The Leech lattice.

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New proofs of several known results about the Leech lattice are given. In particular I prove its existence and uniqueness and prove that its covering radius is the square root of 2. I also give a uniform proof that the 23 "holy constructions" of the Leech lattice all work.

1. History.

In 1935 Witt found the 8- and 16-dimensional even unimodular lattices and more than 10 of the 24-dimensional ones. In 1965 Leech found a 24-dimensional one with no roots, called the Leech lattice. Witt's classification was completed by Niemeier in 1967, who found the twenty four 24-dimensional even unimodular lattices; these are called the Niemeier lattices. His proof was simplified by Venkov (1980), who used modular forms to restrict the possible root systems of such lattices.

When he found his lattice, Leech conjectured that it had covering radius $\sqrt{2}$ because there were several known holes of this radius. Parker later noticed that the known holes of radius $\sqrt{2}$ seemed to correspond to some of the Niemeier lattices, and inspired by this Conway *et al.* (1982a) found all the holes of this radius. There turned out to be 23 classes of holes, which were observed to correspond in a natural way with the 23 Niemeier lattices other than the Leech lattice. Conway (1983) later used the fact that the Leech lattice had covering radius $\sqrt{2}$ to prove that the 26-dimensional even Lorentzian lattice $II_{25,1}$ has a Weyl vector, and that its Dynkin diagram can be identified with the Leech lattice Λ.

So far most of the proofs of these results involved rather long calculations and many case-by-case discussions. The purpose of this paper is to provide conceptual proofs of these results. New proofs of the existence and uniqueness of the Leech lattice and of the fact that it has covering radius $\sqrt{2}$ are given. Finally I prove that the deep holes of Λ correspond to the Niemeier lattices and give a uniform proof that the 23 "holy constructions" of Λ found by Conway & Sloane (1982b) all work.

The main thing missing from this treatment of the Niemeier lattices is a simple proof of the classification of the Niemeier lattices. (The classification is not used in this paper.) By Venkov's results it would be sufficient to find a uniform proof that there exists a unique Niemeier lattice for any root system of rank 24, all of whose components have the same Coxeter number.

Some general results from the papers (Vinberg 1975) and (Venkov 1980) are quoted; apart from this nearly everything used is proved.

2. Preliminary results.

We start with some definitions, some results about Niemeier lattices proved using modular forms, and a description of Vinberg's algorithm.

Definitions.

A *lattice* L is a finitely generated free Z-module with an integer valued bilinear form, written (x, y) for x and y in L. The *type* of a lattice is even (or II) if the *norm* $x^2 = (x, x)$ of every element x of L is even, and odd (or I) otherwise. If L is odd then the vectors in L of even norm form an even sublattice of index 2 in L. L is called *positive definite, Lorentzian, nonsingular,* etc. if the real vector space $L \otimes \mathbf{R}$ is. ($\mathbf{R}$ stands for the reals and $\mathbf{Q}$ for the rationals.)

If L is a lattice then L' denotes its *dual* in $L \otimes \mathbf{R}$; i.e. the vectors of $L \otimes \mathbf{R}$ that have integral inner products with all elements of L. L' contains L and if L is nonsingular then L'/L is a finite abelian group whose order is called the *determinant* of L. (If L is singular we say it has determinant 0.) L is called *unimodular* if its determinant is 1.

Even unimodular lattices of a given signature and dimension exist if and only if there is a real vector space with that signature and dimension and the signature is divisible by 8. Any two indefinite unimodular lattices with the same type, dimension, and signature are isomorphic. $I_{m,n}$ and $II_{m,n}$ ($m \geq 1, n \geq 1$) are the unimodular lattices of dimension $m + n$, signature $m - n$, and type I or II.

A vector v in a lattice L is called *primitive* if v/n is not in L for any $n > 1$. A *root* of a lattice L is a norm 2 vector of L. Reflection in a root r is then an automorphism of L. This reflection maps v in L to $v - (v, r)r$. A *Niemeier lattice* is an even 24-dimensional unimodular lattice.

The norm 2 vectors in a positive definite lattice A form a root system which we call the root system of A. The hyperplanes perpendicular to these roots divide $A \otimes \mathbf{R}$ into regions called *Weyl chambers*. The reflections in the roots of A generate a group called the *Weyl group* of A, which acts simply transitively on the Weyl chambers of A. Fix one Weyl chamber D. The roots r_i that are perpendicular to the faces of D and that have inner product at most 0 with the elements of D are called the simple roots of D. (These have opposite sign to what are usually called the simple roots of D. This is caused by the irritating fact that the usual sign conventions for positive definite lattices are not compatible with those for Lorentzian lattices. With the convention used here something is in the Weyl chamber if and only if it has inner product at most 0 with all simple roots, and a root is simple if and only if it has inner product at most 0 with all simple roots not equal to itself.)

The *Dynkin diagram* of D is the set of simple roots of D. It is drawn as a graph with one vertex for each simple root of D and two vertices corresponding to the distinct roots r, s are joined by $-(r, s)$ lines. (If A is positive definite then two vertices are always joined by 0 or 1 lines. We will later consider the case that A is Lorentzian and then its Dynkin diagram may contain multiple bonds, but these are not the same as the multiple bonds appearing in b_n, c_n, f_4, and g_2.) We use small letters x_n to stand for spherical Dynkin diagrams. The Dynkin diagram of A is a union of components of type a_n, d_n, e_6, e_7 and e_8. The *Weyl vector* ρ of D is the vector in the vector space spanned by roots of A that has inner product -1 with all simple roots of D. It is in the Weyl chamber D and is equal to half the sum of the positive roots of D, where a root is called positive if its inner product with any element of D is at least 0. The *height* of an element a of A is $-(a, \rho)$, so a root of A is simple if and only if it has height 1.

The *Coxeter number* h of A is defined to be the number of roots of A divided by the dimension of A. The Coxeter number of a component of A is defined as the Coxeter number of the lattice generated by the roots of that component. Components a_n, d_n, e_6, e_7 and e_8 have Coxeter numbers $n+1$, $2n-2$, 12, 18, and 30 respectively. For each component R of the Dynkin diagram of D there is an orbit of roots of A under the Weyl group, and this orbit has a unique representative v in D, which is called the *highest root* of that component. We can write $-v = n_1r_1 + n_2r_2 + \cdots$, where the rs are the simple roots of R and the ns are positive integers called the *weights* of the roots r_i. The sum of the ns is $h-1$, where h is the Coxeter number of R, and the height of v is $1-h$. The *extended Dynkin diagram* of A is the simple roots of A together with the highest roots of A, and it is the Dynkin diagram of the positive semidefinite lattice $A \oplus 0$, where 0 is a one-dimensional singular lattice. We say that the highest roots of A have weight one. Any point of weight one of a Dynkin diagram is called a tip of that Dynkin diagram. We write X_n for the extended Dynkin diagram corresponding to the Dynkin diagram x_n. (Note that X_n has $n+1$ points.)

The automorphism group of A is a split extension of its Weyl group by N, where N is the group of automorphisms of A fixing D. N acts on the Dynkin diagram of D and $\mathrm{Aut}(A)$ is determined by its Dynkin diagram R, the group N, and the action of N on R.

If A is Lorentzian or positive semidefinite then we can still talk about its root system and A still has a fundamental domain D for its Weyl group and a set of simple roots. A may or may not have a Weyl vector but does not have highest roots.

Notation. Let L be any Niemeier lattice, and let Λ be any Niemeier lattice with no roots. (It will later be proved that there is a unique such lattice Λ, called the Leech lattice.)

I give some preliminary results about Niemeier lattices, which are proved using modular forms. From this viewpoint the reason why 24-dimensional lattices are special is that certain spaces of modular forms vanish.

Lemma 2.1. *(Conway 1969). Every element of Λ is congruent* mod 2Λ *to an element of norm at most 8.*

We sketch Conway's proof of this. If C_i is the number of elements of Λ of norm i then elementary geometry shows that the number of elements of $\Lambda/2\Lambda$ represented by vectors of norm at most 8 is at least

$$C_0 + C_4/2 + C_6/2 + C_8/(2 \times \dim(\Lambda)).$$

The C_i's can be worked out using modular forms and this sum miraculously turns out to be equal to 2^{24} = order of $\Lambda/2\Lambda$. Hence every element of $\Lambda/2\Lambda$ is represented by an element of norm at most 8. Q.E.D.

(This is the only numerical calculation that we need.)

Several results from Venkov (1980) are now quoted.

Lemma 2.2. *(Venkov 1980) If y is any element of the Niemeier lattice N then*

$$\sum (y,r)^2 = y^2 r^2 n/24,$$

where the sum is over the n elements r of some fixed norm.

This is proved using modular forms; the critical fact is that the space of cusp forms of dimension $\frac{24}{2}+2=14$ is zero-dimensional.

Lemma 2.3. *(Venkov 1980). The root system of N has rank 0 or 24 and all components of this root system have the same Coxeter number h.*

This follows easily from lemma 2.2 as in Venkov (1980). We call h the Coxeter number of the Niemeier lattice N. If N has no roots we put $h=0$.

It is easy to find the 24 root systems satisfying the condition of lemma 2.3. Venkov gave a simplified proof of Niemeier's result that there exists a unique Niemeier lattice for each root system. The next four lemmas are not needed for the proof that Λ has covering radius $\sqrt{2}$.

Lemma 2.4. *(Venkov 1980). N has $24h$ roots and the norm of its Weyl vector is $2h(h+1)$.*

Proof. The root system of N consists of components a_n, d_n, and e_n, all with the same Coxeter number h. For each of these components the number of roots is hn and the norm of its Weyl vector is $\frac{1}{12}nh(h+1)$. The lemma now follows because the rank of N is 0 or 24. Q.E.D.

Lemma 2.5. *If y is in N then*

$$\sum (y,r)^2 = 2hy^2,$$

where the sum is over all roots r of N.

This follows from lemmas 2.2 and 2.4. Q.E.D.

Lemma 2.6. *If ρ is the Weyl vector of N then ρ lies in N.*

Proof. We show that if y is in N then (ρ,y) is an integer, and this will prove that ρ is in N because N is unimodular. We have

$$\begin{aligned}
(2\rho,y)^2 &= (\textstyle\sum r,y)^2 \qquad \text{(The sums are over all positive roots } r.\text{)}\\
&= (\textstyle\sum (r,y))^2\\
&\equiv \textstyle\sum (r,y)^2 \bmod 2\\
&= y^2h \qquad \text{by lemma 2.5}\\
&\equiv 0 \bmod 2 \qquad \text{as } y^2 \text{ is even.}
\end{aligned}$$

The term $(2\rho,y)^2$ is an even integer and (ρ,y) is rational, so (ρ,y) is an integer. Q.E.D.

Lemma 2.7. *Suppose $h\neq 0$ and y is in N. Then*

$$(\rho/h-y)^2 \geq 2(1+1/h)$$

and the y for which equality holds form a complete set of representatives for N/R, where R is the sublattice of N generated by roots.

Proof. $\rho^2 = 2h(h+1)$, so

$$\begin{aligned}(\rho/h - y)^2 - 2(1+1/h) &= [(\rho - hy)^2 - \rho^2]/h^2 \\ &= [hy^2 - 2(\rho, y)]/h \\ &= [\sum (y,r)^2 - \sum (y,r)]/r \quad \text{by lemma 2.5} \\ &= \sum [(y,r)^2 - (y,r)]/h,\end{aligned}$$

where the sums are over all positive roots r. This sum is greater than or equal to 0 because (y,r) is integral, and is zero if and only if (y,r) is 0 or 1 for all positive roots r. In any lattice N whose roots generate a sublattice R, the vectors of N that have inner product 0 or 1 with all positive roots of R (for some choice of Weyl chamber of R) form a complete set of representatives for N/R, and this proves the last part of the lemma. Q.E.D.

Remark. This lemma shows that if N has roots then its covering radius is greater than $\sqrt{2}$. (In fact is is at least $\sqrt{(5/2)}$.) In particular the covering radius of the Niemeier lattice with root system A_2^{12} is at least $\sqrt{(8/3)}$; this Niemeier lattice has no deep hole that is half a lattice vector.

I now describe the geometry of Lorentzian lattices and its relation to hyperbolic space, and give Vinberg's algorithm for finding the fundamental domains of hyperbolic reflection groups.

Let L be an $(n+1)$-dimensional Lorentzian lattice (so L has signature $n-1$). Then the vectors of L of zero norm form a double cone and the vectors of negative norm fall into two components. The vectors of norm -1 in one of these components form a copy of n-dimensional hyperbolic space H_n. The group $\mathrm{Aut}(L)$ is a product $Z_2 \times \mathrm{Aut}_+(L)$, where Z_2 is generated by -1 and $\mathrm{Aut}_+(L)$ is the subgroup of $\mathrm{Aut}(L)$ fixing each component of negative norm vectors. See Vinberg (1975) for more details. (If we wanted to be more intrinsic we could define H_n to be the set of nonzero negative definite subspaces of $L \otimes \mathbf{R}$.)

If r is any vector of L of positive norm then $r^\perp$ gives a hyperplane of H_n and reflection in $r^\perp$ is an isometry of H_n. If r has negative norm then r represents a point of H_n and if r is nonzero but has zero norm then it represents an infinite point of H_n.

The group G generated by reflections in roots of L acts as a discrete reflection group on H_n, so we can find a fundamental domain D for G which is bounded by reflection hyperplanes. The group $\mathrm{Aut}_+(L)$ is a split extension of the reflection group by a group of automorphisms of D.

Vinberg (1975) gave an algorithm for finding a fundamental domain D which runs as follows. Choose a vector w in L of norm at most 0; we call w the *controlling vector*. If r is a root then we will call $-(w,r)/(r,r)$ the *height* of r. (Roughly speaking, the height of r measures the distance of $r^\perp$ from w.) The hyperplanes of G passing through w form a root system that is finite (or affine if $w^2 = 0$); choose a Weyl chamber C for this root system. Then there is a unique fundamental domain D of G containing w and contained in C, and its simple roots can be found as follows.

(1) All simple roots have height at least 0.

(2) A root of height 0 is a simple root of D if and only if it is a simple root of C.

(3) A root of positive height is simple if and only if it has inner product at most 0 with all simple roots of smaller height.

We can now find the simple roots in increasing order of their heights, and we obtain a finite or countable set of simple roots for D. (If $w^2 = 0$ there may be an infinite number of simple roots of some height.) If the hyperplanes of some finite subset of the simple roots of D bound a region of finite volume, then this subset consists of all the simple roots of D.

3. Norm zero vectors in Lorentzian lattices.

Here we describe the relation between norm 0 vectors in Lorentzian lattices L and extended Dynkin diagrams in the Dynkin diagram of L.

Notation. Let L be any even Lorentzian lattice, so it is $II_{8n+1,1}$ for some n. Let U be $II_{1,1}$; it has a basis of two norm 0 vectors with inner product -1.

If V is any $8n$-dimensional positive even unimodular lattice then $V \oplus U \cong L$ because both sides are unimodular even Lorentzian lattices of the same dimension, and conversely if X is any sublattice of L isomorphic to U then $X^\perp$ is an $8n$-dimensional even unimodular lattice. If z is any primitive norm 0 vector of L then z is contained in a sublattice of L isomorphic to U and all such sublattices are conjugate under $\mathrm{Aut}(L)$. This gives 1:1 correspondences between the sets:

(1) $8n$-dimensional even unimodular lattices (up to isomorphism);

(2) orbits of sublattices of L isomorphic to U under $\mathrm{Aut}(L)$; and

(3) orbits of primitive norm 0 vectors of L under $\mathrm{Aut}(L)$.

If V is the $8n$-dimensional unimodular lattice corresponding to the norm 0 vector z of L then $z^\perp \cong V \oplus 0$ where 0 is the one-dimensional singular lattice. The Dynkin diagram of $V \oplus 0$ is the Dynkin diagram of V with all components changed to the corresponding extended Dynkin diagram.

Now choose coordinates (v, m, n) for $L \cong V \oplus U$, where v is in V, m and n are integers, and $(v, m, n)^2 = v^2 - 2mn$ (so V is the set of vectors $(v, 0, 0)$ and U is the set of vectors $(0, m, n)$). We write z for $(0, 0, 1)$ so that z is a norm 0 vector corresponding to the unimodular lattice V. We choose a set of simple roots for $z^\perp \cong V \oplus 0$ to be the vectors $(r^i_j, 0, 0)$ and $(r^0_j, 0, 1)$ where the r^i_j's are the simple roots of the components R_j of the root system of V and r^0_j is the highest root of R_j. We let $\bar{R}_j$ be the vectors $(r^i_j, 0, 0)$ and $(r^0_j, 0, 1)$ so that $\bar{R}_j$ is the extended Dynkin diagram of R_j. Then

$$\sum m_i r^i_j = z \qquad \text{and} \qquad \sum m_i = h_j,$$

where the m_i's are the weights of the vertices of $\bar{R}_j$ and h_j is the Coxeter number of R_j. We can apply Vinberg's algorithm to find a fundamental domain of L using z as a controlling vector and this shows that there is a unique fundamental domain D of L containing z such that all the vectors of the $\bar{R}_j$s are simple roots of D.

Conversely suppose that we choose a fundamental domain D of L and let $\bar{R}$ be a connected extended Dynkin diagram contained in the Dynkin diagram of D. If we put $z = \sum m_i r_i$, where the m_i's are the weights of the simple roots r_i of $\bar{R}$, then z has norm 0 and inner product 0 with all the r_is (because $\bar{R}$ is an extended Dynkin diagram) and has inner product at most 0 with all simple roots of D not in $\bar{R}$ (because all the r_i do), so z is in D and must be primitive because if z' was a primitive norm 0 vector dividing z we could apply the last paragraph to z' to find $z' = \sum m_i r_i = z$. The roots of $\bar{R}$ together with the simple roots of D not joined to $\bar{R}$ are the simple roots of D perpendicular to z and so are a union of extended Dynkin diagrams. This shows that the following 3 sets are in natural 1:1 correspondence:

(1) equivalence classes of extended Dynkin diagrams in the Dynkin diagram of D, where two extended Dynkin diagrams are equivalent if they are equal or not joined;

(2) maximal disjoint sets of extended Dynkin diagrams in the Dynkin diagram of D, such that no two elements of the set are joined to each other;

(3) primitive norm 0 vectors of D that have at least one root perpendicular to them.

4. Existence of the Leech lattice.

In this section I prove the existence of a Niemeier lattice with no roots (which is of course the Leech lattice). This is done by showing that given any Niemeier lattice we can construct another Niemeier lattice with at most half as many roots. It is a rather silly proof because it says nothing about the Leech lattice apart from the fact that it exists.

Leech was the first to construct a Niemeier lattice with no roots (called the Leech lattice), and Niemeier later proved that it was unique as part of his enumeration of the Niemeier lattices. In §6 I will give another proof that there is only one such lattice.

Notation. Fix a Niemeier lattice N with a Weyl vector ρ and Coxeter number h, and take coordinates (y, m, n) for $II_{25,1} = N \oplus U$ with y in N, m and n integers.

By lemma 2.6 the vector $w = (\rho, h, h+1)$ is in $II_{25,1}$ and by lemma 2.4 it has norm 0. We will show that the Niemeier lattice corresponding to w has at most half as many roots as N.

Lemma 4.1. *There are not roots of $II_{25,1}$ that are perpendicular to w and have inner product 0 or ± 1 with $z = (0, 0, 1)$.*

Proof. Suppose that $r = (y, 0, n)$ is a root that has inner product 0 with w and z. Then y is a root of N, and $(y, \rho) = nh$ because $(y, \rho) - nh = ((y, 0, n), (\rho, h, h+1)) = (r, w) = 0$. But for any root y of N we have $1 \le |(y, \rho)| \le h - 1$, so (y, ρ) cannot be a multiple of h.

If $(y, 1, n)$ is any root of $II_{25,1}$ that has inner product -1 with z and 0 with w then

$$(y, \rho) - (h+1) - nh = ((\rho, h, h+1), (y, 1, n)) = (w, r) = 0$$

and

$$y^2 - 2n = (y, 1, n)^2 = r^2 = 2$$

so

$$\begin{aligned}(y - \rho/h)^2 &= y^2 - 2(y, \rho)/h + \rho^2/h^2 \\ &= 2 + 2n - 2(nh + h + 1)/h + 2h(h+1)/h^2 \\ &= 2\end{aligned}$$

which contradicts lemma 2.7. Hence no root in $w^\perp$ can have inner product 0 or -1 with z. Q.E.D.

Remark. In fact there are no roots in $w^\perp$.

Lemma 4.2. *The Coxeter number h' of the Niemeier lattice of the vector w is at most $\frac{1}{2}h$.*

Proof. We can assume $h' \neq 0$. Let R be any component of the Dynkin diagram of $w^\perp$ (so R is an extended Dynkin diagram). The sum $\sum m_i r_i$ is equal to w by §3, where the r_is are the roots of R with weights m_i. Also $\sum m_i = h'$, $(r_i, z) \leq -2$ by lemma 4.1, and

$$(\sum m_i r_i, z) = ((\rho, h, h+1), z) = -h,$$

so $h' \leq \frac{1}{2}h$. Q.E.D.

Theorem 4.3. *There exists a Niemeier lattice with no roots.*

Proof. By lemma 4.2 we can find a Niemeier lattice with Coxeter number at most $\frac{1}{2}h$ whenever we are given a Niemeier lattice of Coxeter number h. By repeating this we eventually get a Niemeier lattice with Coxeter number 0, which must have no roots. Q.E.D.

In §7 it will be shown that the Niemeier lattice of the vector $(\rho, h, h+1)$ never has any roots and so is already the Leech lattice.

5. The covering radius of the Leech lattice.

Here we prove that any Niemeier lattice Λ with no roots has covering radius $\sqrt{2}$.

Notation. We write Λ for any Niemeier lattice with no roots, and put $II_{25,1} = \Lambda \oplus U$ with coordinates (λ, m, n) with λ in Λ, m and n integers and $(\lambda, m, n)^2 = \lambda^2 - 2mn$. We let w be the norm 0 vector $(0, 0, 1)$ and let D be a fundamental domain of the reflection group of $II_{25,1}$ containing w. If we apply Vinberg's algorithm, using w as a controlling vector, then the first batch of simple roots to be accepted is the set of roots $(\lambda, 1, \frac{1}{2}\lambda^2 - 1)$ for all λ in Λ. (Conway proved that no other roots are accepted. See §6.) We identify the vectors of Λ with these roots of $II_{25,1}$, so Λ becomes a Dynkin diagram with two points of Λ joined by a bond of strength 0,1,2,... if the norm of their difference is 4,6,8,.... We can then talk about Dynkin diagrams in Λ; for example an a_2 in Λ is two points of Λ whose distance apart is $\sqrt{6}$.

Here is the main step in the proof that Λ has covering radius $\sqrt{2}$.

Lemma 5.1. *If X is any connected extended Dynkin diagram in Λ then X together with the points of Λ not joined to X contains a union of extended Dynkin diagrams of total rank 24. (The rank of a connected extended Dynkin diagram is one less than the number of its points.)*

Proof. X is an extended Dynkin diagram in the set of simple roots of $II_{25,1}$ of height 1, and hence determines a norm 0 vector $z = \sum m_i x_i$ in D, where the x_is are the simple roots of X of weights m_i. The term z corresponds to some Niemeier lattice with roots, so by lemma 2.3 the simple roots of $z^\perp$ form a union of extended Dynkin diagrams of total rank 24, all of whose components have the same Coxeter number $h = \sum m_i$. Also

$$h = -(z, w)$$

because $z = \sum m_i x_i$ and $(x_i, w) = -1$. By applying Vinberg's algorithm with z as a controlling vector we see that the simple roots of D perpendicular to z form a set of simple roots of $z^\perp$, so the lemma will be proved if we show that all simple roots of D in $z^\perp$ have height 1, because then all simple roots of $z^\perp$ lie in Λ.

Suppose that Z is any component of the Dynkin diagram of $z^\perp$ with vertices z_i and weights n_i so that $\sum n_i z_i = z$. Then

$$-h = (z, w) = (\sum n_i z_i, w) = \sum n_i (z_i, w).$$

Now $(z_i, w) \leq -1$ as there are no simple roots z_i with $(z_i, w) = 0$, and $\sum n_i = h$ as Z has the same Coxeter number h as X, so we must have $(z_i, w) = -1$ for all i. Hence all simple roots of D in $z^\perp$ have height 1. Q.E.D.

It follows easily from this that Λ has covering radius $\sqrt{2}$. For completeness we sketch the remaining steps of the proof of this. (This part of the proof is taken from Conway *et al.* (1982a).)

Step 1. The distance between two vertices of any hole is at most $\sqrt{8}$. This follows easily from lemma 2.1.

Step 2. By a property of extended Dynkin diagrams one of the following must hold for the vertices of any hole.

(i) The vertices form a spherical Dynkin diagram. In this case the hole has radius $(2 - 1/\rho^2)^{\frac{1}{2}}$, where ρ^2 is the norm of the Weyl vector of this Dynkin diagram.

(ii) The distance between two points is greater than $\sqrt{8}$. This is impossible by step 1.

(iii) The vertices contain an extended Dynkin diagram.

Step 3. Let V be the set of vertices of any hole of radius greater than $\sqrt{2}$. By step 2, V contains an extended Dynkin diagram. By lemma 5.1 this extended Dynkin diagram is one of a set of disjoint extended Dynkin diagrams of Λ of total rank 24. These Dynkin diagrams then form the vertices of a hole V' of radius $\sqrt{2}$ whose center is the center of any of the components of its vertices. Any hole whose vertices contain a component of the vertices of V' must be equal to V', and in particular $V = V'$ has radius $\sqrt{2}$. Hence Λ has covering radius $\sqrt{2}$. Q.E.D.

6. Uniqueness of the Leech lattice.

We continue with the notation of the previous section. We show that Λ is unique and give Conway's calculation of $\mathrm{Aut}(II_{25,1})$.

Theorem 6.1. *The simple roots of the fundamental domain D of $II_{25,1}$ are just the simple roots $(\lambda, 1, \frac{1}{2}\lambda^2 - 1)$ of height one.*

Proof (Conway 1983). If $r' = (v, m, n)$ is any simple root of height greater than 1 then it has inner product at most 0 with all roots of height 1. Λ has covering radius $\sqrt{2}$ so there is a vector λ of Λ with $(\lambda - v/m)^2 \leq 2$. But then an easy calculation shows that $(r', r) \geq 1/2m$, where r is the simple root $(\lambda, 1, \frac{1}{2}\lambda^2 - 1)$ of height 1, and this is impossible because $1/2m$ is positive.

Corollary 6.2. *The Leech lattice is unique, i.e. any two Niemeier lattices with no roots are isomorphic.*

(Niemeier's enumeration of the Niemeier lattices gave the first proof of this fact. Also see Conway (1969) and Venkov (1980).)

Proof. Such lattices correspond to orbits of primitive norm 0 vectors of $II_{25,1}$ which have no roots perpendicular to them, so it is sufficient to show that any two such vectors are conjugate under $\mathrm{Aut}(II_{25,1})$, and this will follow if we show that D contains only one such vector. But if w is such a vector in D then theorem 6.1 shows that w has inner product -1 with all simple roots of D, so w is unique because these roots generate $II_{25,1}$. Q.E.D.

Corollary 6.3. $\mathrm{Aut}(II_{25,1})$ *is a split extension* $R.(\cdot\infty)$*, where* R *is the subgroup generated by reflections and* ± 1*, and* $\cdot\infty$ *is the group of automorphisms of the affine Leech lattice.*

Proof (Conway 1983). Theorem 6.1 shows that R is simply transitive on the primitive norm 0 vectors of $II_{25,1}$ that are not perpendicular to any roots, and the subgroup of $\mathrm{Aut}(II_{25,1})$ fixing one of these vectors is isomorphic to $\cdot\infty$. Q.E.D.

7. The deep holes of the Leech lattice.

Conway *et al.* (1982a) found the 23 orbits of "deep holes" in Λ and observed that they corresponded to the 23 Niemeier lattices other than Λ. Conway & Sloane (1982b,c) later gave a "holy construction" of the Leech lattice for each deep hole, and asked for a uniform proof that their construction worked. In this section I give uniform proofs of these two facts.

We continue with the notation of the previous section, so $II_{25,1} = \Lambda \oplus U$. We write Z for the set of nonzero isotropic subspaces of $II_{25,1}$, which can be identified with the set of primitive norm 0 vectors in the positive cone of $II_{25,1}$. Z can also be thought of as the rational points at infinity of the hyperbolic space of $II_{25,1}$. We introduce coordinates for Z by identifying it with the space $\Lambda \otimes Q \cup \infty$ as follows. Let z be a norm 0 vector of $II_{25,1}$ representing some point of Z. If $z = w = (0,0,1)$ we identify it with ∞ in $\Lambda \otimes \mathbf{Q} \cup \infty$. If z is not a multiple of w then $z = (\lambda, m, n)$ with $m \neq 0$ and we identify z with λ/m in $\Lambda \otimes \mathbf{Q} \cup \infty$. It is easy to check that this gives a bijection between Z and $\Lambda \otimes \mathbf{Q} \cup \infty$.

The group $\cdot\infty$ acts on Z. This action can be described either as the usual action of $\cdot\infty$ on $\Lambda \otimes \mathbf{Q} \cup \infty$ (with $\cdot\infty$ fixing ∞) or as the action of $\cdot\infty = \mathrm{Aut}(D)$ on the isotropic subspaces of $II_{25,1}$. We now describe the action of the whole of $\mathrm{Aut}(II_{25,1})$ on Z.

Lemma 7.1. *Reflection in the simple root* $r = (\lambda, 1, \frac{1}{2}\lambda^2 - 1)$ *of* D *acts on* $\Lambda \otimes \mathbf{Q} \cup \infty$ *as inversion in a sphere of radius* $\sqrt{2}$ *around* λ *(exchanging* λ *and* ∞*).*

Proof. Let v be a point of $\Lambda \otimes \mathbf{Q}$ corresponding to the isotropic subspace of $II_{25,1}$ generated by $z = (v, 1, \frac{1}{2}v^2 - 1)$. The reflection in r maps z to $z - 2(z,r)r/r^2$,

$$(z, r) = \frac{1}{2}(z^2 + r^2 - (z - r)^2) = 1 - \frac{1}{2}(v - \lambda)^2,$$

so reflection in $r^\perp$ maps z to

$$z-(1-\frac{1}{2}(v-\lambda)^2)r=(v,1,\frac{1}{2}v^2-1)-([1-\frac{1}{2}(v-\lambda)^2]\lambda,1-\frac{1}{2}(v-\lambda)^2,?)$$
$$\propto(\lambda+2(v-\lambda)/(v-\lambda)^2,1,?),$$

which is a norm 0 vector corresponding to the point $\lambda+2(v-\lambda)/(v-\lambda)^2$ of $\Lambda\otimes\mathbf{Q}\cup\infty$, and this is the inversion of v in the sphere of radius $\sqrt{2}$ about λ. Q.E.D.

Corollary 7.2. *The Niemeier lattices with roots are in natural bijection with the orbits of deep holes of* Λ *under* $\cdot\infty$*. The vertices of a deep hole form the extended Dynkin diagram of the corresponding Niemeier lattice.*

(This was first proved by Conway *et al.* (1982a), who explicitly calculated all the deep holes of Λ and observed that they corresponded to the Niemeier lattices.)

Proof. By lemma 7.1 the point v of $\Lambda\otimes\mathbf{Q}$ corresponds to a norm 0 vector in D if and only if it has distance at least $\sqrt{2}$ from every point of Λ, i.e. if and only if v is the center of some deep hole of Λ. In this case the vertices of the deep hole are the points of Λ at distance $\sqrt{2}$ from v and these correspond to the simple roots $(\lambda,1,\frac{1}{2}\lambda^2-1)$ of D in $z^\perp$.

Niemeier lattices N with roots correspond to the orbits of primitive norm 0 vectors in D other than w and hence to deep holes of $II_{25,1}$. The simple roots in $z^\perp$ form the Dynkin diagram of $z^\perp\cong N\oplus 0$, which is the extended Dynkin diagram of N. Q.E.D.

Conway & Sloane (1982b) gave a construction for the Leech lattice from each Niemeier lattice. They remarked "The fact that this construction always gives the Leech lattice still quite astonishes us, and we have only been able to give a case by case verification as follows. ... We would like to see a more uniform proof". Here is such a proof. Let N be a Niemeier lattice with a given set of simple roots whose Weyl vector is ρ. We define the vectors f_i to be the simple roots of N together with the highest roots of N (so that the f_is form the extended Dynkin diagram of N), and define the glue vectors g_i to be the vectors $v_i-\rho/h$, where v_i is any vector of N such that g_i has norm $2(1+1/h)$. By lemma 2.7 the v_is are the vectors of N closest to ρ/h and form a complete set of coset representatives for N/R, where R is the sublattice of N generated by roots. In particular the number of g_is is $\sqrt{\det(R)}$.

The "holy construction" is as follows.

(i) The vectors $\sum m_if_i+\sum n_ig_i$ with $\sum n_i=0$ form the lattice N.

(ii) The vectors $\sum m_if_i+\sum n_ig_i$ with $\sum m_i+\sum n_i=0$ form a copy of Λ.

Part (i) follows because the vectors f_i generate R and the vectors v_i form a complete set of coset representatives for N/R. We now prove (ii).

We will say that two sets are isometric if they are isomorphic as metric spaces after identifying pairs of points whose distance apart is 0. For example N is isometric to $N\oplus 0$. Let z be a primitive norm 0 vector in D corresponding to the Niemeier lattice N. The sets of vectors f_i and g_i are isometric to the sets of simple roots f_i' and g_i' of D, which have inner product 0 or -1 with z. (This follows by applying Vinberg's algorithm with z as a controlling vector.)

Lemma 7.3. *The vectors f_i' and g_i' generate $II_{25,1}$.*

Proof. The vectors $\sum m_i f_i' + n_i g_i'$ with $\sum n_i = 0$ are just the vectors in the lattice generated by f_i' and g_i', which are in $z^\perp$, and they are isometric to N because the vectors $\sum m_i f_i + \sum n_i g_i$ with $\sum n_i = 0$ form a copy of N. The vector z is in the lattice generated by the f_i's, so the whole of $z^\perp$ is contained in the lattice generated by the vectors f_i and g_i. The vectors g_i' all have inner product 1 with z, so the vectors f_i' and g_i' generate $II_{25,1}$. Q.E.D.

The lattice of vectors $\sum m_i f_i' + \sum n_i g_i'$ with $\sum m_i + \sum n_i = 0$ is $w^\perp$ by lemma 2.7 because $(f_i', w) = (g_i', w) = 1$. $w^\perp$ is $\Lambda \oplus 0$, which is isometric to Λ and therefore isomorphic to Λ because it is contained in the positive definite space $N \otimes \mathbf{Q}$. This proves (ii).

The holy construction for Λ is equivalent to the $(\rho, h, h+1)^\perp$ construction of §4, so that $(\rho, h, h+1)$ is always a norm 0 vector corresponding to Λ.

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