way that the simple roots of $II_{25,1}$ are isometric to the Leech lattice.

We now put everything together to prove the following theorem, which shows that several well-known lattices behave like the Leech lattice.

Theorem 3.3. *Let G be a group of automorphisms of the Leech lattice Λ such that the sublattice Λ' of vectors of Λ fixed by G has no roots, and let R be the Lorentzian lattice that is the sum of Λ' and the two dimensional even unimodular Lorentzian lattice U. Then R has a norm 0 vector c such that the simple roots of the reflection group of R are exactly the roots r of R such that (r,c) is negative and divides (r,v) for all vectors v of R. The group $\mathrm{Aut}(R)$ is isomorphic to a split extension of its reflection group by the group of affine automorphisms of Λ'.*

Proof. The group of affine automorphisms of the Leech lattice Λ can be identified with the group of diagram automorphisms of $II_{25,1}$, so G can be considered to be a subgroup of $\mathrm{Aut}(II_{25,1})$, and it is easy to check that that sublattice of points of $II_{25,1}$ fixed by G is isomorphic to R. The theorem then follows from 2.4, except that we still have to check that the group W' of 2.4 is the full reflection subgroup of R. $II_{25,1}$ has a vector c which has inner product 1 with all simple roots of $II_{25,1}$, and from this it follows that (r,c) divides (r,v) for all simple roots r of W' and all vectors v of R, so $(r,c) \leq |(s,c)|$ for any conjugate s of r. But then by 1.3 r is a simple root for the full reflection group of R, hence W' is the full reflection group of R. Q.E.D.

Remarks. The fact that (r,c) divides (r,v) for all v implies that (r,c) is either (r,r) or $(r,r)/2$; both cases occur. Any root of R has even norm dividing $2|G|$. There is a similar theorem with $II_{25,1}$, Λ and the full reflection group of $II_{25,1}$ replaced by $I_{9,1}$, E_8, and the reflection group of $I_{9,1}$ generated by the reflections of norm 1 vectors; of course all roots of W' will be strong roots.

Examples. $\mathrm{Aut}(\Lambda)$ has elements of orders 2 and 3 whose fixed lattices Λ_{16} and K_{12} have no roots and have dimensions 16 and 12 respectively. (Λ_{16} is the Barnes-Wall lattice, and K_{12} is the Coxeter-Todd lattice; see Conway and Sloane [6, Chap. 4].) These lattices have the same relation to the baby monster and Fi_{24} that Λ has to the monster. (See Conway and Norton [5].) The theorem above shows that there are 18 and 14 dimensional Lorentzian lattices R associated to them whose Dynkin diagrams can be described in terms of the 16 and 12 dimensional lattices. Note that they have roots of norms 4 and 6 as well as roots of norm 2, so the geometry is rather more complicated than that of the Leech lattice. The simple roots of R correspond to some of the vectors of the dual of Λ' (but not necessarily all of them; for example not the ones within $\sqrt{2}$ of a lattice vector.) The vectors of Λ' itself correspond to norm 2 simple roots in R. For example, if we take Λ' to be Λ_{16} of dimension 16 and determinant 256, then R has a simple root of norm 2 for every vector of Λ_{16} and a simple root of norm 4 for every vector in one of 120 cosets of Λ_{16}. If we draw a sphere of radius $\sqrt{2}$ about every point of Λ_{16}, and of radius 1 about every point of these 120 cosets, then these spheres just cover the vector space of Λ_{16} in the same way that spheres of radius $\sqrt{2}$ around points of Λ just cover the vector space of Λ. We also get large numbers of "deep holes" in Λ_{16} (e.g., F_4^4) which behave like the deep holes of Λ. (Likewise K_{12} has deep holes of hype G_2^6 and so on.)

$\mathrm{Aut}(\Lambda)$ also has an element of order 2 whose fixed sublattice is $E_8(2)$ (i.e., the lattice E_8 with the norms of all vectors doubled). In this case Λ' has roots, and the reflection group of R has finite index in $\mathrm{Aut}(R)$. Similarly if R is the sum of $E_8(n)$ and the two dimensional even unimodular Lorentzian lattice U for $2 \leq n \leq 6$ then the reflection group of R has finite index in $\mathrm{Aut}(R)$. (In fact, it follows from 2.4 that this is true for any lattice R fixed by some group of automorphisms of Λ whose roots span its vector space.) For example when $n = 6$ the Dynkin diagram has 4 roots of norm 2, 2 or norm 4, 1 of norm 6, and 10 of norm 12 and its automorphism group has order 4; the theorems of Section 1 are useful for doing these calculations.

If R is a Lorentzian lattice, there are 3 possibilities for the "non-reflection group" of R which is the quotient of the automorphism group of R by the subgroup generated by reflections. (These can be thought of as the "elliptic", "parabolic", and "hyperbolic" cases, although this terminology should not be taken too seriously.)

(1) This group is finite. This case includes many of the lattices whose dimension and determinant are both small.

(2) The non-reflection group if R is infinite, but has a free abelian subgroup of finite index. This case is the one mostly studied in this paper, and seems to be rare. The existence of a free abelian subgroup of finite index is equivalent to the existence of a nonzero vector of norm at most 0 fixed by all automorphisms of a fundamental domain of the reflection group, so $II_{25,1}$ is one example of this case, and 3.3 gives a few other examples.

(3) The general case: everything else. This case seems to include most Lorentzian lattices, possibly all of dimension more than 26. In Borcherds [2] the non-reflection group is calculated for a few unimodular lattices, and in these cases turns out to be a direct limit of a finite number of finite groups. Many of the results of Borcherds [2] still hold when Λ is replaced by some lattice R in class (2) above, so it would be possible to find some more lattices whose non-reflection groups could be presented as a direct limit of finite groups.

The monster Lie algebra (Borcherds, Conway, Queen, and Sloane, [6, Chap. 30], or Borcherds [3]) is a generalized Kac-Moody algebra with root lattice $II_{25,1}$ whose positive simple roots are the simple roots of $II_{25,1}$. (It also has simple roots of norm 0, so it is not a Kac-Moody algebra.) (Note added 1998: this algebra is now called the fake monster Lie algebra.) If G is a finite group of diagram automorphisms of $II_{25,1}$ then by Borcherds [3, theorem 3.1] the subalgebra of the monster Lie algebra fixed by G is still a generalized Kac-Moody algebra. Note that the class of generalized Kac-Moody algebras is invariant under the operation of taking the subalgebra fixed by a finite group of diagram automorphisms, but the class of Kac-Moody algebras is not. We can therefore define the baby monster Lie algebra, the Fi_{24} Lie algebra, and so on to be the subalgebras of the monster Lie algebra fixed by the appropriate group. There algebras are generalized Kac-Moody algebras whose positive simple roots are the simple roots of the corresponding Lorentzian lattice., and which have simple roots of norm 0 which are multiples of the vector c. There is some numerical evidence that the monster Lie algebra has no simple roots of negative norm (note added in 1998: this has been proved), so it is natural to ask if this is true for the new algebras (note added 1998: this is false).

Problems. Find all Lorentzian lattices which have a nonzero vector which has bounded inner product with all simple roots. (Note that this includes as a special case the problem of finding all Lorentzian lattices whose reflection group has finite index.) Are there only essentially a finite number of such lattices of dimension greater than 2? (Note added 1998: this has been proved by Nikulin if we assume that c has norm at most 0.) (Obviously any multiple of such a lattice has the same property.) Is $II_{25,1}$ the only such lattice of dimension at least 26? Is $I_{9,1}$ the only lattice of dimension ≥ 10 such that the simple roots of the reflection group generated by strong roots have bounded inner product with some vector c?

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