Problems in moonshine.

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The talk at the I.C.C.M. was an introduction to moonshine. As there are already several survey articles on what is known about moonshine ([B94], [B99], [D-M94], [D-M96], [F-H98], [J]), this paper differs from the talk and will concentrate mainly on what we do not know about it.

Moonshine is not a well defined term, but everyone in the area recognizes it when they see it. Roughly speaking, it means weird connections between modular forms and sporadic simple groups. It can also be extended to include related areas such as infinite dimensional Lie algebras or complex hyperbolic reflection groups. Also, it should only be applied to things that are weird and special: if there are an infinite number of examples of something, then it is not moonshine.

We first quickly review the original moonshine conjectures of McKay, Thompson, Conway and Norton [C-N]. The classification of finite simple groups shows that every finite simple group either fits into one of about 20 infinite families, or is one of 26 exceptions, called sporadic simple groups. The monster simple group is the largest of the sporadic finite simple groups, and was discovered by Fischer and Griess [G]. Its order is

$$\begin{aligned}&8080, 17424, 79451, 28758, 86459, 90496, 17107, 57005, 75436, 80000, 00000\\ =&2^{46}.3^{20}.5^{9}.7^{6}.11^{2}.13^{3}.17.19.23.29.31.41.47.59.71\end{aligned}$$

(which is roughly the number of elementary particles in the earth). The smallest irreducible representations have dimensions $1, 196883, 21296876, \ldots$. On the other hand the elliptic modular function $j(\tau)$, defined by

$$j(\tau) = \frac{\left(1 + 240\sum_{n>0}\sigma_3(n)q^n\right)^3}{q\prod_{n>0}(1-q^n)^{24}}$$

has the power series expansion

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \ldots$$

where $q = e^{2\pi i\tau}$. John McKay noticed some rather weird relations between coefficients of the elliptic modular function and the representations of the monster as follows:

$$\begin{aligned}1 &= 1\\ 196884 &= 196883 + 1\\ 21493760 &= 21296876 + 196883 + 1\end{aligned}$$

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where the numbers on the left are coefficients of $j(\tau)$ and the numbers on the right are dimensions of irreducible representations of the monster. The term "monstrous moonshine" (coined by Conway) refers to various extensions of McKay's observation, and in particular to relations between sporadic simple groups and modular functions.

McKay and Thompson suggested that there should be a graded representation $V = \oplus_{n\in\mathbf{Z}} V_n$ of the monster, such that $\dim(V_n) = c(n-1)$, where $j(\tau) - 744 = \sum_n c(n)q^n = q^{-1} + 196884q + \cdots$. To characterize V, Thompson suggested looking at the McKay-Thompson series

$$T_g(\tau) = \sum_n Tr(g|V_n)q^{n-1}$$

for each element g of the monster. For example, $T_1(\tau)$ should be the elliptic modular function. Conway and Norton [C-N] calculated the first few terms of each McKay-Thompson series by making a reasonable guess for the decomposition of the first few V_n's into irreducible representations of the monster. They discovered the astonishing fact that all the McKay-Thompson series appeared to be Hauptmoduls for certain genus 0 subgroups of $SL_2(\mathbf{R})$. (A Hauptmodul for a subgroup Γ is an isomorphism from $\Gamma\backslash H$ to $\mathbf{C}$, normalized so that its Fourier series expansion starts off $q^{-1} + O(1)$.)

The module V was constructed explicitly as a representation of the monster group in [F-L-M], and it was shown in [B92] that this module satisfies the moonshine conjectures, using the fact that it has the structure of a vertex algebra. In the rest of this paper we will look at various conjectural ways to generalize this.

Problem 1. Find a "natural" construction for the monster vertex algebra V. All known constructions for it construct it as the sum of several (usually two) different pieces, and it takes a lot of work to show that the vertex algebra structure can be defined on this sum, and to show that the monster acts on it. It would be much nicer to have some sort of construction which gives V as "just one piece".

One idea for doing this might be to "stabilize" the monster vertex algebra by tensoring it with copies of the vertex algebra of the 2-dimensional even Lorentzian lattice $II_{1,1}$. The reason for this is that if we add this lattice to any Niemeier lattice, we get the same answer $II_{25,1}$, so the Niemeier lattices can be recovered from norm 0 vectors in $II_{25,1}$. Moreover the proof of the moonshine conjectures involves taking a tensor product with the vertex algebra of $II_{1,1}$, showing that this is a natural operation. The most naive version of this question would be to ask if the monster vertex algebra tensored with the vertex algebra of $II_{1,1}$ is isomorphic to the vertex algebra of $II_{25,1}$. This seems rather unlikely, though as far as I know it has not been disproved.

Problem 2. Construct a good integral form on the monster vertex algebra V. This should have the property that each of the homogeneous pieces of V has a self dual symmetric bilinear form on it. This integral form might follow from a good answer to problem 1, but might also be possible to prove independently as follows. The usual construction for the monster vertex algebra gives a self dual integral form "up to 2-torsion". If we could carry out a similar construction "up to p-torsion" for some other prime p then it might be possible to splice these together to get a good integral form. There are some constructions [D-M92], [M96] of the monster vertex algebra which look as if they might extend to give a construction up to 3-torsion. One advantage of a good integral form for the monster vertex algebra is that it would make the study of modular moonshine [R], [B-R], [B98] somewhat easier.

Problem 3. Find an affine ind group scheme over the integers for the monster Lie algebra. (Here an affine ind group scheme is roughly a commutative linearly topologized Hopf algebra, in much the same way that an affine group scheme is more or less the same as a commutative Hopf algebra. For infinite dimensional groups, it seems to be necessary to allow linear topologies on the Hopf algebra because there is often no natural map from H to $H \otimes H$, but only to the completion $H \hat{\otimes} H$.) Over the rational numbers it is not hard to find an ind group scheme whose Lie algebra is the monster Lie algebra and whose group of connected components is the monster group. The question is whether one can find a good integral form for this ind group scheme. There are also several intermediate questions: can one find a good integral form for the monster Lie algebra (which would follow from a good integral form for the monster vertex algebra) and can one find a formal group over the integers for the monster Lie algebra? At the moment it is not really clear what one would do with such an ind group scheme. Perhaps it could be used to study infinite dimensional automorphic forms corresponding to the monster Lie algebra, or perhaps its points with values in finite fields might be useful groups. For the related case of the fake monster Lie algebra there is some progress on these questions in [B99a], but the monster Lie algebra seems harder because it is not generated by the root spaces of roots of zero or positive norm.

Problem 4. Prove Norton's generalized moonshine conjectures [N86].

Roughly speaking, these conjectures assign modular functions to pairs of commuting elements of the monster, rather than to elements. The point is that the group $SL_2(\mathbf{Z})$ acts not only on (genus 0) modular functions but also on pairs of commuting elements of any group. Some progress has been made on Norton's conjectures by Dong, Li, and Mason [D-L-M], who proved the generalized moonshine conjectures in the case when g and h generate a cyclic group by reducing to the case when $g = 1$ (the ordinary moonshine conjectures). G. Höhn [H] has made some progress in the harder case when g and h do not generate a cyclic group by constructing the required modules for the baby monster (when g is of type $2A$). It seems likely that his methods would also work for the Fischer group Fi_{24}, but it is not clear how to go further than this. Dong has recently nearly proved the generalized moonshine conjectures. Very roughly speaking, his proof is an extension of Zhu's proof that characters of certain vertex algebras are modular functions to the equivariant case.

Problem 5. Explain the connection (if any) between moonshine and the "Y presentation" of the monster [ATLAS]. The latter is a particularly easy presentation of the monster group conjectured by Conway and Norton and finally proved by Ivanov (see [I]), which looks roughly like the presentation of a Coxeter group with one extra relation. Miyamoto [M95] found a very suggestive relationship between this presentation and the monster vertex algebra, by finding a set of involutions of the monster vertex algebra satisfying the necessary relations. His proof used properties of the E_6^4 Niemeier lattice.

Problem 6. Classify the generalized Kac-Moody algebras that have Weyl vectors and whose denominator functions are automorphic forms (possibly with singularities). The simplest example of such a Lie algebra is the monster Lie algebra itself, with denominator function $j(\sigma) - j(\tau)$. There are plenty of other similar rank 2 Lie algebras, some related to moonshine and sporadic groups, and some which do not seem to be related to sporadic groups. There are also many examples in various ranks up to 26, which is probably the highest possible rank. It is possible to relax the condition on the generalized Kac-Moody algebra that a Weyl vector should exist, and ask just for those whose denominator function is an automorphic form. Gritsenko and Nikulin have classified some of these algebras in low ranks [G-N], [N].

Problem 7. Find all "interesting" hyperbolic reflection groups (this includes the arithmetic ones and some others). This is closely related to the problem of finding interesting generalized Kac-Moody algebras, because the Weyl group of an interesting generalized Kac-Moody algebra is often an interesting hyperbolic reflection group. Nikulin has proved that the number of hyperbolic reflection groups which are close to being arithmetic (in various senses) is finite, and known examples suggest that there are probably a few thousand of them. There seems to be a rather mysterious connection of these hyperbolic reflection groups to genus 0 modular groups; slightly more precisely, most of the interesting hyperbolic reflection groups seem to be associated with certain modular forms with poles of low order at cusps. For example, Conway's reflection group of $II_{1,25}$ is associated with the function $1/\Delta(\tau)$. However this correspondence is not well understood; see [B99b] for some examples.

Problem 8. Find "natural" constructions of the generalized Kac-Moody algebras in the previous problems. All these algebras can be constructed using generators and relations, but this is not a satisfactory way of constructing them; for example, it is very hard to see interesting symmetry groups of these algebras (such as the monster group) in this approach. What we would prefer is a construction in which one can explicitly see the interesting symmetry groups. For example, the monster Lie algebra is constructed from the monster vertex algebra as the Lie algebra of physical states of strings on a 26 dimensional orbifold, so the action of the monster can be seen directly because it acts on the monster vertex algebra. Scheithauer [S] has recently found a similar construction for the "fake monster Lie superalgebra" of rank 10. Harvey and Moore [H-M] have made an exciting suggestion for realizing these algebras using BPS states, or by constructing them using the cohomology groups of moduli spaces of vector bundles of surfaces.

Problem 9. Is there anything like moonshine for the 6 sporadic groups not involved in the monster group? Note that most and probably all the Chevalley groups act naturally on vertex algebras over finite fields, and many of the sporadic groups involved in the monster act naturally on vertex algebras, sometimes over finite fields as in modular moonshine and sometimes over the rational numbers as for the monster. This means that most finite simple groups act naturally on vertex algebras, and an obvious question is whether they all do. Presumably any simple group acting on a natural vertex algebra would have some sort of connection with modular functions. Unfortunately (as far as I know) no one has ever found any serious evidence that the remaining 6 sporadic groups (the Janko groups J_1, J_3, and J_4, the O'Nan group, the Lyons group, and the Rudvalis group) have any connection with vertex algebras or modular functions.

Problem 10. Explain McKay's weird observation (described in [B99] for example) relating the Dynkin diagrams E_8, E_7, and E_6 with the monster, the baby monster, and the group Fi_{24}. The point is that conjugacy classes of pairs of commuting involutions of the monster seem to correspond to vertices of the affine E_8 diagram, and there is a similar connection between the baby monster and the E_7 Dynkin diagram, and between the group $Fi_{24}.2$ and E_6. This may be related to the (well understood) McKay correspondence between Dynkin diagrams and finite rotation groups in 3 dimensions.

Problem 11. Lian and Yau [L-Y] showed that mirror maps for K3 surfaces are sometimes the inverses of Hauptmoduls; for example one of their mirror maps is

$$q - 744q^2 + 356652q^3 + \cdots$$

which is the inverse of the elliptic modular function $j(\tau)$. They found that several other Hauptmoduls of elements turned up. They asked the rather speculative question about whether there is some direct connection between the monster and K3 surfaces. This seems rather wild as there is no obvious way in which the monster could be connected with K3 surfaces, but on the other hand this is what most people said about McKay's original observation connecting the monster and elliptic modular functions!

Problem 12. Is there a monster manifold? More precisely, Hirzebruch asked if there was a 24 dimensional manifold acted on by the monster with Witten genus $j(\tau) - 744$. If so, it might be possible to use this to construct the monster vertex algebra, or at least its underlying space. Hopkins and Mahowald recently constructed a manifold with the correct dimension and Witten genus, but so far it is unclear how to construct an action of the monster on it.

Problem 13. Complex hyperbolic reflection groups. Allcock [A] recently constructed some striking examples of complex hyperbolic reflection groups from the Leech lattice, or more precisely from the complex Leech lattice, a 12 dimensional lattice over the Eisenstein integers. This complex reflection group looks similar in several ways to Conway's real hyperbolic reflection group of the lattice $II_{1,25}$. Allcock also showed that there is an automorphic form on complex hyperbolic space vanishing exactly on the reflection hyperplanes of this reflection group. Several other complex hyperbolic reflection groups found by Allcock are closely related to various moduli spaces, for example the moduli space of cubic surfaces [A-C-T]. Most of these complex hyperbolic reflection groups seem to have something to do with moonshine, though it is hard to be precise about what the relationship is. For example many of them are related to automorphic forms that in turn are related to moonshine. So a general and rather vague question is: what is going on? For some more specific questions we could ask for a classification of arithmetic complex hyperbolic reflection groups, and for each of them we can ask if it is related to some moduli space and some automorphic form. There is some speculation that finite complex reflection groups should be related to some so far unknown algebraic structures provisionally called "spetses" [M99] in the same way that the real reflection groups are related to Lie algebras. As the real hyperbolic reflection groups are closely related to Kac-Moody algebras, an obvious question is to ask if there is an extension of spetses to complex hyperbolic reflection groups.

Problem 14. Is there a "nice" moduli space related to $II_{1,25}$? The denominator function of the rank 10 fake monster Lie superalgebra turns out to be an automorphic form on the period space of Enriques surfaces vanishing exactly along the points of singular Enriques surfaces [B96]. A direct construction of this automorphic form was found by Harvey and Moore [H-M98] and Yoshikawa [Y], following a suggestion of Jorgenson and

Todorov [J-T]. (Note that, as pointed out in [Y], the main theorem stated in [J-T] is not correct.) The fake monster Lie superalgebra and the fake monster Lie algebra seem similar in many ways, so we can ask if there is a similar moduli space related to the fake monster Lie algebra and its root lattice $II_{1,25}$. So the period space should be the hermitian symmetric space corresponding to the lattice $II_{2,26}$, with singular points along the divisors of norm -2 vectors, and the moduli space should be the quotient by the automorphism group of the lattice $II_{2,26}$. One can also ask similar questions about other generalized Kac-Moody algebras. Freitag has suggested that maybe most "interesting" moduli spaces should arise in a similar way, related to automorphic forms on the hermitian symmetric space of $\mathbf{R}^{2,n}$ that have an infinite product expansion and all of whose zeros correspond to roots of some lattice. See [F-H] for some examples.

Problem 15. Modular forms with poles at cusps often turn up in the questions in this paper. Is there some useful analogue for these forms of the L-functions or Dirichlet series of classical cusp forms? Note that the obvious Dirichlet series formed by the coefficients does not converge anywhere, so we have to use a different way to define the "L-function". One possibility is to just use the Mellin transform. This does not converge if there are poles at cusps, so it would be necessary to regularize the integral in some way. Another possibility would be to look at functions with poles at some cusps and zeros at others, and just integrate between two zeros. The resulting Mellin transforms would satisfy some functional equation coming from the functional equation of modular forms in the usual way, but it is hard to see what else one can say about them. Perhaps one could look at a set of modular forms with singularities invariant under the Hecke algebra and ask what the corresponding properties of the Mellin transforms are. (In the case of forms without singularities, the Hecke algebra has eigenfunctions and the action of the Hecke algebra on the Eigenfunctions corresponds to an Euler product decomposition on the L-series. However if the modular forms have cusps then the Hecke algebra tends to act freely because Hecke operators make singularities worse, so there are no eigenfunctions.)

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