Coxeter groups, Lorentzian lattices, and K3 surfaces.  

Richard E. Borcherds,*  
D.P.M.M.S., 16 Mill Lane, Cambridge, CB2 1SB, England.  
e-mail: reb@dpmms.cam.ac.uk  
www home page www.dpmms.cam.ac.uk/~reb

Contents.  
1. Introduction.  
2. Notation and statement of main theorem.  
3. Proof of main theorem.  
4. The structure of $\Gamma_{\Omega}$.  
5. Examples.

1. Introduction.  

The main result of this paper describes the normalizer $N_{W_\Pi}(W_J)$ of a finite parabolic subgroup $W_J$ of a (possibly infinite) Coxeter group $W_\Pi$. More generally we describe $N_{W_\Pi,\Gamma_{\Omega}}(W_J)$ where $\Gamma_{\Pi}$ is a group of diagram automorphisms of the Coxeter diagram $\Pi$ of $W_\Pi$. By taking $\Pi$ to be Conway’s Coxeter diagram of the reflection group of $\text{II}_{1,25}$ we compute the automorphism groups of some Lorentzian lattices and K3 surfaces.

In the case when $W_\Pi$ is a finite Coxeter group (and $\Gamma_{\Pi} = 1$) the normalizer of $W_J$ has been described by Howlett [H]. His result states that $N_{W_\Pi}(W_J)$ is a split extension $W_J.W'_J$, where $W'_J = W_{\Omega}.\Gamma_{\Omega}$ is in turn a split extension with $W_{\Omega}$ a Coxeter group and $\Gamma_{\Omega}$ a more mysterious group acting on $\Omega$. Howlett showed by case by case analysis that if $\Pi$ is connected (and $W_\Pi$ is finite) then $\Gamma_{\Omega}$ is an elementary abelian 2-group and is a subgroup of $\text{Aut}(J)$. When $W_\Pi$ is infinite the normalizer $N_{W_\Pi}(W_J)$ has a similar structure, except that the group $\Gamma_{\Omega}$ can be more complicated. Although there is still a canonical map from $\Gamma_{\Omega}$ to $\text{Aut}(J)$, the kernel can be non trivial, though it has finite cohomological dimension. The kernel is trivial in the case of finite $W_\Pi$ considered by Howlett because any finite group of finite cohomological dimension must be trivial. For example, the case when $J = A_1$, $\Gamma_{\Pi} = 1$ has been done by Brink [Br], who showed that $\Gamma_{\Omega}$ is a free group (and therefore has cohomological dimension at most 1). We will extend Brink’s result to Coxeter diagrams of arbitrary finite reflection groups. More precisely we construct a category $Q_4$ using $\Pi$ and $J$ and prove that the classifying space of this category is a classifying space of the group $\Gamma_{\Omega}$. The main point about this category $Q_4$ is that it is often finite and can often be written down explicitly, in which case we can easily read off a presentation of $\Gamma_{\Omega}$. For example, if $J = A_1$ we show that the classifying space of this category $Q_4$ is 1-dimensional, so its fundamental group is free and we recover Brink’s result. After writing this paper I discovered that Brink and Howlett had previously announced a related description of the normalizer of a parabolic subgroup of a Coxeter group; see [B-H], and see example 2.8 for the relation between their result and theorem 2.7.

---

* Supported by a Royal Society professorship and an NSF grant.
For later applications we need some generalizations as follows. First of all, instead of calculating the normalizer of $W_J$ in a Coxeter group $W$, we calculate the normalizer in an extension $W_{\Pi} \Gamma_{\Pi}$, where $\Gamma_{\Pi}$ is a group of diagram automorphisms. Secondly, we sometimes want to compute not the full normalizer, but a subgroup with image contained in some subgroup $\Gamma_J$ of $\text{Aut}(J)$. Thirdly, we sometimes want to vary the choice of the Coxeter group $W_{\Omega}$, which we do by varying a certain normal subgroup $R$ of $\Gamma_J$. For example, in calculating the automorphism groups of $K3$ surfaces we take $W_{\Omega}$ to be generated by reflections of norm $-2$ vectors rather than by all reflections, so we take $R = 1$.

Section 2 contains a statement of the main result (theorem 2.7) describing a classifying category for the group $\Gamma_{\Omega}$, and section 3 contains the proof of this result. Section 4 contains some more information about the structure of $\Gamma_{\Omega}$. In section 5 we give some applications of theorem 2.7, and in particular show how to describe the automorphism groups of some Lorentzian lattices by embedding them in $II_{1,25}$ and using the description of $\text{Aut}(II_{1,25})$ in [C]. The idea of studying Lorentzian lattices by embedding them as orthogonal complements of root lattices in $II_{1,25}$ comes from Conway and Sloane ([C-S]).

Work of I. Piatetski-Shapiro and I. R. Shafarevich [P-S] shows that there is a map from the automorphism group of a K3 surface to the group of automorphisms of its Picard lattice modulo the group generated by reflections of norm $-2$ vectors which has finite kernel and co-finite image, so in practice if we want to describe the automorphisms of a K3 surface the main step is to calculate the automorphism group of its Picard lattice. Kondo showed in [K] that the automorphism groups of some K3 surfaces could be studied by embedding their Picard lattice as the orthogonal complement of a root lattice in $II_{1,25}$. We use Kondo’s idea to describe the automorphism groups of some K3 surfaces in terms of combinatorics of the Leech lattice. In particular we reprove some results of Vinberg [V] on the “most algebraic” K3 surfaces and extend them to the “next most algebraic” K3 surface. Kondo showed in [K] that the automorphism group of the Kummer surface of a generic genus 2 Jacobian was generated by the classically known automorphisms together with some new automorphisms found by Keum [Ke], and we show how to use Kondo’s results to describe the structure of this group. Kondo and Keum [K-K] have recently proved similar results for some Kummer surfaces associated to the products of two elliptic curves.

Kondo recently found another mysterious connection between automorphism groups of K3 surfaces and Niemeier lattices [K98], and used this to give a short proof of Mukai’s classification [Mu] of the finite groups that act on K3 surfaces.

I would like to thank I. Cherednik, I. Grojnowski, R. B. Howlett, J. M. E. Hyland, S. Kondo, U. Ray, and G. Segal for their help.

2. Notation and statement of main theorem.

This section states the main result (theorem 2.7) describing normalizers of parabolic subgroups of Coxeter groups.

We recall some basic definitions about Coxeter systems. For more about them see [Hi] or [Bo]. A pair $(W, S)$ is called a **Coxeter system** if $W$ is a group with a subset $S$ such that $W$ has the presentation

$$\langle s : s \in S | (ss')^{m_{s,s'}} = 1 \text{ when } m_{s,s'} < \infty \rangle$$
where \( m_{ss'} \in \{1, 2, 3, \ldots, \infty\} \) is the order of \( ss' \), and \( m_{ss'} = 1 \) if and only if \( s = s' \). A diagram automorphism of \( S \) is an automorphism of the set \( S \) that extends to an automorphism of the group \( W \), and \( Aut(S) \) means the group of diagram automorphisms of \( S \). We say that \((W, S)\) is irreducible if \( S \) is not a union of two disjoint commuting subsets. The number of elements of \( S \) is called its rank. The Coxeter system is called spherical if \( W \) has finite order. The irreducible spherical Coxeter diagrams are \( A_n \) \((n \geq 1)\), \( B_n = C_n \) \((n \geq 2)\), \( D_n \) \((n \geq 4)\), \( E_6, E_7, E_8, F_4, G_2^{(n)} = I_2(n) \) \((n \geq 5)\), \( H_3 \), and \( H_4 \). It is also sometimes useful to define the Coxeter diagrams \( B_1 = C_1 = A_1 \), \( D_3 = A_3 \), \( D_2 = A_1^2 \), \( E_5 = D_5, E_4 = A_4, E_3 = A_2 A_1, G_2^{(3)} = B_2 = C_2, G_2^{(3)} = A_2, G_2^{(2)} = A_1^2 \).

If \((W_\Pi, \Pi)\) is a Coxeter system then we write \( V_\Pi \) for the (possibly infinite dimensional) real vector space with a basis of elements \( e_s \) for \( s \in \Pi \), and put a symmetric bilinear form on \( V_\Pi \) by defining

\[
(e_s, e_s') = 2 \cos(\pi/m_{ss'}).
\]

Note that we normalize the roots \( e_s \) so that they have norm \((e_s, e_s) = -2 \) rather than 1; this is done to be consistent with the usual conventions in algebraic geometry.

The Coxeter group \( W_\Pi \) acts on \( V_\Pi \) with the element \( s \in \Pi \subseteq W_\Pi \) acting as the reflection \( v \mapsto v + (v, e_s) e_s \) in the hyperplane \( e_s^+ \). Any subgroup \( \Gamma_\Pi \) of \( Aut(\Pi) \) acts on \( V_\Pi \) by permutations of the elements \( e_s \), so we get an action of \( W \Gamma_\Pi \) on \( V_\Pi \), and hence on the dual space \( V_\Pi^* \). We write \( \Delta^+ \) for the set of positive roots of \( W_\Pi \). We define the fundamental chamber \( C_\Pi \subseteq V_\Pi^* \) of \( W_\Pi \) by

\[
C_\Pi = \{ x \in V_\Pi^* | x(r) \geq 0 \text{ for all } r \in \Pi \text{ and } x(r) > 0 \text{ for almost all } r \in \Delta^+ \}.
\]

(Recall that “almost all” means “all but a finite number of”. ) A theorem due independently to Tits and Vinberg states that no two distinct points of \( C_\Pi \) are conjugate under \( W_\Pi \), and the subgroup of \( W_\Pi \) fixing all points of some subset \( A \) of \( W_\Pi \) is generated by the reflections in the faces of \( C_\Pi \) containing \( A \). In particular \( W_\Pi \) acts simply transitively on the conjugates of \( C_\Pi \). The union \( W_\Pi(C_\Pi) \) of all conjugates of \( C_\Pi \) under \( W_\Pi \) is given by

\[
W_\Pi(C_\Pi) = \{ x \in V_\Pi^* | x(r) > 0 \text{ for almost all } r \in \Delta^+ \}.
\]

In particular \( W_\Pi(C_\Pi) \) is convex and closed under multiplication by positive real numbers. If \( x \in W_\Pi(C_\Pi) \) then the set of roots vanishing on \( x \) is a finite root system.

Note that \( W_\Pi(C_\Pi) \) is usually slightly smaller than the Tits cone, which is defined in the same way except that we omit the condition that \( x(e_s) > 0 \) for all but a finite number of \( s \) in the definition of the fundamental domain. The Tits cone can be thought of as obtained from \( W_\Pi(C_\Pi) \) by “adding some boundary components”. The reason for using \( W_\Pi(C_\Pi) \) rather than the Tits cone is that the cone \( W_\Pi(C_\Pi) \) has the property that the subgroup of \( W_\Pi \) fixing any vector of \( C_\Pi \) is finite.

We fix a spherical subset \( J \) of \( \Pi \). In particular we get a spherical Coxeter system \((W_J, J)\). Suppose \( K \) is an isometry of Coxeter diagrams from \( J \) into \( \Pi \). We write \( W_K \) for the finite reflection group generated by \( K(J) \). There is a natural homomorphism \( p : N_{W_\Pi} \Gamma_\Pi(W_K) \mapsto Aut(J) \). We let \( N_{W_\Pi} \Gamma_\Pi(W_K; \Gamma_J) \) be the subgroup of elements whose image is in a subgroup \( \Gamma_J \) of \( Aut(J) \). We are interested in describing
the group $N_{W_n \Gamma_n}(W_J; \Gamma_J)$. Most of the time we take $\Gamma_J = Aut(J)$ in which case $N_{W_n \Gamma_n}(W_J; \Gamma_J) = N_{W_n \Gamma_n}(W_J)$, but it is occasionally useful to use other values of $\Gamma_J$; see example 5.7 and theorem 4.1.

We define $W'_K$ to be the subgroup of $N_{W_n \Gamma_n}(W_K; \Gamma_J)$ mapping $K(J)$ to itself.

**Lemma 2.1.** $N_{W_n \Gamma_n}(W_K; \Gamma_J) = W_K \cdot W'_K$.

Proof. This follows immediately from the fact that $W_K$ acts simply transitively on the Weyl chambers of $W_K$, and $W'_K$ is the subgroup of $N_{W_n \Gamma_n}(W_K; \Gamma_J)$ fixing a Weyl chamber of $W_K$. This proves lemma 2.1.

The group $W'_K$ acts on the subspace $V^*_W$ of $V^*_W$ of all vectors fixed by $W_K$. We now construct a reflection group $W_{\Omega_K}$ acting on $V^*_W$. We choose a normal subgroup $R$ of $\Gamma_J$. (The subgroup $R$ is used to control the reflection group $W_\Omega$ defined below. Often we want $W_\Omega$ to be as large as possible and we take $R = \Gamma_J$, but sometimes we want to take a smaller $W_\Omega$; see examples 5.3, 5.4 and 5.5.) We define the group $W_{\Omega_K}$ to be the subgroup of $W_{\Omega} \cap W'_K$ generated by elements $w \in W_{\Omega} \cap W'_K$ such that $w$ acts on $V^*_W$ as a reflection and acts on $J$ as an element of $R$. We define $\Omega_K$ to be the Coxeter diagram of $W_{\Omega_K}$ and $C_{\Omega_K}$ to be its fundamental chamber. If $K$ is the identity map from $J$ to $\Omega$, then we write $W_\Omega$, $C_\Omega$, and $\Omega$ instead of $W_{\Omega_K}$, $C_{\Omega_K}$, and $\Omega_K$. Note that $W_{\Omega_K}$ is obviously contained in the inverse image of $R$ in $W_{\Omega} \cap W'_K$, but can be much smaller; see for example the discussion of $D_4$ in example 5.7.

We define the group $\Gamma_{\Omega_K}$ to be the subgroup of $W'_K$ of elements $w$ with $w(C_{\Omega_K}) = C_{\Omega_K}$.

**Lemma 2.2.** The group $W'_K$ is a semidirect product $W'_K = W_{\Omega_K} \cdot \Gamma_{\Omega_K}$.

Proof. We first show that the group $W_{\Omega_K}$ acts faithfully on $V^*_W$. More generally we will show that if $w \in W_{\Omega_K}$ acts trivially on $V^*_W$, then $w = 1$. To see this we observe that $w$ fixes the point $x \in V^*_W$ such that $x(e_s) = 0$ if $s \in K(J)$ and $x(e_s) = 1$ if $s \notin K(J)$. Therefore $w$ is in the subgroup $W'_K$ of $W_{\Omega}$ generated by the simple reflections of $W_{\Omega}$ fixing $x$. On the other hand $w$ maps $K(J)$ into itself as $w \in W'_K$. This implies that $w = 1$ because $1$ is the only element of $W'_K$ mapping $K(J)$ into itself. This proves that the group $W_{\Omega_K}$ acts faithfully on $V^*_W$.

Lemma 2.2 now follows from the fact that $W_{\Omega_K}$ acts simply transitively on the conjugates of $C_{\Omega_K}$ under $W'_K$, and $\Gamma_{\Omega_K}$ is the stabilizer of $C_{\Omega_K}$. This proves lemma 2.2.

Warning: the group $\Gamma_{\Omega_K}$ need not act faithfully on $V^*_W$ (though it does act faithfully on $V^*_W \times J$).

We define a **classifying category** of a group $\Gamma$ to be a category whose geometric realization is a classifying space for $\Gamma$. (Recall from [Q] that the geometric realization of a category is a space with a 0-cell for each object and an $n$-cell for each sequence of $n$ composable morphisms if $n > 0$.) For example, the category with one object whose morphisms are the elements of $\Gamma$ (with composition given by group multiplication) is a classifying category for $\Gamma$.

We have more or less reduced the problem of describing $N_{W_n \Gamma_n}(W_J; \Gamma_J)$ to that of describing $\Gamma_{\Omega_K}$. (The Coxeter diagram $\Omega$ of $W_{\Omega}$ can be described once we know $\Gamma_{\Omega}$.) The main theorem of this paper describes $\Gamma_{\Omega_K}$ by giving an explicit classifying category for it. To define this category we need some more definitions.
Suppose that $S$ is the Coxeter diagram of a finite reflection group $G$ of a finite dimensional vector space with no vectors fixed by $G$. Fix a Weyl chamber $C$ of $G$, so that the walls of $C$ correspond to the points of $S$. Then there is a unique element $\sigma_S$ of $G$ taking $C$ to $-C$ called the opposition involution. The involution $-\sigma_S$ acts on the Coxeter diagram $S$, and its action on $S$ does not depend on the choice of $C$. This action can be described as follows. The points of the Coxeter diagram correspond to the simple roots of $C$. This set of roots is the same as the set of simple roots of $\sigma_S(C) = -C$ multiplied by $-1$. Hence $-\sigma_S$ acts on this set of simple roots, in other words on the Coxeter diagram $S$. The involution $-\sigma_S$ of $S$ can be described explicitly as follows. On diagrams of type $A_n$, $B_n = C_n$, $D_{2n}$, $E_7$, $E_8$, $F_4$, $G_2^{(2n)}$, $H_3$, $H_4$: for any $n \geq 1$ the involution $-\sigma_S$ is the trivial automorphism of $S$, while for diagrams of types $A_{n+1}, D_{2n+1}, E_6, G_2^{(2n+1)}$ for $n \geq 1$ the involution $-\sigma_S$ is the unique nontrivial automorphism of the Coxeter diagram $S$. Finally if the diagram $S$ is a union of connected components then $-\sigma_S$ acts on each connected component as described above.

Suppose that $J$ and $S$ are Coxeter diagrams. Suppose that $K$ and $K'$ are two isometries from $J$ into $S$. We define $K$ and $K'$ to be adjacent if there is a point $s$ of $S$ not in $K(J)$ such that $K(J) \cup s$ is spherical and $\sigma_{K(J) \cup s}^s \sigma_{K(J)}$ takes $K$ to $K'$. If $K$ is adjacent to $K'$ then $K'$ is adjacent to $K$. We define two isometries $K$, $K'$ of $J$ into $S$ to be associate if $K(J)$ is a union of connected components then $-\sigma_S$ acts on each connected component as described above.

**Example 2.3.** Suppose $J$ is $A_1$ and $S$ is $A_3$, with the isometries from $J$ into $S$ labeled as $K_1$, $K_2$, $K_3$ in the obvious way. Then the isometry $K_1$ is adjacent to $K_2$ as $\sigma_{K_1(J) \cup K_2(J)}^s \sigma_{K_1}$ takes $K_1$ to $K_2$. Similarly $K_2$ is adjacent to $K_3$, and $K_1$ is adjacent to $K_3$. The isometries $K_1$ and $K_3$ are adjacent to themselves and $K_2$ is not adjacent to itself, so the isometry $K_2$ is not $R$-reflective but the isometries $K_1$ and $K_3$ are. So $K_2$ is not $R$-reflective but the equivalence class $\overline{K_2}$ is.

**Example 2.4.** Suppose $K$ is $D_5$ and $S$ is $D_6$. Then there are exactly two isometries $K_1, K_2 : J \hookrightarrow S$ of $S$, which are adjacent to each other but not to themselves. These two isometries are exchanged by the nontrivial automorphism of $D_5$. Hence if $R$ contains just $1 \in Aut(D_5)$ then $K_1$, $K_2$, and the equivalence class $\overline{K_1} = \{K_1, K_2\}$ are not $R$-reflective, but if $R$ is the whole of $Aut(D_5) = \mathbb{Z}/2\mathbb{Z}$ then all of them are $R$-reflective.

**Example 2.5** Suppose $K$ is $A_3$ and $S$ is $D_5$. Then there are 8 isometries $K : J \hookrightarrow S$. These form two equivalence classes under the relation of being associate, one of size 2 and
one of size 6. This shows that two isometries from a connected diagram $J$ into $S$ need not be associated to conjugates of each other under $\text{Aut}(J)$.

**Example 2.6** Suppose $J$ is $A_2$. If $S$ is $A_n$ for $n \geq 2$ then there are two equivalence classes of isometries $K : J \mapsto S$, which are exchanged by the nontrivial automorphism of $A_2$. However if $S$ is $D_n$ ($n \geq 4$, $E_6$, $E_7$, or $E_8$) then there is only one equivalence class, as the $A_2$ can be reversed by doing a “three point turn” around the point of valence 3 in $S$.

We define a poset $P^+_3$ as follows. The objects of $P^+_3$ are pairs $(S, K)$ consisting of a spherical subdiagram $S$ of $\Pi$ and an equivalence class $K$ of isometries from $J$ into $S$. We define the partial order on $P^+_3$ by putting $(S, K) \leq (S', K')$ if $S \subseteq S'$ and $K \subseteq K'$. We define $P_3$ to be the sub-poset of $P^+_3$ of elements $(S, K)$ such that $K$ is not $R$-reflective.

Note that the condition that $K$ is not $R$-reflective is quite restrictive and implies that $S$ is usually not much larger than $J$ and in any case has at most twice the rank of $J$. In particular $S$ cannot contain any root orthogonal to $K(J)$ as this implies that $K$ is $R$-reflective for any $R$.

Suppose that $P$ is a poset acted on by a group $G$. We define the homotopy quotient of $P$ by $G$ to be the following category $Q$. The objects of $Q$ are the elements of $P$. The morphisms from $p_1 \in P$ to $p_2 \in P$ correspond to the group elements $g \in G$ such that $g(p_1) \leq p_2$, and composition of morphisms is given by multiplication of group elements. If $G$ is trivial this is the usual category associated to the poset $P$, and if $P$ has just one point this is the usual category with one object associated to the group $G$. If we take a full set of representatives of the orbits of $G$ on $P$, then the full sub category of $Q$ with these objects is a skeleton of the category $Q$.

The poset $P_3$ is acted on by the group $\Gamma_J \times \Gamma_{II} \subseteq \text{Aut}(J) \times \text{Aut}(\Pi)$. We define $Q_4$ to be the homotopy quotient of $P_3$ by $\Gamma_J \times \Gamma_{II}$, and we construct the category $Q_4$ as a skeleton of $Q_4^+$ as above. In other words the objects of $Q_4$ are a complete set of representatives for the orbits of $\Gamma_J \times \Gamma_{II}$ on the elements of $P_3$ and the morphisms from $(S, K)$ to $(S', K')$ correspond to group elements $g \in \Gamma_J \times \Gamma_{II}$ such that $g((S, K)) \leq (S', K')$.

The main result of this paper is the following description of the classifying space of $\Gamma_{II}$.

**Theorem 2.7.** Suppose we are given the following objects.

1. $(W_{II}, \Pi)$ A Coxeter system.
2. $\Gamma_{II}$ A subgroup of $\text{Aut}(\Pi)$.
3. $(W_J, J)$ A spherical Coxeter system with $J \subseteq \Pi$.
4. $\Gamma_J$ A subgroup of $\text{Aut}(J)$.
5. $R$ A normal subgroup of $\Gamma_J$.

Define $W_{\Omega}$, $\Gamma_{\Omega}$, and $Q_4$ as above, so that the group $N_{W_{II}, \Gamma_{II}}(W_J; \Gamma_J)$ has the structure

$$W_J, W_{\Omega}, \Gamma_{\Omega}$$

where $W_{\Omega}$ is a Coxeter group. Then the component of $Q_4$ containing the object $(J, \overline{\lambda(J)})$ is a classifying category for the group $\Gamma_{\Omega}$.

Theorem 2.7 gives a presentation of the group $\Gamma_{\Omega}$ because $\Gamma_{\Omega}$ is the fundamental group of the category $Q_4$ with respect to the basepoint $(J, \overline{\lambda(J)})$, and it is easy to write down a presentation of the fundamental group of any connected category $Q$ as follows.
Choose a spanning tree $T$ for the underlying 1-complex of $Q$ (which has a point for each object of $Q$ and a 1-cell for each morphism). Then the fundamental group $\Gamma_\Omega$ of $Q$ has a presentation as follows. The group $\Gamma_\Omega$ has a generator $\gamma$ for each morphism $g$ of $Q$. The relations are $\gamma h = \gamma \overline{h}$ whenever $gh$ is defined, and $g = 1$ for $g$ in the spanning tree.

**Example 2.8.** Suppose we put $R = \Gamma J = \text{Aut}(J)$ and $\Gamma \Pi = 1$. Then we see from theorem 2.7 that $N_{W_\Pi}(W_J) = W_J W_\Omega \Gamma_\Omega$, where $\Gamma_\Omega$ is the fundamental group of the component of $Q_4$ corresponding to $J$. So in particular theorem 2.7 describes normalizers of finite parabolic subgroups of Coxeter groups. More generally, Brink and Howlett [B-H] have described a presentation for normalizers of possibly infinite parabolic subgroups of Coxeter groups. It is not trivial to see that the presentation given by [B-H] is equivalent to the one given by theorem 2.7 (though of course this follows from the fact that they are both presentations of the same group). Howlett pointed out to me that their result only requires considering subdiagrams $S$ of $\Pi$ whose rank is at most $2 + \text{rank}(J)$ to get the relations of $\Gamma_\Omega$, and of rank at most $1 + \text{rank}(J)$ to get the generators of $\Gamma_\Omega$. It seems possible that a similar simplification could be made to theorem 2.7 if all that is required is a presentation rather than a classifying space. Perhaps the natural map from $\pi_1(Q_4)$ to $\pi_1(Q_4)$ is an isomorphism for $i < j$ and an epimorphism for $i = j$, where $Q_4$ is the full sub-category of $Q_4$ whose objects are the elements $(S, K)$ such that $\text{rank}(S) \leq \text{rank}(J) + j$. If so, the map from $\pi_1(Q_4)$ to $\pi_1(Q_4)$ would be an isomorphism, so this would give a closer connection to the presentation of Brink and Howlett. Their result also suggests that theorem 2.7 could be generalized by allowing $W_J$ to be infinite and modifying the definition of $Q_4$ to allow subdiagrams $S$ such that $W_J$ has finite index in $W_S$.

3. Proof of main theorem.

This section gives the proof of the theorem 2.7. The idea of the proof is to construct categories and functors according to the following diagram.

$$Q_1 \leftarrow Q_2 \rightarrow Q_3 \leftarrow Q_4$$

It is easy to show that a component of $Q_1$ is a classifying category for $\Gamma_\Omega$. We also show that the functors between the categories are all homotopy equivalences, so a component of $Q_4$ is a classifying category for $\Gamma_\Omega$, which is what we wanted to prove.

We define an isometry from $J$ into the roots of $W_\Pi$ to be primitive if it is conjugate under $W_\Pi$ to an isometry from $J$ into $\Omega$. An example of a non-primitive isometry is an isometry from $A_1^2$ into the roots of $D_4$.

We define a category $Q_1$ as follows. We define the poset $P_1$ to be the poset of pairs $(C, K)$ where $K$ is a primitive isometry from $J$ into the (possibly non-simple) roots of $W_\Pi$, and $C$ is a Weyl chamber of the reflection group $W_{\Pi^K}$ of $W_\Pi^{W_K}$. The partial order on $P_1$ is the trivial one with $(C_1, K_1) \leq (C_2, K_2)$ if and only if $(C_1, K_1) = (C_2, K_2)$. The objects of $P_1$ are acted on in the natural way by the group $W_\Pi \Gamma_\Pi$ via its action on $W_\Pi$, and by the group $\Gamma_J$ via its action on $J$. We define the category $Q_1$ to be the homotopy quotient of $P_1$ by the group $\Gamma_J \times W_\Pi \Gamma_\Pi$.

**Lemma 3.1.** The component of $Q_1$ containing the object $id_J : J \mapsto J \subseteq \Pi$ is a classifying category for the group $\Gamma_\Pi$.
Proof. This follows immediately from the fact that $Q_1$ is a groupoid such that the automorphism group of the object $K$ is the group $\Gamma_{\Pi_K}$. This proves lemma 3.1.

Let $C_\Pi$ be the Weyl chamber of $W_\Pi$ defined in section 2. By a face of $C_\Pi$ we mean a nonempty intersection of $C_\Pi$ with some of the hyperplanes bounding $C_\Pi$. The faces of $C_\Pi$ of codimension $n$ correspond to the spherical subdiagrams of $\Pi$ of rank $n$. We define a $\Pi$-cell to be a conjugate of a face of $C_\Pi$ under $W_\Pi$. The cone $X$ is the union of all $\Pi$-cells, and the intersection of two $\Pi$-cells is either empty or another $\Pi$-cell.

We define a category $Q_2$ and posets $P_2, P_2^+$ as follows. The objects of the poset $P_2^+$ are the pairs $(D, K)$ where $K$ is a primitive isometry from $J$ into the roots of $W_\Pi$, and $D$ is a $\Pi$-cell contained in $V_\Pi^{W_K}$. We define the partial order on $P_2^+$ by saying $(D_1, K_1) \leq (D_2, K_2)$ if $D_1 \subseteq D_2$ and $K_1 = K_2$. We define $P_2$ to be the sub-poset of $P_2^+$ of elements $(D, K)$ such that $D$ is not contained in a reflection hyperplane of $W_\Gamma_K$. The category $Q_2$ is defined to be the homotopy quotient of $P_2$ by $\Gamma_J \times W_\Pi.\Gamma_\Pi$.

**Lemma 3.2.** Suppose $G$ is a group, $P_1$ and $P_2$ are $G$-posets, and $f$ is a morphism of $G$-posets from $P_2$ to $P_1$. Also suppose that for any $Y \in P_1$ the poset $f^{-1}(Y)$ is contractible (in other words the corresponding simplicial complex is contractible). Then the functor induced by $f$ between the homotopy quotient categories $Q_2, Q_1$ of the posets $P_2$ and $P_1$ by the group $G$ is a homotopy equivalence.

Proof. If $f$ is a functor from a category $Q_2$ to a category $Q_1$ and $Y$ is an object of $Q_1$ then we write $f^{-1}(Y)$ for the fiber of $f$ over $Y$, in other words the sub category of $Q_2$ whose morphisms are those mapped to the identity of $Y$ by $f$. We write $Y \backslash f$ for the category consisting of pairs $(X, v)$ with $v : Y \rightarrow f(X)$, where a morphism from $(X, v)$ to $(X', v')$ is a morphism $w : X \rightarrow X'$ such that $f(w)v = v'$. Then a result due to Quillen (the corollary to theorem A on page 9 of [Q]) states that $f$ is a homotopy equivalence provided that for all $Y$ in $Q_1$ the poset $f^{-1}(Y)$ is contractible and the functor from $f^{-1}(Y)$ to $Y \backslash f$ taking $X$ to $(X, id_Y)$ has a right adjoint. (Here $id_Y$ is the identity morphism of $Y$.)

We will use Quillen’s result to show that $f$ is a homotopy equivalence. For any object $Y$ of $P_1$ the category $f^{-1}(Y)$ is just the category of the poset $f^{-1}(Y)$, which is contractible by assumption. So it only remains to check the condition about the existence of a right adjoint from $Y \backslash f$ to $f^{-1}(Y)$. The category $Y \backslash f$ has as objects pairs $(X, v)$ with $v \in G$, $v(Y) \leq f(X)$ and there is a morphism from $(X, v)$ to $(X', v')$ if and only if $v^{-1}(X) = v'^{-1}(X')$, in which case the morphism is unique. We define a functor $g$ from $Y \backslash f$ to $f^{-1}(Y)$ on objects by $g((X, v)) = v^{-1}(X)$. It is easy to check that this extends in a unique way to morphisms. It is a right adjoint to $f$ because $Y \leq g((X, v))$ if and only if there is a morphism (necessarily unique) from $f(Y)$ to $(X, v)$, both conditions being equivalent to $v(Y) \leq X$. This shows that the conditions of Quillen’s result are satisfied, so $f$ is a homotopy equivalence. This proves lemma 3.2.

**Lemma 3.3.** The functor $f$ is a homotopy equivalence from $Q_2$ to $Q_1$.

Proof. By lemma 3.2 it is sufficient to check that for each $Y \in P_1$, the sub poset $f^{-1}(Y)$ of $P_2$ is contractible. The poset $f^{-1}(Y)$ is the poset of a cell decomposition of a convex cone in a real vector space. As any convex set is contractible, the poset $f^{-1}(Y)$ is also contractible. This proves lemma 3.3.
Lemma 3.4. Suppose \((W, S)\) is a spherical Coxeter system acting on the vector space \(V_S\) with Weyl chamber \(C\). Suppose \(K\) is an isometry from \(J\) into \(S\). Let \(V_S^{\text{WK}}\) be the subspace of \(V^*\) fixed by \(W_K\), where \(W_K\) is the reflection group whose simple roots are the points \(K(J)\). The walls of \(C \cap V_S^{\text{WK}}\) correspond to the points in \(S\) not in the image of \(K\); let \(s\) be one of these points and let \(s^+ \cap V_S^{\text{WK}}\) be the wall in \(V_S^{\text{WK}}\) corresponding to \(s\). Choose \(w \in W\) so that \(w(C)\) is the (unique) Weyl chamber of \(W\) such that \(w(C) \cap V_S^{\text{WK}}\) is the cell in \(V_S^{\text{WK}}\) on the other side of \(s^+ \cap V_S^{\text{WK}}\) to \(C \cap V_S^{\text{WK}}\) and such that \(C\) and \(w(C)\) are both in the same Weyl chamber of \(W_K\). Then

\[w = \sigma_{K(J)\cup s} \sigma_{K(J)}.\]

Proof. We can reduce to the case when \(S = K(J) \cup s\), so that \(V_S^{\text{WK}}\) is one dimensional and \(s^+ \cap V_S^{\text{WK}}\) is just the point 0. Then \(\sigma_s(\sigma_{K(J)}(C)) = -\sigma_{K(J)}(C)\) which contains \(-C \cap V_S^{\text{WK}}\), so \(\sigma_s(\sigma_{K(J)}(C)) \cap V_S^{\text{WK}}\) is a cell on the other side of \(s^+ \cap V_S^{\text{WK}}\) to \(C \cap V_S^{\text{WK}}\). Moreover \(\sigma_{K(J)}(C)\) is in the opposite Weyl chamber of \(K(J)\) to \(C\), and \(\sigma_s(\sigma_{K(J)}(C))\) is in the opposite Weyl chamber to \(\sigma_{K(J)}(C)\), so \(\sigma_s(\sigma_{K(J)}(C))\) is in the same Weyl chamber of \(K(J)\) as \(C\). This shows that the element \(w\) of the lemma is \(\sigma_s \sigma_{K(J)}\). This proves lemma 3.4.

A result similar to the following lemma (using subsets of \(S\) rather than isometries \(K : J \mapsto S\)) is given in [H, lemma 5] when \(S\) is finite and in [D] for arbitrary \(S\).

Lemma 3.5. Suppose \((W, S)\) is a Coxeter system, \(J\) is a spherical Coxeter diagram, and \(K\) and \(K'\) are two isometries from \(J\) into \(S\). Then \(K\) and \(K'\) are conjugate under \(W\) if and only if they are associate.

Proof. First suppose that \(K\) and \(K'\) are adjacent. Then \(\sigma_{K \cup K'}(\sigma_K(K)) = K'\), so \(K\) and \(K'\) are conjugate under \(W\). Next suppose \(K\) and \(K'\) are associate. Then by definition we can find a sequence \(K = K_1, K_2, \ldots, K_n = K'\) such that \(K_i\) and \(K_{i+1}\) are adjacent for all \(i\). Hence \(K = K_1\) and \(K' = K_n\) are also conjugate under \(W\). So associate isometries from \(J\) into \(S\) are conjugate under \(W\).

Conversely, suppose that \(K\) and \(K'\) are conjugate by an element \(w \in W\). Consider the subspace \(V_S^{\text{WK}}\) of \(V^*_S\), and the codimension 0 cells in it of the form \(V_S^{\text{WK}} \cap C\) for some Weyl chamber \(C\) of \(W\). For any two such cells, for example \(D = V_S^{\text{WK}} \cap C\) and \(D' = V_S^{\text{WK}} \cap w(C)\), we can find a sequence \(D = D_1, D_2, \ldots, D_n = D'\) such that \(D_i\) and \(D_{i+1}\) are adjacent by a face of codimension 1 in \(V_S^{\text{WK}}\). For each \(i\) let \(C_i\) be the (unique) Weyl chamber whose intersection with \(V_S^{\text{WK}}\) is \(D_i\) and that is contained in the Weyl chamber of \(W_K\). We identify each \(C_i\) with \(C_{ii}\) using \(w_i\). The set \(K(J)\) is a subset of the simple roots of \(C_i\), so \(K_i = w^{-1}_i(K)\) maps \(J\) to the simple roots of \(C_{ii}\). The isometries \(K_i\) and \(K_{i+1}\) are adjacent for all \(i\), because the element \(w_{i+1}w_i^{-1}\) mapping \(C_i\) to \(C_{i+1}\) is equal to \(\sigma_{K(J)\cup s_i} \sigma_{K(J)}\), where \(s_i\) is the simple root of \(C_i\) orthogonal to \(D_i \cap D_{i+1}\) but not to \(D_i\). Therefore \(K = K_1\) and \(K' = K_n\) are adjacent. This proves lemma 3.5.

Suppose that \(P\) is a \(W\)-poset for a group \(W\) with the property that if \(p \leq w(p)\) for \(w \in W, p \in P\) then \(p = w(p)\). We define the quotient \(W/P\) of \(P\) by \(W\) to be the poset whose elements are the orbits \(Wp\) of \(W\) acting on \(P\), where we put \(Wp \leq Wq\) if \(w(p) \leq q\) for some \(w \in W\). This should not be confused with the homotopy quotient of \(P\) by \(W\).
Lemma 3.6. The $\Gamma_J \times \Gamma_\Pi$ posets $P_2^+$ and $W_\Pi \backslash P_2^+$ are isomorphic.

Proof. We will construct an isomorphism $f$ of posets from $W_\Pi \backslash P_2^+$ to $P_2^+$. Suppose $(D, K)$ is an element of $P_2^+$ representing an element of $W_\Pi \backslash P_2^+$. We can find an element $w$ of $W_\Pi$ such that $w(D) \subseteq C_\Pi$ and $w(K(J)) \subseteq \Pi$. We define $f((D, K))$ to be $(S, w(K))$, where $S$ is the set of simple roots of $C_\Pi$ orthogonal to $w(D)$. We check that this is well defined even though $w$ is not unique. To prove this we can assume that $D \subseteq C_\Pi$ and $K(J) \subseteq \Pi$. Then the different possibilities for $w$ are elements of the group generated by the reflections fixing $D$ and the Weyl chamber of $K(J)$. But these elements take $K$ to an associated isometry $K : J \mapsto S$, so the equivalence class $K$ is well defined by lemma 3.5. These elements also take $S$ to $S$, so $S$ is well defined. This proves that $(S, K)$ is uniquely defined.

The isomorphism $f$ of posets from $W_\Pi \backslash P_2^+$ to $P_2^+$ obviously preserves the $\Gamma_J \times \Gamma_\Pi$ action on both posets. This proves lemma 3.6.

Lemma 3.7. Suppose $(W, S)$ is a spherical Coxeter system, $K$ is an isometry $K : J \mapsto S$, and $s$ is a point of $S$ not in $K(J)$. Then there is an element of $W$ mapping $V_s^{*W_K}$ to itself and acting on $V_s^{*W_K}$ as reflection in $s^\perp \cap V_s^{*W_K}$ if and only if $-\sigma_{K(J) \cup s}$ maps $K(J)$ to itself. If such an element of $W$ exists, then there is a unique such element $w$ mapping $K$ to itself, given by $w = \sigma_{K(J) \cup s} \sigma_{K(J)}$.

Proof. If an element of $w$ maps $V_s^{*W_K}$ to itself then there is a unique element of $W$ with the same action on $V_s^{*W_K}$ and mapping $K(J)$ to itself because $W_K$ acts simply transitively on its Weyl chambers, so we may assume that $w$ maps $K(J)$ to itself. If in addition $w$ acts on $V_s^{*W_K}$ as reflection in $s^\perp \cap V_s^{*W_K}$ then by lemma 3.4 $w$ must be $\sigma_{K(J) \cup s} \sigma_{K(J)}$.

Conversely if $-\sigma_{K(J) \cup s}$ maps $K(J)$ to itself then $\sigma_{K(J) \cup s} \sigma_{K(J)}$ maps $K(J)$ to itself and acts on $V_s^{*W_K}$ as reflection in $s^\perp \cap V_s^{*W_K}$. This proves lemma 3.7.

Lemma 3.8. Suppose $K$ is an isometry from $J$ into a spherical subdiagram $S$ of $\Pi$. Then $K : J \mapsto S$ is $R$-reflective if and only if the $\Pi$-cell $S^\perp \cap C_\Pi$ is contained in a reflection hyperplane of $W_{\Omega_K}$ of the form $s^\perp \cap V_s^{W_K}$ for $s \in \Pi$.

Proof. First suppose that $K : J \mapsto S$ is $R$-reflective. Then there is a point $s \in S$ not in $K(J)$ such that $w = \sigma_{K(J) \cup s} \sigma_{K(J)}$ acts on $K(J)$ as an element of $R$. So $w$ is a reflection of $W_\Omega$ corresponding to the hyperplane $s^\perp \cap V_s^{W_K}$, and this hyperplane contains $S^\perp \cap C_\Pi$.

Conversely, suppose that $S^\perp \cap C_\Pi$ is contained in a reflection hyperplane of $w \in W_{\Omega_K}$ of the form $s^\perp \cap V_s^{W_K}$ for $s \in \Pi$. Then we must have $s \in S$ because $S^\perp \cap C_\Pi \subseteq s^\perp$. The element $w$ must be equal to $\sigma_{K(J) \cup s} \sigma_{K(J)}$, and this element acts on $K$ as an element of $R$ because $w \in W_\Omega$. Therefore $K : J \mapsto S$ is $R$-reflective. This proves lemma 3.8.

Lemma 3.9. Suppose $K$ is an isometry from $J$ into a spherical subdiagram $S$ of $\Pi$. Then the equivalence class $K$ is $R$-reflective if and only if the $\Pi$-cell $S^\perp \cap C_\Pi$ of $V_s^{W_K}$ corresponding to $S$ is contained in a reflection hyperplane of $W_{\Omega_K}$.

Proof. Suppose the cell of $V_s^{W_K}$ corresponding to $S$ is contained in a reflection hyperplane of $W_{\Omega_K}$. Choose a Weyl chamber $C_\Pi$ for $W_\Pi$ such that this reflection hyperplane
is a wall of $C_k \cap W_k$. Then by lemma 3.8 the corresponding isometry $K' : J \mapsto S$ is $R$-reflective, and by lemma 3.5 is associate to $K$. So $\mathcal{R}$ is $R$-reflective.

Conversely suppose that $\mathcal{R}$ is $R$-reflective. Then $w(K)$ is $R$-reflective and has image in $S$ for some $w \in W_{\Omega}$. By lemma 3.8 this implies that $w(S_k \cap C_k)$ is contained in a reflection hyperplane of $W_{\Omega}$, so the same is true of $S_k \cap C_k$. This proves lemma 3.9.

Lemma 3.10. Suppose $P$ is a $G$-poset for some group $G$. Suppose that $W$ is a normal subgroup of $G$ such that if $p \leq q$ and $w(p) \leq q$ for some $w \in W$, $p, q \in P$, then $w = 1$. Then the homotopy quotient $Q_2$ of $P$ by $G$ is equivalent to the homotopy quotient $Q_3$ of $W \setminus P$ by $G/W$.

Proof. Recall from [M, theorem 1, page 91] that if $f$ is any functor from a category $Q_2$ to a category $Q_3$, then $f$ is an equivalence if the following two conditions are satisfied:
1. Any object of $Q_3$ is isomorphic to some object in the image of $f$.
2. For any two objects $p, q$ of $Q_2$, $f$ induces an isomorphism from $\text{Mor}(p, q)$ to $\text{Mor}(f(p), f(q))$.

We will apply this to show that our categories $Q_2$ and $Q_3$ are equivalent. We define $f$ on objects by $f(p) = Wp$, and define $f$ on morphisms using the obvious homomorphism from $G$ to $G/W$. Condition 1 above is satisfied because every element of $P_3$ is the image of an element of $P_2$, so every object of $Q_3$ is the image of an object of $Q_2$. Suppose $p$ and $q$ are objects of $P_2$. The set of morphisms of $Q_2$ from $p$ to $q$ can be identified with the set of elements $g$ of $G$ such that $g(p) \leq q$, and $\text{Mor}_{Q_2}(f(p), f(q))$ can also be identified with the set of elements $g$ of $G$ such that $g(p) \leq q$, so condition 2 above is satisfied. This shows that $f$ is an equivalence and proves lemma 3.10.

Lemma 3.11. If $p \leq q$ and $w(p) \leq q$ for some $w \in W_{\Omega}$, $p, q \in P_2$, then $w = 1$.

Proof. Suppose that $p = (D, K)$. Then $q = (D_1, K)$ for some $D_1$ containing $D$ as $p \leq q$. But then $w(D) \subseteq D_1$, so $w(D) = D$ as no two distinct subsets $D$, $w(D)$ of $D_1$ are conjugate under $W_{\Omega}$, as $D_1$ is contained in a fundamental domain of $W_{\Omega}$. Hence we can assume that $w$ fixes $D$ as well as $K$.

The subgroup of $W_{\Omega}$ fixing $D$ is a finite reflection group $W_D$ generated by the reflections of $W_{\Omega}$ fixing $D$ because $D \subseteq C_{\Omega}$. The subgroup of $W_{\Omega}$ fixing $K$ is generated by the reflections fixing all elements of $K(J)$. Any such reflection is in $W_{\Omega,K}$ because $K$ contains $1$. However, the condition that $p = (D, K) \in P_2$ implies that there are no reflections of $W_{\Omega,K}$ fixing $D$. Hence the subgroup of $W_D$ fixing $K$ is trivial. So any element $w \in W_{\Omega}$ such that $w(p) \leq q$ is trivial. This proves lemma 3.11.

Lemma 3.12. The categories $Q_2$ and $Q_3$ are equivalent.

Proof. Lemmas 3.10 and 3.11 show that there is an equivalence of categories from $Q_2$ to the homotopy quotient of $W_\Omega \setminus P_2$ by $\Gamma_j \times \Gamma_\Omega$. Lemma 3.6 shows that the $\Gamma_j \times \Gamma_\Omega$ posets $P_j^\Omega$ and $W_\Omega \setminus P_2^\Omega$ are isomorphic. Lemma 3.9 shows that the subset $P_3$ of $P_2^\Omega$ corresponds under this isomorphism to the subset $W_\Omega \setminus P_2$ of $W_\Omega \setminus P_2^\Omega$, so the $\Gamma_j \times \Gamma_\Omega$ posets $P_3$ and $W_\Omega \setminus P_2$ are isomorphic. Therefore the category $Q_2$ is equivalent to the homotopy quotient of $P_3$ by $\Gamma_j \times \Gamma_\Omega$, which is just $Q_3$. This proves lemma 3.12.
Lemma 3.13. The natural injection from $Q_4$ to $Q_3$ is an equivalence of categories.

Proof. This follows from the fact that $Q_4$ is a skeleton of $Q_3$, so the natural injection is an equivalence of categories. This proves lemma 3.13.

We can now prove theorem 2.7. By lemmas 3.3, 3.12, and 3.13, the categories $Q_4$ and $Q_1$ are homotopy equivalent. So by lemma 3.1, the component of $Q_4$ containing $(J, \mathbf{id}_J)$ is a classifying category for $\Gamma_\Omega$. This proves theorem 2.7.

4. The structure of $\Gamma_\Omega$.

Theorem 4.1. The kernel of the natural map from $\Gamma_\Omega$ to $\Gamma_J \times \Gamma_\Pi$ has finite cohomological dimension.

Proof. The classifying category of the kernel is just a component of the category of the poset $P_3$. A case by case check on possible Coxeter diagrams shows that the lengths of chains in $P_3$ are bounded (by $\text{rank}(J) + 1$ for example), so the corresponding simplicial complex has finite dimension at most $\text{rank}(J)$. Therefore the kernel has cohomological dimension at most $\text{rank}(J)$. This proves theorem 4.1.

Corollary 4.2. If $\Gamma_\Pi$ has finite virtual cohomological dimension, then so does $\Gamma_\Omega$.

Proof. This follows immediately from theorem 4.1 and the fact that $\Gamma_J$ is finite and standard properties of the virtual cohomological dimension.

Howlett showed that if $W_\Pi$ is finite then the group $\Gamma_\Omega$ is a subgroup of $\text{Aut}(J) \times \text{Aut}(\Pi)$. We can deduce this from theorem 4.1 as follows. If $W_\Pi$ is finite then so is the kernel of the map from $\Gamma_\Omega$ to $\text{Aut}(J) \times \text{Aut}(\Pi)$. On the other hand this kernel has finite cohomological dimension by theorem 4.1. But any finite group of finite cohomological dimension must be trivial, so the natural map from $\Gamma_\Omega$ to $\text{Aut}(J) \times \text{Aut}(\Pi)$ is injective. If $W_\Pi$ is infinite then this kernel is usually infinite, as can be seen from most of the examples below. The fact that this kernel no longer vanishes is the main reason why normalizers of parabolic subgroups of Coxeter groups are more complicated to describe when the Coxeter group is infinite.

5. Examples.

Example 5.1. Suppose that $J$ is $A_1$ and the group $\Gamma_\Pi$ is trivial. In this case Brink [Br] gave an elegant description of the group $\Gamma_\Omega$ as follows. Form the graph obtained from the Coxeter graph $\Pi$ by keeping only the edges of odd order. Then for any point $J = A_1$ of this new graph, the centralizer of the corresponding reflection (which is the group $\Gamma_\Omega$ corresponding to $J$) is the fundamental group of this graph with basepoint the chosen point, and in particular $\Gamma_\Omega$ is a free group.

We now check that this is equivalent to the description given by theorem 2.7. The only subdiagrams $S$ with a non-$R$-reflective class $\overline{K}$ are the points of the Coxeter graph or the edges of odd order together with their endpoints. So the category $Q_4$ has an object for each point or odd order edge of the Coxeter graph. The only non-identity morphisms correspond to inclusions of points in edges. The classifying space of this category is just the first barycentric subdivision of Brink’s graph. In particular the fundamental group of this category with some object as basepoint is canonically isomorphic to the fundamental
number 12 is the number of terms of the Leech lattice as follows: the group of order 2 is the group of automorphisms by Vinberg in [V]. In fact we can describe the various parts of Vinberg’s description in is the group \( G \) a normal subgroup isomorphic to the free product of 12 copies of \( \mathbb{Z}/6 \mathbb{Z} \). This is equivalent to the description of this group given by Vinberg in [V].

The following lemma can often be used to find the fundamental group of a category with at most 2 objects.

**Lemma 5.2.** Suppose \( A \) and \( B \) are subgroups of a group. Let \( Q \) be the category with 2 objects \( p \) and \( q \) such that \( \text{Mor}(p, p) = A \), \( \text{Mor}(p, q) = BA \), \( \text{Mor}(q, q) = B \), \( \text{Mor}(q, p) = \emptyset \), with composition defined in the obvious way. Then \( \pi_1(Q) = A \ast_{A \cap B} B \).

**Proof.** This can be proved by writing down a set of generators and relations for the fundamental group, and checking that they are equivalent to a set of generators and relations for \( A \ast_{A \cap B} B \). We will leave the details to the reader.

For most of the examples below we will take \( W_\Pi \Gamma_\Pi \) to be the group of automorphisms of the even 26 dimensional Lorentzian lattice \( H_{1,25} \) not exchanging the two cones of norm 0 vectors. According to Conway [C], the Coxeter group \( W_\Pi \) has a simple reflection \( r_\lambda \in \Pi \) for each vector \( \lambda \) of the Leech lattice \( \Lambda = \Pi \), and the order of \( r_\lambda \) is 1, 2, 3, or \( \infty \) according to whether \( (\lambda - \mu)^2 \) is 0, 4, 6, or greater than 6. The group \( \Gamma_\Pi \) is the automorphism group \( \Lambda.\text{Aut}(\Lambda) = \Lambda.(\mathbb{Z}/2\mathbb{Z}).C_01 \) of the affine Leech lattice, where \( \Lambda \) is the subgroup of translations and \( C_01 \) is Conway’s largest sporadic simple group.

If \( J \) is a spherical subdiagram of \( \Lambda \) then there is a homomorphism from \( N_{W_\Pi \Gamma_\Pi}(W_J)/W_J \) to the automorphism group of the lattice \( J^{\perp} \). This has finite kernel and co-finite image, so theorem 2.7 can usually be used to describe the automorphism group of the lattice \( J^{\perp} \). Most of the remaining examples in this section use this idea.

**Example 5.3.** Suppose \( L \) is the even Lorentzian lattice of dimension 20 and determinant 3. Let \( W^{(2)}(L) \) be the subgroup of \( \text{Aut}(L) \) generated by reflections of norm -2 vectors of \( L \). Vinberg showed in [V] that \( \text{Aut}(L)^+/W^{(2)}(L) \) was the automorphism group of a certain K3 surface modulo a cyclic subgroup, and also showed that this group was an extension of a group of order 72 by a free product of 12 groups of order 2. We will show how to recover Vinberg’s description of \( \text{Aut}(L)^+/W^{(2)}(L) \) from theorem 2.7.

We take \( J \) to be an \( E_6 \subset \Lambda \) so that \( L = J^{\perp} \), and take \( R = 1 \subset \Gamma_J = \text{Aut}(E_6) = \mathbb{Z}/2\mathbb{Z} \). The category \( Q_4 \) contains exactly two objects, corresponding to an \( E_6 \) and an \( E_7 \) in \( \Lambda \). For each subdiagram \( X \) of \( \Lambda \) we write \( G(X) \) for the automorphisms of \( \Pi = \Lambda \) mapping \( X \) into itself. The morphisms from the \( E_6 \) object to itself form a group \( G(E_6) \) of order 72. The morphisms from the \( E_7 \) object to itself form a group \( \mathbb{Z}/2\mathbb{Z} \times G(E_7) \) of order 2 \( \times 6 = 12 \) (where the \( \mathbb{Z}/2\mathbb{Z} \) comes from the group \( \Gamma_J \)). The morphisms from \( E_6 \) to \( E_7 \) can be identified with \( \mathbb{Z}/2\mathbb{Z} \times G(E_6) \). By lemma 5.2 the group \( \Gamma_\Omega \) is isomorphic to \( G(E_6) \ast_{G(E_6)} (G(E_7) \times \mathbb{Z}/2\mathbb{Z}) \). If \( A \), \( B \), and \( C \) are any groups with \( B \subset A \) then \( A \ast_B (B \times C) \) is a semidirect product of a normal subgroup isomorphic to the free product of \( |A|/|B| \) copies of \( C \) and with the quotient by this normal subgroup isomorphic to \( A \). Hence \( \Gamma_\Omega \) has a normal subgroup isomorphic to the free product of 12 copies of \( \mathbb{Z}/2\mathbb{Z} \), and the quotient is the group \( G(E_6) \) of order 72. This is equivalent to the description of this group given by Vinberg in [V]. In fact we can describe the various parts of Vinberg’s description in terms of the Leech lattice as follows: the group of order 2 is the group of automorphisms of \( E_6 \), the group of order 72 is the subgroup of \( \text{Aut}(\Lambda) \) mapping an \( E_6 \) into itself, and the number 12 is the number of \( E_7 \)’s of \( \Lambda \) containing an \( E_6 \).
The group $\text{Aut}(L)^+$ also contains reflections in norm $-6$ vectors. The quotient by the full reflection group is finite of order 72, isomorphic to $G(E_6)$. In this case the category $Q_4$ contains just one point. Note that the reflections of norm $-6$ vectors induce the nontrivial automorphism of $E_6$. The 12 elements of order 2 in the paragraph above are in fact reflections of norm $-6$ vectors. See [V] or [B] for more details of this case.

If $L$ is the even Lorentzian lattice of determinant 4 and dimension 20, which is again the Picard lattice of a K3 surface, then Vinberg gave a similar description of the automorphism group as an extension $((\mathbb{Z}/2\mathbb{Z}) \ast (\mathbb{Z}/2\mathbb{Z}) \ast (\mathbb{Z}/2\mathbb{Z}) \ast (\mathbb{Z}/2\mathbb{Z})) \ast S_3$ of the symmetric group $S_3$ by the free product of 5 group of order 2. This group can also be calculated using theorem 2.7. The corresponding category $Q_4$ has 2 objects, corresponding to a $D_6$ or $D_7$ in $\Lambda$, the group $S_3$ is the subgroup of $\text{Aut}(\mathbb{A})$ mapping the $D_6$ to itself, the group $Z/2Z$ is the group of automorphisms of the $D_6$ diagram, and the number 5 of copies of $Z/2Z$ is the number of $D_7$'s containing a $D_6$.

**Example 5.4** As a more complicated example we will describe the automorphism group of the Picard lattice $L$ of the “next most algebraic K3 surface”, in other words $L$ is the 20 dimensional even Lorentzian lattice of determinant 7. We take $\Pi = \Lambda$, $\Gamma_{\Pi} = \Lambda.\text{Aut}(\mathbb{A})$, $J = A_6$, $R = 1$, $\Gamma_J = \text{Aut}(A_6) = Z/2Z$. Then $\Gamma_{\Pi}$ is the subgroup of elements of $\text{Aut}(L)$ fixing a Weyl chamber of the reflection group generated by the reflections of norm $-2$ vectors. By theorem 2.7 the group $\Gamma_{\Pi}$ is the fundamental group of the category $Q_4$. The category $Q_4$ has exactly 5 objects, corresponding to the 5 orbits of Coxeter diagrams $A_6$, $A_7$, $D_7$, $E_7$, and $D_8$ with a non $R$-reflective isometry from $A_6$ into them. (The group $\Lambda.\text{Aut}(\mathbb{A})$ acts transitively on any of these Coxeter diagrams into $\Lambda$.) Note that for $D_8$ and $E_7$ there is only one equivalence class of isometries from $A_6 = J$ into it, while for $A_6$, $A_7$, and $D_7$ there are two classes, which are exchanged by $\text{Aut}(J) = Z/2Z$.

The category $Q_4$ looks like this.

$$
\begin{array}{cccc}
A_7(48) & & & \\
E_7(12) & \leftarrow & (672) & \rightarrow & (192) \\
& \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \rightarrow \\
& A_6(336) & D_8(16) & (672) & (672) & (2016) & (48) \\
& & & D_7(24) & & & \\
\end{array}
$$

Here the numbers are the numbers of morphisms between pairs of objects in $Q_4$.

**Example 5.5.** Kondo in [K] studied the automorphism group of a generic Jacobian Kummer surface by embedding its Picard lattice $L$ as the orthogonal complement of a certain $J = A_3A_1^6$ in $II_{1,25}$, and used this to describe a generating set for the automorphism group. (Note that the Leech lattice contains more than 1 orbit of subdiagrams of the form $A_3A_1^6$, the one used by Kondo has largest possible stabilizer in $\text{Aut}(\Lambda)$.) By results of Nikulin [N] the automorphism group of the K3 surface is the subgroup of $\text{Aut}(L)^+ / W(2)(L)$ of elements acting on $L/L$ as ±1. This is just the group $\Gamma_{\Pi}$ of theorem 2.7 where we take $\Pi = \Lambda$, $\Gamma_{\Pi} = \Lambda.\text{Aut}(\mathbb{A})$, $J$ to be Kondo’s $A_3A_1^6$, $R = 1$, and $\Gamma_J$ to be the subgroup of order 2 of $\text{Aut}(J)$ generated by the nontrivial automorphism of the $A_3$. The category $Q_4$ is not connected and the component containing $J$ seems quite complicated. Some partial calculations I have done suggest that the automorphism group of the generic Jacobian Kummer surface might be

$$(W.(Z/2Z)^5) \ast (Z/2Z) \ast (Z/2Z) \ast (Z/2Z) \ast (Z/2Z) \ast (Z/2Z)$$
where $W$ is a Coxeter group of rank $32+60$ generated by the 16 projections, 16 correlations, sixty Cremona transformations and the $(Z/2Z)^5$ is generated by sixteen translations and a switch $[K]$, but I have not proved this rigorously.

It is much easier to work out the group $\Gamma_\Omega$ for $R$ and $\Gamma_J$ replaced by $\text{Aut}(J) = Z/2Z \times S_6$. In this case the group $\Gamma_\Omega$ has a normal subgroup of index $|S_6|$ isomorphic to the quotient of the automorphism group of a generic Jacobian Kummer surface by the Coxeter group generated by reflections in norm $-4$ vectors. Theorem 2.7 describes the $\Gamma_\Omega$ as the fundamental group of a component of the finite category $Q_4$. This component has just two elements, corresponding to the Coxeter diagrams $A_3A_1^3$ and $A_5A_1^3$. Using lemma 5.2 and the results in $[K]$ we see that

$$\Gamma_\Omega = ((Z/2Z)^5.S_6) *_{S_5} (S_5 \times Z/2Z)$$

where $(Z/2Z)^5.S_6$ is the subgroup of $\Gamma_\Pi$ fixing $A_3A_1^6$ ([K, lemma 4.5]), and $S_5 \times Z/2Z$ is the subgroup of $\Gamma_\Pi$ fixing $A_3A_1^3$. In particular $\Gamma_\Omega$ has a normal subgroup which is the free product of 192 groups of order 2, and the quotient by this normal subgroup is $(Z/2Z)^5.S_6$. If we change $R$ to 1 but keep $\Gamma_J = \text{Aut}(J)$ then $\Gamma_\Omega$ becomes the group $(\text{Aut}(L))^+/W(2)(L)$ which appears to be

$$(W.(Z/2Z)^5.S_6) *_{S_5} (S_5 \times (Z/2Z))$$

though I have not proved this rigorously.

For more examples of automorphism groups of Kummer surfaces, corresponding to the cases $J = D_4, D_4A_3, D_4A_2, \text{ or } A_3^2$, see [K-K].

**Example 5.6** Suppose that $J$ in theorem 2.7 contains no components of types $A_n$ ($n \geq 1$) or $D_5$, and assume that $R = \Gamma_J = \text{Aut}(J)$. Then the map from $\Gamma_\Omega$ to $\Gamma_J \times \Gamma_\Omega$ is injective. This follows because a case by case check over all irreducible spherical Coxeter diagrams shows that any isometry from $J$ into a strictly larger spherical Coxeter diagram is $R$-reflective.

**Example 5.7** We show how to explain Vinberg’s result [V, V-K] that the reflection group of $I_{1,n}$ has finite index if and only if $n \leq 19$. Following Conway and Sloane [C-S] we write the even sublattice $L$ of $I_{1,n}$ as $D_{25-n}$ in $I_{1,25}$ for $n \leq 23$, where $D_3 = A_3$ and $D_2 = A_1^2$. In theorem 2.7 we take $\Pi = \Lambda$, $\Gamma_\Pi = \Lambda.\text{Aut}(\Lambda)$, $J = D_{25-n}$, $R = \Gamma_J$ a subgroup of order 2 of $\text{Aut}(J)$ (which is equal to $\text{Aut}(J)$ for $n \neq 21$). Then the quotient of $\text{Aut}(L)^+$ by its reflection subgroup is the group $\Gamma_\Omega$. For $n \leq 19$ the group $\Gamma_\Omega$ is finite by example 5.6. For $n = 20$ this argument breaks down because $J$ is the “exceptional” case $D_5$ of example 5.6. For $20 \leq n \leq 23$ we can still describe the group $\Gamma_\Omega$ explicitly using theorem 2.7; see [B, theorem 6.6] for details. When $n = 21$ this gives a natural example with $\Gamma_J \neq \text{Aut}(J)$.

If we take $J$ to be $D_4$ and take $R = \Gamma_J$ to be the symmetric group $S_4 = \text{Aut}(D_4)$ instead of a group of order 2 then $W_0, \Gamma_\Omega$ is the automorphism group of the even sublattice of $I_{1,21}$, and $\Gamma_\Omega$ is a finite group. See [B, p. 149] for details.

**Example 5.8.** The groups $\Gamma_\Omega$ have many of the properties of arithmetic groups; for example, they often have finite classifying categories. (It follows easily from [S] that arithmetic groups have this property.) It is natural to ask when they are arithmetic. There seems to be no obvious general way of deciding this. The following argument can often
be used to show that $\Gamma_{\Omega}$ is not arithmetic. First of all a theorem due to Margulis [Ma, page 3] implies that if a group is an arithmetic subgroup of a group of rank at least 2, then all normal subgroups are either in the center or of finite index. Secondly, a theorem of Borel and Serre [B-S, 11.4.4] says that if a group $\Gamma$ is arithmetic in a Lie group $G$ then $d = r + vcd(\Gamma)$ where $d$ is the dimension of the symmetric space of $G$ and $r$ is the rank of $G$ and $vcd(\Gamma)$ is the virtual cohomological dimension of $\Gamma$. Let us use these results to prove that the group $\Gamma_{\Omega}$ of example 5.3 (the automorphism group of a K3 surface) is not arithmetic in any Lie group $G$. The group $G$ must have rank 1 by the theorem of Margulis, as $\Gamma_{\Omega}$ has non abelian free subgroups of finite index and so cannot be an arithmetic subgroup of a group of rank at least 2. Its virtual cohomological dimension is one, so by the theorem of Borel and Serre [B-S, 11.4.4] the symmetric space of $G$ must have dimension $1 + 1 = 2$. But any finite subgroup of $\Gamma_{\Omega}$ must fix a point of this symmetric space, and therefore acts faithfully on the 2 dimensional tangent space of this point. But $\Gamma_{\Omega}$ has finite subgroups that are too large to have 2 dimensional faithful representations. Hence $\Gamma_{\Omega}$ is not arithmetic. Note that $\Gamma_{\Omega}$ has subgroups of finite index that are free and therefore arithmetic.

References.


[K98] S. Kondo, Niemeier lattices, Mathieu groups and finite groups of symplectic automor-

[M] S. MacLane, Categories for the working mathematician. Graduate Texts in Mathe-

[Ma] G. A. Margulis, “Discrete subgroups of semisimple Lie groups”. Ergebnisse der Math-
12179-X.


[N] V. V. Nikulin, An analogue of the Torelli theorem for Kummer surfaces of Jacobians.

[P-S] I. R. Shafarevich, I. I. Piatetski-Shapiro, Torelli’s theorem for algebraic surfaces of
translation in pages 516-557 of Collected mathematical papers by Igor R. Shafarevich.


1–21.

[V-K] È. B. Vinberg, I. M. Kaplinskaja, The groups $O_{18,1}(Z)$ and $O_{19,1}(Z)$. (Russian) Dokl.