Sporadic groups and string theory.

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This is an expanded version of my talk at the ECM.

1. Some classical infinite product identities.

I will start by giving some well known product identities. The first is

$$\sum_{n\in\mathbf{Z}}(-1)^n q^{3(n+1/6)^2/2} = q^{1/24}\prod_{n>0}(1-q^n).$$

This identity was found by Euler while investigating the partition function; the right hand side is essentially the inverse of the generating function $\prod(1-q^n)^{-1}=\sum p(n)q^n$ of the partition function, and the left hand side is a theta function. A similar identity due to Gauss is

$$\sum_{n\in\mathbf{Z}} q^{n^2} = (1+q)^2(1-q^2)(1+q^3)^2(1-q^4)\dots$$

Both of these are special cases of Jacobi's triple product identity

$$\sum_{n\in\mathbf{Z}}(-1)^n q^{n^2}z^n = \prod_{n>0}(1-q^{2n})(1-q^{2n-1}z)(1-q^{2n-1}z^{-1})$$

if we choose z to be some fixed power of q. Finally Weyl's denominator formula for finite dimensional Lie algebras is

$$e^\rho\sum_{w\in W}\det(w)e^{-w(\rho)} = \prod_{\alpha>0}(1-e^\alpha)$$

where W is the Weyl group, ρ is the Weyl vector, and the product is over all positive roots α. This is a special case of Weyl's character formula which says that the character of a finite dimensional representation is equal to a sum similar to the left hand side divided by the product on the right hand side; for the 1-dimensional representation the character is 1 so the sum is equal to the product.

The Weyl character formula looks rather like the other product formulas. This similarity was explained by Macdonald and Kac as follows. Macdonald observed that the Weyl denominator formula was just a statement about finite root systems, and generalized this to affine root systems, producing a collection of identities called the Macdonald identities. The Macdonald identity for the simplest affine root system is just the Jacobi triple product identity. Kac observed that the Macdonald identities were just the denominator formulas for the Kac-Moody algebras that he and Moody had discovered a few years before, and went on to prove a character formula for representations of these algebras generalizing the Weyl character formula. The Weyl-Kac denominator formula [12] for Kac-Moody algebras is

$$e^\rho\sum_{w\in W}\det(w)e^{-w(\rho)} = \prod_{\alpha>0}(1-e^\alpha)^{\mathrm{mult}(\alpha)}$$

where $\mathrm{mult}(\alpha)$ is the multiplicity of the root α, which is just the dimension of the vector space corresponding to the root α. For finite dimensional Lie algebras the multiplicity of a root is always 1 which is why this expression does not appear in the Weyl denominator formula. For affine Lie algebras such as $SL_n(\mathbf{R}[z,z^{-1}])$ this formula is just a Macdonald identity; for example the Jacobi triple product identity is just the Weyl-Kac denominator formula for $SL_2(\mathbf{R}[z,z^{-1}])$.

I will now briefly describe a proof of the Weyl-Kac character formula using Lie algebra cohomology due to Bott, Kostant, Garland and Lepowsky [10, 14]. Perhaps the best way to motivate Lie algebra cohomology is by analogy with the de Rham cohomology of a compact Riemannian manifold M. The de Rham cohomology of M is defined to be the cohomology groups of the sequence

$$\Lambda^0\longrightarrow\Lambda^1\longrightarrow\Lambda^2\longrightarrow\dots$$

where Λ^n is the vector space of smooth n-forms over M and the arrows are given by the exterior derivative d. The Riemannian manifold on M defines inner products on all the spaces Λ^n which we can use to define the adjoint d^* of d. The Laplace operator is defined as $\Omega = dd^* + d^*d$, and the main theorem of Hodge theory says that the n'th cohomology group H^n is isomorphic to the kernel of Ω on Λ^n. If we have a complex like the one above, the Euler-Poincaré principle states that the alternating sum $\Lambda^* = \Lambda^0 - \Lambda^1 + \Lambda^2 - \ldots$ is equal to the alternating sum $H^* = H^0 - H^1 + H^2 - \ldots$. Strictly speaking, this does not make sense unless the groups Λ^n are all finite dimensional (which they are not) and almost all 0. I will deal with this problem by ignoring it.

Lie algebra cohomology is very similar except that we replace the space of n forms on a manifold by the n'th exterior power of the dual of some Lie algebra, and replace the Riemannian metric on M by an invariant (or contravariant) bilinear form on the Lie algebra. Then, at least for the Lie algebras I will be talking about, the analogues of Hodge theory and the Euler-Poincaré principle are both true.

The Weyl-Kac character formula now turns out to be the Euler-Poincaré principle applied to the cohomology of a suitable Lie algebra. Any Kac-Moody algebra can be written as a (vector space) sum of subalgebras $E \oplus H \oplus F$ where H is the Cartan subalgebra, and E and F are the subalgebras associated to the positive and negative roots. For example, for $SL_n(\mathbf{R})$ the algebras E, F, and H are the upper triangular, lower triangular, and diagonal matrices. The Weyl-Kac denominator formula for the Kac Moody algebra is the Euler-Poincaré principle applied to the cohomology of the subalgebra E, and the cohomology groups can be worked out using Hodge theory. To see this, we use the facts that $\Lambda^*(A \oplus B) = \Lambda^*(A) \otimes \Lambda^*(B)$ for any vector spaces A and B, and $\Lambda^*(A) = 1 - A$ if A is 1-dimensional, so if a vector space is a sum of 1-dimensional spaces A_α, then $\Lambda^*(A)$ is just the product of the factors $(1 - A_\alpha)$. This is where the product in the Weyl-Kac denominator formula comes from. It is more difficult to see why the cohomology gives a sum over the Weyl group; I will just state that the dimension of the n'th cohomology group is the number of elements of the Weyl group of length n. This makes it plausible that the alternating sum over the Weyl group has something to do with the alternating sum of the cohomology groups.

The Weyl-Kac denominator formula is true for non affine Kac-Moody algebras, but unfortunately there are no known examples, other than the finite dimensional and affine algebras, for which the root multiplicities are known explicitly, so this does not really give any new identities. (It is of course always possible to calculate the root multiplicities by using the Weyl-Kac formula or something equivalent to it, but there is no simpler description of them.) However there is a class of Lie algebras, called generalized Kac-Moody algebras [2], which are similar to Kac-Moody algebras, and we can find some examples of these for which the root multiplicities are known. For a finite dimensional Lie algebra all roots have positive norm. For a Kac-Moody algebra, roots can have zero or negative norm, but all simple roots have positive norm, while generalized Kac-Moody algebras can have simple roots of negative norm. Generalized Kac-Moody algebras can also be characterized as Lie algebras with an almost positive definite contravariant bilinear form. More precisely, a Lie algebra G is a generalized Kac-Moody algebra if it satisfies the following conditions:

1. G is graded as $G = \oplus_{n \in \mathbf{Z}} G_n$ such that G_n is finite dimensional for $n \neq 0$.
2. G has an invariant bilinear form compatible with this grading.
3. G has an involution ω preserving $(,)$ and mapping G_n to G_{-n} which acts as -1 on G_0.
4. If we define the contravariant bilinear form $(,)_0$ on G by $(g,h)_0 = (g, \omega(h))$, then $(,)_0$ is positive definite on G_n for $n \neq 0$.

(If we strengthen condition 4 slightly by saying that $(,)_0$ should be positive definite on the whole of G, then the algebras we get are essentially sums of finite dimensional and affine Kac-Moody algebras.)

Many examples of generalized Kac-Moody algebras can be constructed using string theory.

2. String theory.

I will describe how to quantize a string moving in spacetime. The space of states of a quantized string is sometimes a generalized Kac-Moody algebra. See [15] for a more detailed account of the things in this section.

A classical closed string moving in spacetime can be represented by a map from $\mathbf{S}^1 \times \mathbf{R}^1$ to $\mathbf{R}^{n,1}$, where $\mathbf{S}^1$ is the string, $\mathbf{R}^1$ is time on the string, and $\mathbf{R}^{n,1}$ is $n+1$-dimensional spacetime. The classical equations of motion for the string say that the action must be stationary, where the action is some function defined on the space of all maps from $\mathbf{S}^1 \times \mathbf{R}^1$ to $\mathbf{R}^{n,1}$. The original action is the Nambu-Goto action, which is equal to the area of the image of $\mathbf{S}^1 \times \mathbf{R}^1$. (It does not matter that this area is usually infinite, because we are only interested in the change in the area under small local perturbations, and this is well defined. Also it is easier to use a slightly different action called the Polyakov action, but this produces the same solutions to the classical equations of motion.)

Phase space is defined to be the space of all solutions to the classical equations of motion, possibly quotiented out by the action of some group. (It is often defined to be the cotangent space to "configuration space", because this is more or less the same thing, because an element of this cotangent space is roughly the same as giving the position and momentum of everything, and this often determines a unique solution to the classical equations of motion.) The phase space of some classical system can often be made into a symplectic manifold in a canonical way, so that it has a nondegenerate closed 2-form on it. On any symplectic manifold the vector fields preserving the 2-form are called the Hamiltonian vector fields, and can be thought of as the Lie algebra of the group of automorphisms of the symplectic manifold. This Lie algebra has a canonical central extension equal to the Lie algebra of all smooth functions on the manifold under the Poisson bracket (which takes the functions f and g to $\langle df, dg\rangle$). In other words we have the following exact sequence of Lie algebras (at least if the symplectic manifold is compact and simply connected):

$$0 \longrightarrow \mathbf{R} \longrightarrow \Lambda^0 \xrightarrow{d} \Lambda^1_{\text{closed}} \longrightarrow 0,$$

where Λ^0 is the Lie algebra of smooth functions under Poisson bracket, and $\Lambda^1_{\text{closed}}$ is the Lie algebra of closed 1-forms, which are identified with the Hamiltonian vector fields if we use the 2-form to identify 1-forms with vector fields.

The quantization of the string is roughly a representation of some Lie subalgebra of the Lie algebra of functions on phase space. Which subalgebra and which representation we take are up to the person doing the quantization. (This is a gross simplification of what is usually meant by quantization.)

The result of quantizing a (parameterized chiral) string is sometimes a vertex algebra V [1]. Vertex algebras (with some minor extra structure, such as a grading or a bilinear form or an action of the Virasoro algebra) are also called vertex operator algebras [8], meromorphic conformal field theories, W-algebras, and chiral algebras.

To construct the space of states of an unparameterized string, we would like to take the subspace of the vertex algebra V fixed by the diffeomorphism group of the circle $\mathbf{S}^1$, which is $H^0(Diff(\mathbf{S}^1), V)$. This does not work because $Diff(\mathbf{S}^1)$ does not quite act on V; instead we only get a projective action of it on V, so the space of fixed vectors is 0. We can get around this problem by defining the space of physical states of an unparameterized string to be the semi-infinite cohomology group $H^{\infty+1/2}(Vir, V)$ [9]. Here Vir is the Virasoro algebra, spanned by elements L_n, $n \in Z$, and c with the relations

$$\begin{aligned}[L_m, L_n] &= (m-n)L_{m+n} + \delta_m^{-n}(m^3 - m)c/12\\ [L_m, c] &= 0\end{aligned}$$

and is a central extension of the Lie algebra of polynomial vector fields on the circle.

This cohomology group $H^{\infty+1/2}(Vir, V)$ is the generalized Kac-Moody algebra we were trying to construct. It turns out to be nonzero only if spacetime is 26-dimensional; the mysterious number 26 appears because it is minus the value of c when Vir acts on the space of semi-infinite forms over Vir.

3. The monster Lie algebra.

We can apply the denominator formula $H^*(E) = \Lambda^*(E)$ of section 1 to some of the generalized Kac-Moody algebras constructed using string theory in section 2. For one of these algebras, called the monster Lie algebra, this leads to a proof of Conway and Norton's moonshine conjectures for the monster simple group.

The monster Lie algebra [4] is the simplest example of a Lie algebra of physical states of a chiral string on some orbifold. In this case the orbifold is a quotient of the 26-dimensional torus $\mathbf{R}^{25,1}/II_{25,1}$ by an involution, where $II_{25,1}$ is the unique 26-dimensional even unimodular Lorentzian lattice. This algebra can be described in terms of the graded representation $V = \oplus V_n$ of the monster constructed by Frenkel, Lepowsky and Meurman [8], with the property that $\dim(V_n) = c(n-1)$, the coefficient of q^n in the elliptic modular function $j(q) - 744 = \sum c(n)q^n = q^{-1} + 196884q + \ldots$. It follows from the Goddard-Thorn no-ghost theorem [11] that the monster Lie algebra is a $\mathbf{Z}^2$-graded Lie algebra $\oplus_{m,n\in\mathbf{Z}} M_{m,n}$, whose piece $M_{m,n}$ of degree $(m,n) \in \mathbf{Z}^2$ is isomorphic as a module over the monster to V_{mn} if $(m,n) \neq (0,0)$ and to $\mathbf{R}^2$ if $(m,n) = (0,0)$. For small degrees it looks like

$$\begin{array}{cccccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 0 & 0 & 0 & 0 & V_3 & V_6 & V_9 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & V_2 & V_4 & V_6 & \cdots \\
\cdots & 0 & 0 & V_{-1} & 0 & V_1 & V_2 & V_3 & \cdots \\
\cdots & 0 & 0 & 0 & \mathbf{R}^2 & 0 & 0 & 0 & \cdots. \\
\cdots & V_3 & V_2 & V_1 & 0 & V_{-1} & 0 & 0 & \cdots \\
\cdots & V_6 & V_4 & V_2 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & V_9 & V_6 & V_3 & 0 & 0 & 0 & 0 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

In the decomposition of this algebra as $E\oplus H\oplus F$, the Cartan subalgebra H is the 2-dimensional algebra of degree 0, E is the sum of everything lying to the right of H (in the diagram above), and F is the sum of everything to the left of H. The real roots correspond to the two 1-dimensional spaces V_{-1}, and the simple roots are the ones in the column to the right of H.

To work out the denominator formula $H^*(E) = \Lambda^*(E)$ for E explicitly, we need to know how the Laplace operator Ω acts on $\Lambda^n(E)$. This is easy to describe: it just multiplies any element of degree $(m,n) \in \mathbf{Z}^2$ by $(m-1)n$, so that $H^n(E)$ is just the subspace of elements of $\Lambda^n(E)$ of degree $(1,n)$ or $(m,0)$. This implies that $H^0(E) = \mathbf{R}$, $H^1(E) = \sum_n V_n pq^n$, $H^2(E) = \sum_{m>0} V_m p^{m+1}$, and $H^n = 0$ for $n > 2$, where we use powers of p and q to keep track of the $\mathbf{Z}^2$ grading.

If we substitute these values into the formula $H^*(E) = \Lambda^*(E)$ and take dimensions of both sides, we obtain the product formula for the j function

$$p^{-1} \prod_{m>0,n\in\mathbf{Z}} (1-p^m q^n)^{c(mn)} = j(p) - j(q)$$

which is the denominator formula of the monster Lie algebra, in the same way that the Macdonald identities are the denominator formulas of the affine Lie algebras.

The monster group acts on the monster Lie algebra by diagram automorphisms. (It is not the full group of diagram automorphisms; its has the special property that $M_{a,b} = M_{c,d}$ as representations of the monster whenever $ab = cd$, $(a,b) \neq (0,0) \neq (c,d)$.) By taking traces of various elements of the monster on the formula $H^*(E) = \Lambda^*(E)$ we obtain the following product formulas involving the coefficients of the Thompson series $T_g(q) = \sum_{n\in\mathbf{Z}} Tr(g|V_n)q^n$:

$$\begin{aligned}
&\sum_{m\in\mathbf{Z}} Tr(g|V_m)p^m - \sum_{n\in\mathbf{Z}} Tr(g|V_n)q^n \\
=&p^{-1}\exp(-\sum_{i>0}\sum_{m>0,n\in\mathbf{Z}} Tr(g^i|V_{mn})p^{mi}q^{ni}/i).
\end{aligned}$$

For $g = 1$ this is just the product formula for the elliptic modular function. These formulas can be rewritten as recursion formulas for the coefficients $Tr(g|V_n)$, and the same recursion formulas for certain modular functions were conjectured by Norton [16] and proved by Koike [13]. This can be used [4] to prove Conway and Norton's moonshine conjectures [5] that the Thompson series $T_g(q)$ are all modular functions (more precisely, normalized Hauptmoduls of genus 0 groups). Norton's generalizations of these conjectures in [17] are still not proved.

4. Modular forms on $O_{n+2,2}(\mathbf{R})$.

The denominator formulas of the examples of many generalized Kac-Moody algebras are modular forms on $O_{n+2,2}(\mathbf{R})$, which is the group of rotations of $\mathbf{R}^{n+2,2}$, and is also the group of conformal transformations of the conformal completion of $\mathbf{R}^{n+1,1}$.

For $n=-1$, $O_{n+2,2}(\mathbf{R})$ is locally isomorphic to $SL_2(\mathbf{R})$ so modular forms on this group are essentially the same as ordinary modular forms. One example is the theta function $\sum q^{n^2}$, which can be written as an infinite product by Gauss's identity.

For $n=0$, $O_{n+2,2}(\mathbf{R})$ is locally isomorphic to $SL_2(\mathbf{R})\times SL_2(\mathbf{R})$. Modular forms on this group are usually Hilbert modular forms, but the examples we give are not, because the discrete subgroup we use is of the form $\Gamma_1\times\Gamma_2$, where each Γ_1 is a subgroup of $SL_2(\mathbf{R})$. (Informally, we could think of them as "degenerate Hilbert modular forms" for the "degenerate real quadratic number field" $Q(\sqrt{1})$.) An example of one of these modular functions is $j(p)-j(q)$, which can be written as an infinite product using the denominator formula of the monster Lie algebra.

For $n=1$, $O_{n+2,2}(\mathbf{R})$ is locally isomorphic to $Sp_4(\mathbf{R})$, so modular forms on this group are essentially Siegel modular forms of genus (or degree) 2. An example of such a form is the Siegel theta function $\sum_{i,j\in\mathbf{Z}} p^{i^2}q^{j^2}r^{ij}$. We will show how to write this as an infinite product in the next section.

I know of modular forms on $O_{n,2}(\mathbf{R})$ which can be written as interesting infinite products for $n=1$, 2, 3, 4, 6, 8, 10, 14, 18, and 26. It is curious that these seem to be exactly the integers for which there is a lacunary modular form of weight $n/2$ which is a product of eta functions (for example $\eta(q)^{26}$); see Serre [18] and Dyson [6]. (The number $n=18$ does not appear in Serre's paper, but the form $\eta(q)^9\eta(q^2)^9$ of weight 18/2 seems to be lacunary.)

5. A superalgebra of rank 3.

I will give a fairly typical example of one of the generalized Kac-Moody superalgebras of higher rank that are (sometimes only conjecturally) the spaces of physical states of some sort of string on some sort of orbifold. Gauss's product identity for the theta function can be written as

$$\sum_{n\in\mathbf{Z}}(-1)^n q^{n^2}=\prod_{n>0}\frac{1-q^n}{1+q^n}.$$

I write it like this to bring out the analogy with the product formula for Siegel's theta function of degree 2, which is

$$\sum_{m,n\in\mathbf{Z}}(-1)^{m+n}p^{m^2}q^{n^2}r^{mn}=\prod_{a+b+c>0}\left(\frac{1-p^aq^cr^b}{1+p^aq^cr^b}\right)^{f(ac-b^2)}$$

where $f(n)$ is defined by $\sum f(n)q^n=1/(\sum_n(-1)^nq^n)=1+2q+4q^2+8q^3+14q^4+\dots$. (The product does not converge for all values of p, q, and r for which the sum is defined.) This is the denominator formula for a generalized Kac-Moody superalgebra of rank 3. This superalgebra is graded by $\mathbf{Z}^3$, and the subspace of degree (a,b,c) had dimension 3 if $(a,b,c)=(0,0,0)$, and $f(ac-b^2)|f(ac-b^2)$ otherwise. (The symbol $m|n$ for the dimension of a superspace means that it is the sum of an ordinary part of dimension m and a super part of dimension n.) This product formula can be proved using the theory of Jacobi forms [7].

The product is a product over all positive semidefinite binary quadratic forms $ax^2+2bxy+cy^2$, and the identity above is equivalent to the following curious fact about such forms. Let $\mathbf{Z}^n$ be a lattice spanned by n pairwise orthogonal vectors of norm 1, and let $r_{a,b,c}(n)$ be the number of representations of the quadratic form $ax^2+2bxy+cy^2$ in the lattice $\mathbf{Z}^n$ which are not contained in any sublattice of the form $\mathbf{Z}^{n-1}$. Then $\sum_{n>0}(-1)^n r_{a,b,c}(n)/n$ depends only on the discriminant and the highest common factor of the coefficients of the form $ax^2+2bxy+cy^2$. This result is unusual because it is usually only possible to say something about the average number of representations of forms of fixed discriminant.

The Mellin transform of Siegel's theta function of degree 2 is essentially a real analytic Eisenstein series (see Zagier [18] for example). Unfortunately it seems to be very difficult to use the product formula for Siegel's theta function to say anything about real analytic Eisenstein series. For that matter, it seems to be just as difficult to use Gauss's product formula for the theta function to say anything about its Mellin transform, the Riemann zeta function.

6. A Lie algebra of rank 26.

I will finish by describing the largest interesting generalized Kac-Moody algebra that I know of [3]. It is the space of physical states of a string moving on a 26-dimensional torus, and is very closely related to the monster Lie algebra, which is the space of states of a string moving on a $\mathbf{Z}_2$ orbifold of the same torus.

The denominator formula for this Lie algebra is

$$e^{\rho}\prod_{r\in\Pi^+}(1-e^r)^{p_{24}(1-r^2/2)}=\sum_{w\in W}\det(w)w(e^{\rho}\prod_{n>0}(1-e^{n\rho})^{24}).$$

Here both sides are elements of a completion of the group ring of $II_{1,1}$. The vector ρ is the Weyl vector of this lattice which has inner product -1 with all simple roots of the Weyl group W, and $p_{24}(n)$ is the number of partitions of n into parts of 24 colours.

The Lie algebra itself is graded by $II_{1,1}$, and the piece of degree $r\in II_{1,1}$ has dimension 26 if $r=0$, and $p_{24}(1-r^2/2)$ otherwise. The algebra is acted on by a group $\Lambda.Aut(\Lambda)$ where Λ is the Leech lattice and $Aut(\Lambda)$ is the double cover of Conway's largest simple group. We can write down a twisted denominator identity similar to the one above for every conjugacy class of this group, by taking traces in the formula $H^*(E)=\Lambda^*(E)$.

The function

$$f(v)=\sum_{w\in W}\det(w)e^{(2\pi i w(\rho),v)}\prod_{n>0}(1-e^{(2\pi i n w(\rho),v)})^{24}.$$

is an automorphic form of the group $O_{26,2}(\mathbf{R})$ with respect to the discrete subgroup $II_{26,2}$. This follows from the functional equation

$$f(2v/(v,v))=-((v,v)/2)^{12}f(v).$$

I will give a brief sketch of the proof of this functional equation. If we just consider purely imaginary values of v, then f is a solution of the wave equation, and this implies that $((v,v)/2)^{26/2-1}f(2v/(v,v))$ is also a solution by the transformation of the wave operator under the conformal transformation $v\to 2v/(v,v)$. On the other hand, it is easy to show that $f(v)$ vanishes whenever v is imaginary and has norm 2, because the series for $\log(f(v))$ has positive terms and therefore has a singularity on the edge of its region of convergence; this singularity is at all points of the surface C, so f must vanish there because it it regular and $\log(f)$ is not. The fact that f vanishes on this surface easily implies that $f(v)$ and $-((v,v)/2)^{12}f(2v/(v,v))$ both have the same partial derivatives of order at most 1 on this surface. These two functions both satisfy the wave equation and have the same partial derivatives of order at most 1 on a non-characteristic surface, so by the Cauchy-Kovalevsky theorem they must be equal.

The function $f(v)$ also satisfies the trivial functional equations $f(v)=f(v+\lambda)$ for $\lambda\in II_{25,1}$, and $f(w(v))=\det(w)f(v)$ for w an automorphism of $II_{25,1}$ of spinor norm 1. Together these transformations generate a group isomorphic to the subgroup of index 2 of $Aut(II_{26,2})$ of elements of spinor norm 1. This means that f is essentially a modular form on the group $O_{26,2}(\mathbf{R})$ with respect to the discrete subgroup $Aut(II_{26,2})$. There are many other examples of Lie algebras or superalgebras (for example, two superalgebras of superstrings on a 10 dimensional torus) whose denominator functions are modular forms on $O_{n+2,2}(\mathbf{R})$ for some n. I do not know whether there are an infinite number of such examples.

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