The Gross-Kohnen-Zagier theorem in higher dimensions. *1 Oct and 31 Dec 1997, 4 March 2000.*

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1. Introduction.

The Gross-Kohnen-Zagier theorem [G-K-Z] says roughly that the Heegner divisors of a modular elliptic curve are given by coefficients of a vector valued modular form of weight 3/2. We will give another proof of this (see theorem 4.5 and example 5.1), which extends to some more general quotients of hermitian symmetric spaces of dimensions b^- and shows that formal power series whose coefficients are higher dimensional generalizations of Heegner divisors are vector valued modular forms of weight $1 + b^-/2$.

The main idea of the proof of theorem 4.5 is easy to state. One of the main results of [B] is a correspondence from modular forms of weight $1 - b^-/2$ with singularities to automorphic forms with known zeros and poles, which give relations between Heegner divisors and their higher dimensional generalizations. On the other hand Serre duality for modular forms says that the only obstructions to finding modular forms of weight $1-b^-/2$ with given singularities are given by modular forms of weight $1 + b^-/2$. In other words the only obstructions to finding relations between Heegner divisors are given by certain modular forms of weight $1 + b^-/2$. It is a formal consequence of this that the Heegner divisors themselves are the coefficients of a modular form of weight $1 + b^-/2$. The idea of using the results of [B] to prove relations between Heegner divisors was suggested to me by R. L. Taylor.

Most of the more interesting special cases of theorem 4.5 in low dimensions are already known, though it does at least simplify and unify several previous proofs of known results. For modular curves the theorem is more or less the same as the main result (Theorem C) of [G-K-Z] stating that Heegner divisors on modular curves are given by coefficients of a Jacobi form of weight 2. (See example 5.1.) The main difference is that we prove the result for all Heegner divisors while the authors of [G-K-Z] restrict to the case of Heegner divisors of discriminant coprime to the level for simplicity, though their method could probably be extended to cover all Heegner divisors. The only reason this has not been done before (as far as I know) seems to be that it would take a lot of extra work for a rather small improvement to the result. Hayashi [H] has extended the results of [G-K-Z] to some of the other discriminants.

There is a similar result for CM points on Shimura curves (example 5.3). The abelian varieties in the Jacobians of Shimura curves are also in the Jacobians of modular curves, so it is likely that the result for Shimura curves is a formal consequence of the result of [G-K-Z] for modular curves.

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In two dimensions theorem 4.5 shows that Hirzebruch-Zagier cycles on a Hilbert modular surfaces are coefficients of a modular form of weight $1+b^-/2=2$ (example 5.5). This is the main result of [H-Z]. There are several other higher dimensional generalizations of Hirzebruch and Zagier's result about intersection products of cycles; see for example [K].

In 3 dimensions we recover the result of van de Geer [G] that Humbert surfaces on a Siegel modular variety of dimension 3 are the coefficients of a modular form of weight $1+b^-/2=5/2$. There is also a 4 dimensional example due to Hermann [He], who showed that the degrees of some 3 dimensional modular varieties for the group $Sp_4(\mathbf{R})$ in a modular variety for the group $U(2,2)$ were the coefficients of a modular form of weight $1+b^-/2=3$.

Most previous proofs of the results above prove relations between divisors by considering an inner product on a space of divisors, either the Néron-Tate inner product in the case of modular curves, or the intersection number in the case of Hilbert modular surfaces. The proof in this paper says nothing about inner products, but instead proves relations between divisors by explicitly constructing automorphic forms with known poles and zeros. Unfortunately this means that there is no obvious way to use the methods of this paper to prove the Gross-Zagier result relating Heegner divisors to the vanishing of the derivative of an L series.

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2. Modular forms.

In this section we summarize some standard results about modular forms and set up notation for the rest of the paper.

Recall that the group $SL_2(\mathbf{R})$ has a double cover $Mp_2(\mathbf{R})$ called the metaplectic group whose elements can be written in the form

$$\left(\begin{pmatrix} ab \\ cd \end{pmatrix}, \pm\sqrt{c\tau+d}\right)$$

where $\binom{ab}{cd} \in SL_2(\mathbf{R})$ and $\sqrt{c\tau+d}$ is considered as a holomorphic function of τ in the upper half plane whose square is $c\tau+d$. The multiplication is defined so that the usual formulas for the transformation of modular forms work for half integer weights, which means that

$$(A, f(\cdot))(B, g(\cdot)) = (AB, f(B(\cdot))g(\cdot))$$

for $A, B \in SL_2(\mathbf{R})$ and f, g suitable functions on H. The group $Mp_2(\mathbf{Z})$ is the discrete subgroup of $Mp_2(\mathbf{R})$ of elements of the form $\left(\binom{ab}{cd}, \pm\sqrt{c\tau+d}\right)$ with $\binom{ab}{cd} \in SL_2(\mathbf{Z})$.

Suppose that Γ is a subgroup of $Mp_2(\mathbf{R})$ commensurable with $Mp_2(\mathbf{Z})$, and suppose that ρ is a representation of Γ on a finite dimensional complex vector space V_ρ which factors through a finite quotient of Γ such that $\rho = \sigma_k$ on $\Gamma \cap K$. Choose $k \in \frac{1}{2}\mathbf{Z}$. We define a modular form of weight k and type ρ to be a holomorphic function f on the upper half plane H with values in the vector space V_ρ such that

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \sqrt{c\tau+d}^{2k} \rho_M\left(\left(\begin{pmatrix} ab \\ cd \end{pmatrix}, \sqrt{c\tau+d}\right)\right) f(\tau)$$

for elements $\left(\binom{ab}{cd}, \sqrt{c\tau+d}\right)$ of Γ. (We allow singularities at cusps.)

A modular form has a Fourier expansion at the cusp at infinity as follows. The Fourier coefficients $c_{n,\gamma} \in \mathbf{C}$ of f are defined by

$$f(\tau) = \sum_{n\in\mathbf{Q}} \sum_{\gamma} c_{n,\gamma} q^n e_\gamma$$

where q^n means $e^{2\pi i n\tau}$ and where the sum runs over a basis e_γ of V_ρ consisting of eigenvectors of T. Note that n is not necessarily integral; more precisely, $c_{n,\gamma}$ is nonzero only if $n \equiv \lambda_\gamma \bmod 1$, where the eigenvalue of T on e_γ is $e^{2\pi i\lambda_\gamma}$. We say that f is meromorphic at the cusp $i\infty$ if $c_{n,\gamma} = 0$ for $n << 0$, and we say f is meromorphic at the cusp a/c if $f((a\tau+b)/(c\tau+d))$ is meromorphic at $i\infty$ for $\binom{ab}{cd} \in SL_2(\mathbf{Z})$. We say that f is holomorphic at cusps if the coefficients of the Fourier expansions at all cusps vanish for $n < 0$. We will write $ModForm(\Gamma, k, \rho)$ for the space of modular forms of weight k and representation ρ for Γ which are meromorphic at cusps, and $HolModForm(\Gamma, k, \rho)$ for the subspace of modular forms which are holomorphic at all cusps.

A particularly important example ρ_M of a representation ρ as above can be constructed as follows. We let M be a nonsingular even lattice of signature (b^+, b^-), with dual M'. The quotient M'/M is a finite group whose order is the absolute value of the discriminant of the lattice M. The mod 1 reduction of $(\lambda,\lambda)/2$ is a $\mathbf{Q}/\mathbf{Z}$-valued quadratic form on M'/M, whose associated $\mathbf{Q}/\mathbf{Z}$-valued bilinear form is the mod 1 reduction of the bilinear form on M'. We let the elements e_γ for $\gamma \in M'/M$ be the standard basis of the group ring $\mathbf{C}[M'/M]$, so that $e_\gamma e_\delta = e_{\gamma+\delta}$. The Grassmannian $G(M)$ of M is defined to be the space of all b^+-dimensional positive definite subspaces of $M \otimes \mathbf{R}$. It is a symmetric space acted on by the orthogonal group $O_M(\mathbf{R})$, and if $b^+ = 2$ it is a hermitian symmetric space.

Recall that there is a unitary representation ρ_M of the double cover $Mp_2(\mathbf{Z})$ of $SL_2(\mathbf{Z})$ on $V_{\rho_M} = \mathbf{C}[M'/M]$ defined by

$$\rho_M(T)(e_\gamma) = e^{2\pi i(\gamma,\gamma)/2} e_\gamma$$

$$\rho_M(S)(e_\gamma) = \frac{\sqrt{i}^{b^- - b^+}}{\sqrt{|M'/M|}} \sum_{\delta\in M'/M} e^{-2\pi i(\gamma,\delta)} e_\delta$$

where $T = (\binom{11}{01}, 1)$ and $S = (\binom{0-1}{1\ \ 0}, \sqrt{\tau})$ are the standard generators of $Mp_2(\mathbf{Z})$, with $S^2 = (ST)^3 = Z$, $Z = (\binom{-1\ \ 0}{\ \ 0-1}, i)$, $\rho_M(Z)(e_\gamma) = i^{b^- - b^+} e_{-\gamma}$, $Z^4 = 1$. The representation ρ_M factors through the double cover $Mp_2(\mathbf{Z}/N\mathbf{Z})$ of the finite group $SL_2(\mathbf{Z}/N\mathbf{Z})$, where N is the smallest integer such that $N(\gamma,\delta)$ and $N\gamma^2/2$ are integers for all $\gamma, \delta \in M'$. In particular the representation ρ_M factors through a finite quotient of $Mp_2(\mathbf{Z})$. Also note that there is only one cusp in this case, so that a modular form is holomorphic if and only if it is holomorphic at $i\infty$.

3. Serre duality.

In this section we show that the space of obstructions to finding modular forms with given singularities at cusps is dual to a space of holomorphic modular forms. We prove this by identifying both spaces with cohomology groups of line bundles over modular curves; the result we want then follows immediately from Serre duality.

The space $HolModForm(\Gamma, k, \rho)$ can be identified with the space of holomorphic sections of the following vector bundle $\mathcal{V}_{k,\rho}$ on $\Gamma\backslash H$, and similarly $ModForm(\Gamma, k, \rho)$ can be identified with the space of sections which are holomorphic outside the cusps. We first do the special case when ρ is the trivial representation, when $\mathcal{V}_{k,\rho}$ becomes a line bundle $\mathcal{L}_k$. Let K be the inverse image of $SO(2)$ in $Mp_2(\mathbf{R})$, and, for $k \in \frac{1}{2}\mathbf{Z}$, define the character σ_k of K by

$$\sigma_k(\theta) = (\pm\sqrt{ci+d})^{2k}$$

for $\theta = (\binom{ab}{cd}, \pm\sqrt{c\tau + d})$. Note that

$$H \simeq SL_2(\mathbf{R})/SO(2) \simeq Mp_2(\mathbf{R})/K,$$

so that there is a homogeneous holomorphic line bundle $\mathcal{L}_k$ on H defined by

$$\mathcal{L}_k = \Big(Mp_2(\mathbf{R}) \times \mathbf{C}\Big)/K,$$

where $\theta \in K$ acts by $(g, z)\theta = (g\theta, \sigma_k(\theta)^{-1}z)$. For a representation ρ of Γ on V_ρ such that $\rho = \sigma_k^{-1}$ on $\Gamma \cap K$, define a holomorphic vector bundle $\mathcal{V}_{k,\rho}$ over $\Gamma\backslash H$ by

$$\mathcal{V}_{k,\rho} = \Gamma\backslash\Big(Mp_2(\mathbf{R}) \times V_\rho\Big)/K,$$

where K acts as before, and $\gamma \in \Gamma$ acts by $\gamma(g, v) = (\gamma g, \rho(\gamma)v)$.

If κ is a cusp of Γ, let q_κ be a uniformizing parameter at κ on $\Gamma\backslash H$. For a representation ρ on V_ρ, let V_ρ^* denote the dual. Let

$$PowSer_\kappa(\Gamma, \rho) = \mathbf{C}[[q_\kappa]] \otimes V_\rho$$

be the space of formal power series in q_κ with coefficients in V_ρ, let

$$Laur_\kappa(\Gamma, \rho) = \mathbf{C}[[q_\kappa]][q_\kappa^{-1}] \otimes V_\rho$$

be the space of formal Laurent series in q_κ with coefficients in V_ρ, and let

$$Sing_\kappa(\Gamma, \rho) = \frac{Laur_\kappa(\Gamma, \rho)}{q_\kappa PowSer_\kappa(\Gamma, \rho)}$$

be the space of possible singularities and constant terms of V_ρ valued Laurent series at κ. The two spaces $PowSer_\kappa(\Gamma, \rho^*)$ and $Sing_\kappa(\Gamma, \rho)$ are paired into $\mathbf{C}$ by taking the residue

$$\langle f, \phi\rangle = \mathrm{Res}\big(f\phi\, q_\kappa^{-1}\, dq_\kappa\big),$$

for $f \in PowSer_\kappa(\Gamma, \rho^*)$ and $\phi \in Sing_\kappa(\Gamma, \rho)$. Here the product of f and ϕ is defined using the pairing of V_ρ and V_ρ^*.

Then the spaces

$$Sing(\Gamma,\rho) = \oplus_\kappa Sing_\kappa(\Gamma,\rho)$$

and

$$PowSer(\Gamma,\rho^*) = \oplus_\kappa PowSer_\kappa(\Gamma,\rho^*),$$

where κ runs over the Γ-inequivalent cusps, are paired by the sum of the local pairings at the cusps.

There are maps

$$\lambda : HolModForm(\Gamma,k,\rho^*) \longrightarrow PowSer(\Gamma,\rho^*)$$

and

$$\lambda : ModForm(\Gamma,2-k,\rho) \longrightarrow Sing(\Gamma,\rho),$$

defined in the obvious way by taking the Fourier expansions of their nonpositive part at the various cusps.

We define the space $Obstruct(\Gamma,k,\rho)$ of obstructions to finding a modular form of type ρ and weight k which is holomorphic on H and has given meromorphic singularities and constant terms at the cusps to be the space

$$Obstruct(\Gamma,k,\rho) = \frac{Sing(\Gamma,\rho)}{\lambda(ModForm(\Gamma,k,\rho))}.$$

Theorem 3.1. *Suppose that $k \in \frac{1}{2}\mathbf{Z}$, Γ is a subgroup of $Mp_2(\mathbf{R})$ which is commensurable with $Mp_2(\mathbf{Z})$ and ρ is a finite dimensional complex representation of Γ factoring through a finite quotient of Γ such that $\rho = \sigma_k$ on $\Gamma \cap K$. Then the space of obstructions $Obstruct(\Gamma,2-k,\rho)$ is finite dimensional and dual to the space $HolModForm(\Gamma,k,\rho^*)$. The pairing between them is induced by the above pairing between $Sing(\Gamma,\rho)$ and $PowSer(\Gamma,\rho^*)$. In other words,*

$$\lambda\Big(ModForm(\Gamma,2-k,\rho)\Big) = \lambda\Big(HolModForm(\Gamma,k,\rho^*)\Big)^\perp,$$

and also, since the pairing is nondegenerate,

$$\lambda\Big(HolModForm(\Gamma,k,\rho^*)\Big) = \lambda\Big(ModForm(\Gamma,2-k,\rho)\Big)^\perp.$$

Proof. Suppose first that the group Γ acts freely on the upper half plane H, and the representation ρ is one dimensional and trivial. For any $k \in \frac{1}{2}\mathbf{Z}$ we let $\mathcal{L}_k$ be the line bundle defined above whose sections are holomorphic modular forms of weight k. We let $\mathcal{L}_{cusp}$ be the line bundle corresponding to the divisor which is the union of all cusps of the compactification of $\Gamma\backslash H$. The canonical line bundle is isomorphic to $\mathcal{L}_2 \otimes \mathcal{L}^*_{cusp}$ because holomorphic 1-forms are essentially the same as cusp forms of weight 2. By Serre duality we see that the space $H^0(\mathcal{L}_k)$ of holomorphic forms of weight k is dual to $H^1(\mathcal{L}_2 \otimes \mathcal{L}^*_{cusp} \otimes \mathcal{L}^*_k) = H^1(\mathcal{L}_{2-k} \otimes \mathcal{L}^*_{cusp})$. If $\mathcal{L}$ is any line bundle over a compact Riemann surface then the cohomology group $H^1(\mathcal{L})$ can be identified with the space of obstructions to finding a meromorphic section of $\mathcal{L}$ with given singularities at some fixed nonempty finite set of points and holomorphic elsewhere. Hence $H^1(\mathcal{L}_{2-k} \otimes \mathcal{L}^*_{cusp})$ is the space of obstructions to finding a meromorphic section of $\mathcal{L}_{2-k} \otimes \mathcal{L}^*_{cusp}$ with given singularities at cusps and holomorphic elsewhere (since there is at least one cusp). This in turn is the space of obstructions $Obstruct(\Gamma, 2-k, \mathbf{C})$ to finding a meromorphic section of $\mathcal{L}_{2-k}$ with given singularities and constant terms at cusps and holomorphic elsewhere. The pairing in Serre duality is the one above given by taking the sum of residues of the product. This proves theorem 3.1 in the case that Γ acts trivially on ρ and fixed point freely on H.

In the general case choose a finite index subgroup Γ_0 of Γ such that Γ_0 acts trivially on ρ and acts fixed point freely on H. Then the cohomology groups H^1 and H^0 for Γ are just the fixed points under Γ/Γ_0 of the corresponding cohomology groups for Γ_0. The theorem for Γ now follows from the fact that if we have a perfect duality between finite dimensional complex vector spaces that is invariant under some finite group Γ/Γ_0 acting on these vector spaces, then we also get a perfect duality between the fixed points of Γ/Γ_0 (as follows from the fact that finite dimensional complex representations of finite groups are completely reducible). This proves theorem 3.1.

4. Heegner divisors as Fourier coefficients.

In this section we prove the main theorem 4.5, saying roughly that Heegner divisors are the Fourier coefficients of a modular form. We do this by using the modular forms with singularities constructed in section 3 to find a large number of relations between Heegner divisors.

We let M be an even lattice of signature $(2, b^-)$. Suppose that Γ is a discrete group acting on the Grassmannian $G(M)$ of M. We define a divisor on $X_\Gamma = \Gamma\backslash G(M)$ to be a locally finite Γ invariant divisor on $G(M)$ whose support is a locally finite union of a finite number of Γ-orbits of irreducible codimension 1 subvarieties of $G(M)$. This definition is a sort of crude substitute for the definition of a divisor on the "orbifold" (or algebraic stack) X_Γ but is adequate for the purposes of this paper. Note that this is not quite the same as a divisor on the complex analytic space X_Γ; for example, if $G(M)$ is the upper half plane H and Γ is $SL_2(\mathbf{Z})$ then the image of the point i represents a divisor in $\Gamma\backslash H$, which is twice a divisor in the orbifold $\Gamma\backslash H$ but not in the complex manifold $\Gamma\backslash H$.

Recall that for any negative norm vector $v \in M \otimes \mathbf{R}$ there is a divisor $v^\perp$ of $G(M)$, equal to the points of the Grassmannian represented by 2-planes orthogonal to v. If n is any negative rational number and $\gamma \in M'/M$ then we define the Heegner divisor $y_{n,\gamma}$ to be the sum of the divisors of all norm $2n$ vectors of $M+\gamma$. Note that $y_{n,\gamma} = y_{n,-\gamma}$ because if $v \in M+\gamma$ then $-v \in M-\gamma$. We define the group $Heeg(X_\Gamma)$ of Heegner divisors to be the direct sum of a copy of $\mathbf{Z}$ generated by a symbol $y_{0,0}$, and the subgroup of the group of divisors generated by the Heegner divisors $y_{n,\gamma}$. If $n > 0$ or $n = 0, \gamma \neq 0$ then we define $y_{n,\gamma}$ to be 0. We define a Heegner divisor to be principal if it is of the form $c_{0,0}y_{0,0} + D$, where D is the divisor of a meromorphic automorphic form of weight $c_{0,0}/2$ for some integer $c_{0,0}$ and some unitary character of finite order of the subgroup $\mathrm{Aut}(M)$ fixing all elements of M'/M. Here the weight of an automorphic form is the weight used in theorem 13.3 of [B]. We write $PrinHeeg(X_\Gamma)$ for the subgroup of principal Heegner divisors, and $HeegCl(X_\Gamma)$ for the group $Heeg(X_\Gamma)/PrinHeeg(X_\Gamma)$ of Heegner divisor classes. (The published version of this paper omitted the condition that the character have finite order. J. Bruinier pointed out to me that there are sometimes "too many" automorphic forms with characters of infinite order (see [F]), so that if infinite order characters are allowed the group of Heegner divisor classes would sometimes collapse.)

There is a surjective linear map

$$\xi : Sing(Mp_2(\mathbf{Z}), \rho_M) \longrightarrow Heeg(X_\Gamma) \otimes_{\mathbf{Z}} \mathbf{C}$$

taking $q^n e_\gamma$ to $y_{n,\gamma}$. For a subring F of $\mathbf{C}$ let $Sing(Mp_2(\mathbf{Z}), \rho_M)_F$ be the F-submodule of $Sing(Mp_2(\mathbf{Z}), \rho_M)$ for which the coefficients of $q^n e_\gamma$ for $n \leq 0$ are in F, and let

$$ModForm(Mp_2(\mathbf{Z}), 1 - b^-/2, \rho_M)_{\mathbf{Z}} \subseteq ModForm(Mp_2(\mathbf{Z}), 1 - b^-/2, \rho_M)$$

be the $\mathbf{Z}$-submodule whose image under λ lies in $Sing(Mp_2(\mathbf{Z}), \rho_M)_{\mathbf{Z}}$.

Theorem 4.1. *Suppose M is an even lattice of signature $(2, b^-)$ and f is a modular form of weight $1 - b^-/2$ and representation ρ_M which is holomorphic on H and meromorphic at cusps and whose coefficients $c_{n,\gamma}$ are integers for $n \leq 0$. Then $\sum_{n,\gamma} c_{n,\gamma} y_{n,\gamma}$ is a principal Heegner divisor. In other words,*

$$\xi\Big(\lambda\big(ModForm(Mp_2(\mathbf{Z}), 1 - b^-/2, \rho_M)_{\mathbf{Z}}\big)\Big) \subseteq PrinHeeg(X_\Gamma).$$

Proof. Theorem 13.3 of [B] implies that if M and f satisfy the conditions of theorem 4.1 then there is a meromorphic function Ψ on $G(M)$ with the following properties.

1. Ψ is an automorphic form of weight $c_{0,0}/2$ for some unitary character of the subgroup of $\mathrm{Aut}(M)$ fixing all elements of M'/M.
2. The only zeros or poles of Ψ lie on the divisors $\lambda^\perp$ for $\lambda \in M$, $\lambda^2 < 0$ and are zeros of order
$$\sum_{\substack{0 < x \in \mathbf{R} \\ x\lambda \in M'}} c_{x^2\lambda^2/2, x\lambda}$$
(or poles if this number is negative).

Theorem 4.1 is just a restatement of this result in terms of the divisors $y_{n,\gamma}$, provided we show that the characters of the automorphic forms have finite order. This can be shown as follows. For $O_{2,n}(\mathbf{R})$ with $n > 2$ this follows because these Lie groups have no almost simple factors of real rank 1, and if G is a lattice in a connected Lie group with no simple factors of rank 1 then the abelianization of G is finite. (See [M page 333, proposition 6.19].) So any character of G has finite order.

For the cases $n = 1$ and $n = 2$ we use the embedding trick ([B98, lemma 8.1]) to see that if f is an infinite product of $O_{2,n}(\mathbf{R})$ then f is the restriction of an infinite product g of $O_{2,24+n}(\mathbf{R})$. The infinite product g is not necessarily single valued; however, a look at the proof of lemma 8.1 shows that if f is constructed from a vector valued modular form with integral coefficients, then g^{24} has zeros and poles of integral order and is therefore a meromorphic automorphic form for some unitary character. By the previous paragraph this character has finite order, and therefore so does the character of f. This proves theorem 4.1.

We remark that the proof of theorem 13.3 of [B] is quite long but most of the proof is the calculation of the Fourier expansion of Ψ (or rather of its logarithm), which we do not use in this paper. The proof of the parts of theorem 13.3 that we use here is much shorter and is mainly in section 6 of [B].

Lemma 4.2. *There is a number field F of finite degree over $\mathbf{Q}$ such that the finite dimensional space $HolModForm(Mp_2(\mathbf{Z}), 1 + b^-/2, \rho_M^*)$ has a basis whose Fourier coefficients all lie in F, i.e., such that $\lambda(f) \in PowSer(Mp_2(\mathbf{Z}), \rho_M^*)_F$.*

Proof. It follows from [S, section 3.5] that the space of modular forms of level N has a base of forms whose Fourier expansions at all cusps have coefficients in some algebraic number field of finite degree. (Shimura covers the case of integral weight at least 2, but the other cases can easily be reduced to this by multiplying forms by a power of $\eta(\tau)$; note that if we are given a basis for the space of modular forms of level N all of whose Fourier expansions have coefficients in F then we can find a similar basis for the space of forms which vanish to given orders at various cusps.) Each of the $|M'/M|$ components of $HolModForm(Mp_2(\mathbf{Z}), 1 + b^-/2, \rho_M^*)$ is a modular form of level N for some N, and the representation ρ_M is obviously defined over an algebraic number field of finite degree. This implies that the space $HolModForm(Mp_2(\mathbf{Z}), 1 + b^-/2, \rho_M^*)$ has a basis of elements all of whose coefficients lie in some finite algebraic number field, because this space is the $Mp_2(\mathbf{Z}/N\mathbf{Z})$-invariant subspace of the space of modular forms of level N with coefficients in $\mathbf{C}[M'/M]^*$. This proves lemma 4.2.

Lemma 4.3. *Let $Gal(\bar{\mathbf{Q}}/\mathbf{Q}) \cdot \lambda(HolModForm(Mp_2(\mathbf{Z}), 1 + b^-/2, \rho_M^*))$ be the space of $Gal(\bar{\mathbf{Q}}/\mathbf{Q})$ conjugates of the q-expansions of elements of $HolModForm(Mp_2(\mathbf{Z}), 1 + b^-/2, \rho_M^*)$. (This is well defined and finite dimensional by lemma 4.2.) Then*

$$\lambda\big(ModForm(Mp_2(\mathbf{Z}), 1 - b^-/2, \rho_M)_{\mathbf{Z}}\big) \otimes \mathbf{C}$$
$$= \Big(Gal(\bar{\mathbf{Q}}/\mathbf{Q}) \cdot \lambda\big(HolModForm(Mp_2(\mathbf{Z}), 1 + b^-/2, \rho_M^*)\big)\Big)^{\perp}.$$

Moreover, this space has finite index in $Sing(Mp_2(\mathbf{Z}), \rho_M)$.

Proof. It is obvious using theorem 3.1 that the first space is contained in the second. To show that the second space is contained in the first, choose a basis for $HolModForm(Mp_2(\mathbf{Z}), 1 + b^-/2, \rho_M^*)$ consisting of forms with coefficients in some algebraic number field F of finite degree, which we can do by lemma 4.2. But then the image of this space under $Gal(\bar{\mathbf{Q}}/\mathbf{Q})$ is spanned by a finite number of functions all of whose Fourier coefficients are rational. So the orthogonal complement is defined by rational linear relations and therefore has a basis consisting of modular forms whose image under λ has rational coefficients. By multiplying by a common denominator we can assume the coefficients are integral. By applying 3.1 again this proves that the second space is contained in the first, so both spaces are equal.

The fact that the space has finite index in $Sing(Mp_2(\mathbf{Z}), \rho_M)$ follows because it is the orthogonal complement of the sum of a finite number of $Gal(\bar{\mathbf{Q}}/\mathbf{Q})$ conjugates of a finite dimensional space. This proves lemma 4.3.

Lemma 4.4. *The complex vector space $HeegCl(X_\Gamma)\otimes\mathbf{C}$ generated by the Heegner divisor classes is finite dimensional.*

Proof. By Theorem 4.1, the surjective map

$$Sing(Mp_2(\mathbf{Z}),\rho_M)\longrightarrow Heeg(X_\Gamma)\otimes_{\mathbf{Z}}\mathbf{C}\longrightarrow HeegCl(X_\Gamma)\otimes\mathbf{C}$$

factors through the space

$$Sing(Mp_2(\mathbf{Z}),\rho_M)/\lambda(ModForm(Mp_2(\mathbf{Z}),1-b^-/2,\rho_M)_{\mathbf{Z}})\otimes\mathbf{C}.$$

which is finite dimensional by lemma 4.3. This proves lemma 4.4.

Theorem 4.5. *If the Heegner divisors $y_{n,\gamma}$ are considered as elements of the Heegner divisor class group $HeegCl(X_\Gamma)\otimes\mathbf{C}$ then*

$$\sum_{n\in\mathbf{Q}}\sum_{\gamma\in M'/M} y_{-n,\gamma}q^n e_\gamma$$

is a modular form, and more precisely it lies in the space

$$\Big(HeegCl(X_\Gamma)\otimes\mathbf{C}\Big)\otimes_{\mathbf{C}}\Big(Gal(\bar{\mathbf{Q}}/\mathbf{Q})\cdot\lambda\big(HolModForm(Mp_2(\mathbf{Z}),1+b^-/2,\rho_M^*)\big)\Big).$$

Proof. By lemma 4.4 we know that

$$\sum_{n\in\mathbf{Q}}\sum_{\gamma\in M'/M} y_{-n,\gamma}q^n e_\gamma\in\Big(HeegCl(X_\Gamma)_{\mathbf{C}}\Big)\otimes_{\mathbf{C}} PowSer(Mp_2(\mathbf{Z}),\rho_M^*).$$

By Theorem 4.1, the pairing

$$\sum c_{n,\gamma}y_{n,\gamma}=\xi(\lambda(\sum c_{n,\gamma}q^n e_\gamma))$$

of this series with any element $\sum c_{n,\gamma}q^n e_\gamma$ of

$$\lambda\big(ModForm(Mp_2(\mathbf{Z}),1-b^-/2,\rho_M)_{\mathbf{Z}}\big)$$

is zero. Theorem 4.5 therefore follows from lemma 4.3 above,

In the few examples I have checked the space of modular forms of weight $1+b^-/2$ and type ρ_M^* has a basis of forms with rational coefficients, so that the Heegner divisors are the coefficients of a modular form of type ρ_M^*. I do not know whether or not such a basis always exists.

Modular forms of type ρ_M^* and weight $1+b^-/2$ can be identified with certain Jacobi forms in several variables of weight $1+b^-$ as in [E-Z theorem 5.1], so theorem 4.5 says that the Heegner divisors are coefficients of a Jacobi form of weight $1+b^-$.

We can often work out the dimensions of the spaces of vector valued modular forms using the Riemann-Roch theorem. For example, if ρ is a d-dimensional representation of

$Mp_2(\mathbf{Z})$ on which Z acts as $e^{-\pi ik}$ for some $k \geq 2$ with $k \in \frac{1}{2}\mathbf{Z}$ then the dimension of the space of holomorphic modular forms of type ρ and weight k is equal to

$$d + dk/12 - \alpha(e^{\pi ik/2}S) - \alpha((e^{\pi ik/3}ST)^{-1}) - \alpha(T)$$

where $\alpha(X)$ is the is the sum of the numbers β_j, $1 \leq j \leq d$, where the eigenvalues of X are $e^{2\pi i\beta_j}$ and $0 \leq \beta_j < 1$. This formula cannot be applied directly to ρ_M^* as the condition on Z is not satisfied, but we can apply it to the subspace of ρ_M^* on which Z acts as $e^{-\pi ik}$; for $k = 1 + b^-/2$ this is the subspace spanned by the elements $e_\gamma^* + e_{-\gamma}^*$. The equality $y_{n,\gamma} = y_{n,-\gamma}$ implies that the modular form in theorem 4.5 lies in the space on which Z acts as $e^{-\pi ik}$.

5. Examples.

In this section we give some examples to illustrate theorem 4.5. In most cases we will describe the lattice M and the Heegner divisors.

Example 5.1 We work out theorem 4.5 in the case of modular curves; see [Z84] and [G-K-Z] for more about this case. We fix N to be any positive integer (called the level).

We let M be the 3 dimensional even lattice of all symmetric matrices $v = \begin{pmatrix} C/N & -B/2N \\ -B/2N & A/N \end{pmatrix}$ with $A/N, B/2N, C$ integers, with the norm (v,v) defined to be $-2N\det(v) = (B^2 - 4AC)/2N$. The dual lattice is the set of matrices as above with $A/N, B, C \in \mathbf{Z}$, and M'/M can be identified with $\mathbf{Z}/2N\mathbf{Z}$ by mapping a matrix of M' to the value of $B \in \mathbf{Z}/2N\mathbf{Z}$. The lattice M splits as the direct sum of the 2 dimensional hyperbolic unimodular even lattice $II_{1,1}$ and a lattice generated by an element of norm $2N$. The group $\Gamma_0(N) = \{\binom{ab}{cd} \in SL_2(\mathbf{Z}) | c \equiv 0 \bmod N\}$ acts on the lattice M by $v \mapsto XvX^t$ for $X \in \Gamma_0(N)$, and under this action it fixes all elements of M'/M.

We identify the upper half plane with points in the Grassmannian $G(M)$ by mapping $\tau \in H$ to the 2 dimensional positive definite space spanned by the real and imaginary parts of the norm 0 vector $\begin{pmatrix} \tau^2 & \tau \\ \tau & 1 \end{pmatrix}$. For each $n \in \mathbf{Q}$ and $\gamma \in M'/M = \mathbf{Z}/2N\mathbf{Z}$ the Heegner divisor $y_{n,\gamma}$ is the union of the points orthogonal to norm $2n$ vectors of $M + \gamma$. In terms of points on H this Heegner divisor consists of all points $\tau \in H$ such that

$$A\tau^2 + B\tau + C = 0$$

for some integers A, B, C (not necessarily coprime) with $N|A$, $B \equiv \gamma \bmod 2N$, $B^2 - 4AC = 4Nn$. (Warning: we omit the condition $A > 0$ which is sometimes included in the definition of Heegner divisors. The points with $A > 0$ are exchanged with the points with $A < 0$ under the Fricke involution $\tau \mapsto -1/N\tau$, so the difference is not important as long as we quotient out by the Fricke involution.) The Heegner divisor $y_{n,\gamma}$ is invariant under the group $\Gamma_0(N)$ and the Fricke involution, because the corresponding set of vectors is invariant under $\Gamma_0(N)$ and changes sign under the Fricke involution.

The element $y_{0,0}$ is 0 (modulo torsion) as follows easily from the existence of the modular form Δ with no zeros on H, and this is why $y_{0,0}$ does not appear explicitly in this case. However $y_{0,0}$ is usually nonzero in other cases; see example 5.4. The vanishing of $y_{0,0}$ seems to be related to the failure of the Koecher boundedness principle.

To compare theorem 4.5 with theorem C of [G-K-Z] which is stated in terms of Jacobi forms rather than vector valued modular forms we recall that according to [E-Z, theorem 5.1] the space $J_{k,N}$ of Jacobi forms of weight k and index N is isomorphic to the space of modular forms of weight $k-1/2$ and type ρ_M^*. By [E-Z theorem 9.3] the spaces of Jacobi forms $J_{2,N}$ have bases consisting of forms with rational coefficients, so by the remarks after theorem 4.5 the series

$$\sum_{n,\gamma} y_{-n,\gamma} q^n e_\gamma$$

is a modular form of weight 3/2 and type ρ_M^*. This almost implies the main result theorem C of [G-K-Z], and (assuming the Gross-Zagier theorem [G-Z]) is essentially the "ideal statement" of theorem C conjectured on page 503 of [G-K-Z]. (But note that there is a small technical difference between our definitions and those of [G-Z], because we allow nontrivial unitary characters of finite order in the definition of principal Heegner divisors. This makes no difference, because some finite positive power of an automorphic form with character of finite order has trivial character, so our definitions are equivalent to those of [G-Z].)

We have implicitly quotiented out by the divisors of cusps. We can ask what happens if we leave them in, which essentially comes down to looking at the power series whose coefficients are given by the degrees of Heegner divisors. These degrees are given by various sorts of class numbers (e.g. Hurwitz class number in the case of level 1) and Zagier showed ([Z], [H-Z chapter 2]) that these are the coefficients of certain non holomorphic modular forms of weight 3/2.

Example 5.2. We can ask if all relations between Heegner divisors are given by theorem 4.5. The answer seems to be that usually they are but there are some exceptions where there are extra relations. (Remark added 2000: this has mostly been proved by J. Bruinier [Br].) For example, by the Gross-Zagier theorem [G-Z] this happens whenever there is a modular elliptic curve of odd rank at least 3, because for such elliptic curves all Heegner points vanish. For $b^- > 2$ it might be possible to prove that there are no further relations between Heegner points as follows. For each Heegner point we can construct a real analytic function with a logarithmic singularity along the Heegner divisor by applying the singular theta correspondence to a possibly non holomorphic modular form. For any linear combination of Heegner divisors we get a function with logarithmic singularities along these divisors, which is the logarithm of an automorphic form when there is a relations between the Heegner divisors given by theorem 4.5. When there is no relation it should be possible to prove that there is no automorphic form giving this relation, as otherwise the difference of the log of this form and the function we have constructed would be a nonzero harmonic function vanishing at infinity, which is not possible. This argument breaks down for $b^- =$ 1 or 2 because in attempting to construct non holomorphic modular forms with given singularities at cusps we run into problems with the Green's functions having poles in the critical strip. These poles are presumably related to the derivative of L functions at $s = 1$, and it might be possible to prove a weak version of the Gross-Zagier theorem along these lines, saying that certain spaces spanned by Heegner points have rank 0 if and only if the derivative of some L series vanishes at $s = 1$.

Example 5.3. We can do the same as in example 5.1 with the Shimura curves associated with quaternion algebras over the rationals. Suppose that K is a non split 4 dimensional central simple algebra over the rationals which is split at infinity and R is some order in K. We let M be the lattice of elements of R orthogonal to 1, with the inner product given by minus that of R. Then M is a lattice in $\mathbf{R}^{2,1}$, and the group of units of R acts on M by conjugation. The Grassmannian $G(M)$ is isomorphic to the upper half plane H and the group Γ acts on it with the quotient being a compact Riemann surface (or more precisely an orbifold) called a Shimura curve. The points on the Shimura curve associated to vectors of M are called CM points. So theorem 4.5 implies that certain divisors associated to CM points are coefficients of modular forms of weight 3/2. The case of modular curves for $\Gamma_0(N)$ is really a special case of the construction above where we take K to be the split central simple algebra $M_2(\mathbf{Q})$.

Example 5.4. In the case of the Gross-Kohnen-Zagier theorem the modular form we get is a cusp form and $y_{00} = 0$. We give an example to show that in higher dimensions the modular forms we get are not necessarily cusp forms and y_{00} can be nonzero. We take $M = M'$ to be the lattice $II_{25,1}$. Then for each positive integer n we have a Heegner divisor $y_{-n,0}$ (which is irreducible if and only if n is square free). The representation ρ_M is trivial, so vector valued modular forms of type ρ_M^* are the same as modular forms of level 1 and in particular we can find a base of forms with rational coefficients. By theorem 4.5 the sum $\sum q^n y_{-n,0}$ is then a modular form of weight 14, so must be a multiple of $E_{14} = 1 - 24\sum_n \sigma_{13}(n)q^n$. Therefore $y_{-n,0} = -24\sigma_{13}(n)y_{0,0}$. The Koecher boundedness principle implies that no positive integral multiple of any $y_{-n,0}$ for any positive integer n is zero. This example also shows that $y_{0,0}$ is not necessarily in the integral lattice generated by Heegner divisors. The automorphic form giving the relation $y_{-1,0} = -24y_{0,0}$ is unusual because it is a holomorphic form of singular weight and is also the denominator function of the fake monster Lie algebra.

Example 5.5. Hirzebruch and Zagier showed in [H-Z] that a power series with coefficients given by modular curves on a Hilbert modular surface was a modular form of weight 2. We will show that similar results can be deduced from theorem 4.5. (It seems likely that these results imply Hirzebruch and Zagier's result, but I have not checked this in detail.)

For background about Hilbert modular surfaces see [G] or [H-Z]. We fix a real quadratic field K of discriminant $D > 0$ and ring of integers R. We denote the conjugate of $\lambda \in K$ by λ'. We will do the case of Hilbert modular surfaces of level 1 associated to the trivial ideal class; it should be obvious how to choose M to extend this to higher levels and other ideal classes.

We let M be the even lattice of matrices of the form $v = \begin{pmatrix} C & -B \\ -B' & A \end{pmatrix}$ with $A, C \in \mathbf{Z}$, $B \in R$, with norm given by $-2\det(v)$. The group $SL_2(K)$ acts on the vector space $M \otimes \mathbf{Q}$ of hermitian matrices by $v \to XvX'^t$ for $X \in SL_2(K)$ and $v \in M \otimes \mathbf{Q}$. The group $SL_2(R)$ maps M to itself under this action. The lattice M is the direct sum of the lattice $II_{1,1}$ and a lattice of determinant D.

We identify the product of two copies of the upper half plane H with the Grassmannian of M by mapping $(\tau_1, \tau_2) \in H^2$ to the space spanned by the real and imaginary parts of the norm 0 vector $\begin{pmatrix} \tau_1\tau_2 & \tau_1 \\ \tau_2 & 1 \end{pmatrix}$. This induces the usual action of $SL_2(K)$ on H^2 given by

$$\begin{pmatrix} ab \\ cd \end{pmatrix}((\tau_1, \tau_2)) = \left(\frac{a\tau_1 + b}{c\tau_1 + d}, \frac{a'\tau_2 + b'}{c'\tau_2 + d'} \right).$$

If v is a negative norm vector of M' then we define the modular curve T_v to be the orthogonal complement of v in the Grassmannian of M. We can describe T_v explicitly as follows. If v is the matrix $\begin{pmatrix} C & -B \\ -B' & A \end{pmatrix}$ then T_v is the set of points $(\tau_1, \tau_2) \in H^2$ such that

$$A\tau_1\tau_2 + B'\tau_1 + B\tau_2 + C = 0.$$

If $0 > n \in \mathbf{Q}$ and $\gamma \in M'/M$ then $y_{n,\gamma}$ is the union of all the curves T_v for norm $2n$ vectors $v \in M + \gamma$. Then theorem 4.5 implies that for some choice of $y_{0,0}$ the power series

$$\sum_{n,\gamma} y_{-n,\gamma} q^n e_\gamma$$

is a modular form of weight 2.

In [H-Z, Theorem 1] Hirzebruch and Zagier show that for certain divisors T_N for $N > 0$ the power series $\sum_N T_N^c q^N$ is a modular form of weight 2. (Here T_N^c is the homology class of the Hilbert modular surface given by the projection of the homology class of T_N into the orthogonal complement of the subspace generated by homology cycles of the curves of the cusp resolutions.) This result is closely related to the one above, because the divisors T_N are the unions of the divisors $y_{-N,\gamma}$ for all $\gamma \in M'/M$. There are several minor differences between the results here and the result in [H-Z], as follows. For simplicity Hirzebruch and Zagier only treat the case of K having discriminant D a prime p congruent to 1 mod 4, but their methods could probably be extended to all positive discriminants. The lattice used by [H-Z] is pM' of discriminant p^3 rather than M of discriminant p. The definition of a principal Heegner divisor here is slightly different because we allow nontrivial unitary characters (though it it likely that all these characters have finite order, in which case there is no essential difference). There are also some other small changes of notation; for example, we use hermitian matrices rather than skew hermitian matrices to emphasize the similarity with example 5.1.

This example is closely related to the case of modular curves in 5.1; in fact in [Z84] Zagier shows how to deduce the results about modular curves from the result on Hilbert modular surfaces in many special cases.

Example 5.6. If we take M to be a 5 dimensional lattice then the symmetric space of M is isomorphic to the Siegel upper half plane of genus 2. The divisors on this Siegel upper half plane associated to vectors of M (or rather their images in the quotient) are just the so-called Humbert surfaces. Theorem 4.5 implies that power series with coefficients given by certain Humbert divisors are vector valued modular forms of weight 5/2. A similar result is mentioned in [G, p. 213].

Example 5.7. If we take M to be a 6 dimensional lattice then the symmetric space of M is isomorphic to the Hermitian upper half space of complex dimension 4. The divisors associated to vectors of M are quotients of the Siegel upper half space of complex dimension 3. Hermann [He] found an example of a modular form of weight $1 + b^-/2 = 3$ associated to this case. Note that the groups $Sp_4(\mathbf{R})$ and $SU(2,2)$ used in [He] are locally isomorphic to the groups $O_{2,3}(\mathbf{R})$ and $O_{2,4}(\mathbf{R})$ used in this paper, and in particular the Hermitian half space of [He] is isomorphic to the Grassmannian of $\mathbf{R}^{2,4}$.

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