An automorphic form related to cubic surfaces. First draft.

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In the paper [A-C-T] the authors showed that the moduli space of cubic surfaces was $(CH^4\backslash H)/G$, where H is the union of the reflection hyperplanes of the reflection group G. The purpose of this note is to construct an automorphic form, called the discriminant, on complex hyperbolic space CH^4 whose zeros are exactly the reflection hyperplanes of G, each with multiplicity 1.

The complex hyperbolic space of $E^{1,4}$ embeds naturally in the Grassmannian of 2 dimensional positive definite subspaces of the underlying even integral lattice M of $E^{1,4}$. We will construct the automorphic form on complex hyperbolic space by constructing an automorphic form Ψ on the Grassmannian and restricting it to complex hyperbolic space. The construction of Ψ is similar to that of example 13.7 of [B]. The lattice M is isomorphic to $A_2 \oplus A_2^4(-1)$, so it has dimension 10 and determinant 3^5. We let the elements e_γ for $\gamma \in M'/M$ stand for the obvious basis of the group ring $C[M'/M]$. We will construct a modular form of weight -3 and type ρ_M for $SL_2(Z)$ which is holomorphic on the upper half plane and meromorphic at cusps, by which we mean a holomorphic vector valued function $f = \sum_{\gamma\in M'/M} e_\gamma f_\gamma(\tau)$ such that

$$f_\gamma(\tau+1) = e^{\pi i \gamma^2} f_\gamma(\tau), \qquad f_\gamma(-1/\tau) = -i3^{-5/2}\tau^{-3} \sum_{\delta\in M'/M} e^{-2\pi i(\delta,\gamma)} f_\delta(\tau)$$

Recall that if $\gamma, \delta \in M'/M$ then γ^2 is well defined mod 2 and (γ, δ) is well defined mod 1. There are 4 orbits of vectors $\gamma \in M'/M$ under $\mathrm{Aut}(M)$, which we name as follows: a nonzero vector γ with $\gamma^2/2 \equiv n/3 \bmod 1$ will be called a vector of type n ($n = 0, 1, 2$), and the zero vector will be called a vector of type 00.

We will construct a modular form f such that the components f_γ of f are given by functions f_{00}, f_0, f_1, or f_2 depending only on the type of γ. We now work out the conditions that these functions have to satisfy for f to transform correctly. In order to check that f is a modular form under $\tau \mapsto -1/\tau$ we need to know, given some fixed vector u, how many vectors v there are with given type and given inner product with u. These numbers are given in the following table.

type of u	00	00	00	00	0	0	0	0	1	1	1	1	2	2	2	2
type of v	00	0	1	2	00	0	1	2	00	0	1	2	00	0	1	2
$(u,v)\equiv 0$	1	80	90	72	1	26	36	18	1	32	24	24	1	20	30	30
$(u,v)\equiv 1/3$	0	0	0	0	0	27	27	27	0	24	33	24	0	30	30	21
$(u,v)\equiv 2/3$	0	0	0	0	0	27	27	27	0	24	33	24	0	30	30	21

Using this table we see that f_{00}, f_0, f_1, and f_2 have to satisfy the equations

$$f_{00}(\tau+1) = f_{00}(\tau), \quad f_{00}(-1/\tau) = -i3^{-5/2}\tau^{-3}(f_{00}(\tau) + 80f_0(\tau) + 90f_1(\tau) + 72f_2(\tau))$$
$$f_0(\tau+1) = f_0(\tau), \quad f_0(-1/\tau) = -i3^{-5/2}\tau^{-3}(f_{00}(\tau) - f_0(\tau) + 9f_1(\tau) - 9f_2(\tau))$$
$$f_1(\tau+1) = e^{2\pi i/3} f_1(\tau), \quad f_1(-1/\tau) = -i3^{-5/2}\tau^{-3}(f_{00}(\tau) + 8f_0(\tau) - 9f_1(\tau))$$
$$f_2(\tau+1) = e^{4\pi i/3} f_2(\tau), \quad f_2(-1/\tau) = -i3^{-5/2}\tau^{-3}(f_{00}(\tau) - 10f_0(\tau) + 9f_2(\tau))$$

One solution of these equations is given as follows.

$$\begin{aligned} f_{00}(\tau) &= 24\eta(3\tau)^3\eta(\tau)^{-9} = 24(1+9q+54q^2+O(q^3)) \\ f_0(\tau) &= -3\eta(3\tau)^3\eta(\tau)^{-9} = -3+O(q) \\ f_1(\tau) &= 0 \\ f_2(\tau) &= \eta(\tau/3)^3\eta(\tau)^{-9} + 3\eta(3\tau)^3\eta(\tau)^{-9} = q^{-1/3}+14q^{2/3}+92q^{5/3}+O(q^{8/3}) \end{aligned}$$

Most of the transformations follow formally from the functional equations $\eta(\tau+1) = e^{2\pi i/24}\eta(\tau)$ and $\eta(-1/\tau) = \sqrt{\tau/i}\eta(\tau)$ of η. The only one which takes slightly more work is the transformation of f_2 under $\tau \mapsto \tau+1$, and this follows from the identity $\eta(\tau)^3 = \sum_{n\in Z}(4n+1)q^{(4n+1)^2/8}$ and its consequence

$$\eta(\tau/3)^3\eta(\tau)^{-9} + \eta((\tau+1)/3)^3\eta(\tau+1)^{-9} + \eta((\tau+2)/3)^3\eta(\tau+2)^{-9} = -9\eta(3\tau)^3\eta(\tau)^{-9}.$$

By theorem 13.3 of [B] there is an automorphic form Ψ on the symmetric space of M with the following properties. It has weight 12 = (coefficient of q^0 in f_{00})/2. The zeros of Ψ correspond to the negative powers of q in f, so are zeros of order 1 orthogonal to all the norm $-2/3$ vectors of M'. Ψ is holomorphic on the symmetric space, and therefore holomorphic at cusps as well by the Koecher boundedness principle. Ψ is an automorphic form for some one dimensional representation of $\mathrm{Aut}(M)$.

By restricting Ψ to complex hyperbolic space we get an automorphic form which has zeros of order 3 along the reflection hyperplanes (because f has zeros of order 1, but every reflection hyperplane is the restriction of 3 hyperplanes of the symmetric space of M). So by taking the cube root of the restriction of Ψ we get an automorphic form for a one dimensional character of G whose zeros are exactly the reflection hyperplanes with multiplicity 1.

References.

[A-C-T] D. J. Allcock, J. Carlson, D. Toledo, A Complex Hyperbolic Structure for Moduli of Cubic Surfaces, alg-geom/9709016

[B] R. E. Borcherds, Automorphic forms with singularities on Grassmannians, alg-geom/9609022, to appear in Invent. Math.