Classification of positive definite lattices. *22 Feb 2000*

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Contents.

1. Classification of positive norm vectors.

In this paper we describe an algorithm for classifying orbits of vectors in Lorentzian lattices. The main point of this is that isomorphism classes of positive definite lattices in some genus often correspond to orbits of vectors in some Lorentzian lattice, so we can classify some positive definite lattices. Section 1 gives an overview of this algorithm, and in section 2 we describe this algorithm more precisely for the case of $II_{1,25}$, and as an application we give the classification of the 665 25-dimensional unimodular positive definite lattices and the 121 even 25 dimensional positive definite lattices of determinant 2 (see tables 1 and 2). In section 3 we use this algorithm to show that there is a unique 26 dimensional unimodular positive definite lattice with no roots. Most of the results of this paper are taken from the unpublished manuscript [B], which contains more details and examples. For general facts about lattices used in this paper see [C-S], especially chapters 15–18 and 23–28.

Some previous enumerations of unimodular lattices include Kneser's list of the unimodular lattices of dimension at most 16 [K], Conway and Sloane's extension of this to dimensions at most 23 [C-S chapter 16], and Niemeier's enumeration [N] of the even 24 dimensional ones. All of these used some variation of Kneser's neighborhood method [K], but this becomes very hard to use for odd lattices of dimension 24, and seems impractical for dimension at least 25 (at least for hand calculations; computers could probably push this further). The method used in this paper works well up to 25 dimensions, could be pushed to work for 26 dimensions, and does not seem to work at all beyond this.

We use the "$(+,-,-,\cdots,-)$" sign convention for Lorentzian lattices L, so that the reflection we are interested in are (usually) those of *negative* norm vectors of L. We fix one of the two cones of positive norm vectors and call it the positive cone. The norm 1 vectors in the positive cone form a copy of hyperbolic space in the usual way. We assume that we are given a group G of automorphisms of a Lorentzian lattice L, such that G is the semidirect product of a normal subgroup R generated by reflections of some negative

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norm vectors, and a group $\mathrm{Aut}(D)$ of automorphisms preserving a fundamental domain domain D of R in hyperbolic space. We assume that all elements of $u \in L$ having non-negative inner product with all simple roots of R have norm (u, u) at least 0 (this is just to eliminate some degenerate cases). If L is a lattice then $L(-1)$ is the lattice L with all norms multiplied by -1. We use Conway's convention of using small letters a_n, d_n, e_n for the spherical Dynkin diagrams, and capital letters A_n, D_n, E_n for the corresponding affine Dynkin diagrams. The Weyl vector of a root system is the vector ρ such that $(\rho, r) = -r^2/2$ for any simple root r.

We want to find the orbits of positive norm vectors of the positive cone of L under the group G. Every positive norm vector of the positive cone of L is conjugate under R to a unique vector in D, so it is enough to classify orbits of vectors u in D under $\mathrm{Aut}(D)$.

The algorithm works by trying to reduce a vector u of D to a vector of smaller norm by adding a root of $u^\perp$ to u. There are three possible cases we need to consider:

(1) There are no roots in $u^\perp$.
(2) There is a root r in $u^\perp$ such that $u + r \in D$.
(3) There is at least one root in $u^\perp$, but if r is a root in $u^\perp$ then $u + r$ is never in D.

We try to deal with these three cases as follows.

If there are no roots in $u^\perp$, then we assume that D contains a non-zero vector w such that $(r, w) \le (r, u)$ for any simple root r and any vector $u \in L$ in the interior of D. Then $u - w$ has inner product at least 0 with all simple roots, so it also lies in D and has smaller norm than u unless u is a multiple of w and $w^2 = 0$. So we can reduce u to a vector of smaller norm in D. The existence of a vector w with these properties is a very strong condition on the lattice L.

Example 1.1. The lattices $II_{1,9}$ and $II_{1,17}$ have properties 1 and 2; this follows easily from Vinberg's description [V85] of their automorphism groups. Conway showed that the lattice $II_{1,25}$ also has these properties; see the next section. The lattices $II_{1,8n+1}$ for $n \ge 4$ do not have these properties; but the Minkowski-Siegel mass formula shows that these lattices have such vast numbers of orbits of positive norm vectors that there seems little point in classifying them.

Example 1.2. It follows from [B90] that several lattices that are fixed points of finite groups acting on $II_{1,25}$ also have a suitable vector w. For example the lattice $II_{1,1} \oplus BW(-1)$, where BW is the Barnes-Wall lattice, has this property. Some of the norm 0 vectors correspond to the 24 lattices in the genus of BW classified in [S-V]; the remaining orbits of norm 0 vectors should not be hard to find.

Example 1.3. Take L to be the lattice $I_{1,9}$ and R to be the group generated by reflections of norm -1 vectors. (This has infinite index in the full reflection group.) Then the lattice has a Weyl vector for the reflection group as in [B90], so we can apply the algorithm to this reflection group. (However it is not entirely clear what the point of doing this is, as it is easier to use the full reflection group of the lattice!)

Next we look at the second case when $u^\perp$ has a root r such that $v = u - r$ is in D. Then $-r$ is in the fundamental domain of the finite reflection group of $u^\perp$, so r is a sum of the simple roots of $u^\perp$ with the usual multiplicities.

For u in D we let $R_i(u)$ be the simple roots u of D that have inner product $i(r, r)/2$ with u, so $R_i(u)$ is empty for $i < 0$ and $R_0(u)$ is the Dynkin diagram of $u^\perp$. We write

$S(u)$ for $R_0(u) \cup R_1(u) \cup R_2(u)$. Then given $S(v)$ we can find all vectors u of D that come from v as in (3) above, and $S(u)$ is contained in $S(v)$. By keeping track of the action of $\mathrm{Aut}(D, v)$ on $S(v)$ for vectors v of D we can find all possible vectors v constructed in this way from v, together with the sets $S(u)$.

Finally, the third case, when there is at least one root in $u^\perp$, but if r is a root in $u^\perp$ then $v - r$ is never in D, has to be dealt with separately for each lattice L. In practice it does not present too much difficulty for lattices with a vector w as in case 1. See the next section for the example of $L = II_{1,25}$.

The following two lemmas will be used later to prove some properties of the root systems of 25 dimensional lattices.

Lemma 1.4. *Suppose that reflection in $u^\perp$ is an automorphism of L. Then there is an automorphism σ of L (of order 1 or 2) with the following properties:*
(1) σ fixes D.
(2) If σ fixes w, then w is a linear combination of u and the roots of L in $u^\perp$.

Proof. There is an automorphism of L acting as 1 on u and as -1 on $u^\perp$, given by the product of -1 and reflection in $u^\perp$. As this automorphism fixes $u \in D$, we can multiply it by some (unique) element of the reflection group of $u^\perp$ so that the product σ fixes D. The element σ acts as -1 on the space orthogonal to u, z, and all roots of R in $u^\perp$, which implies assertion (2) of the lemma 1.4.

Lemma 1.5. *Suppose that there is a norm 0 vector z such that $(z, u) = 2$, where u is a vector in D. Then there is an automorphism σ of L with the following properties:*
(1) σ fixes D.
(2) If σ fixes w, then w is a linear combination of u, z, and the roots of L in $u^\perp$.

Proof. If M is the lattice spanned by z and u then M has the property that all elements of M'/M have order 1 or 2. So there is an automorphism of L acting as 1 on M and -1 on $M^\perp$. The result now follows as in the proof of lemma 1.4. This proves lemma 1.5.

Remark. It is usually easy to classify all orbits of negative norm vectors u in Lorentzian lattices, because this is closely related to the classification of the *indefinite* lattices $u^\perp$, and by Eichler's theorem [E] indefinite lattices in dimension at least 3 are classified by the spinor genus (which in practice is often determined by the genus). For example, it is easy to give a proof along these lines that if $n > 0$ and $m > 0$ then $II_{1,8n+1}$ has a unique orbit of primitive vectors of norm $-2m$.

2. Vectors in the lattice $II_{1,25}$.

In this section we specialize the algorithm of the previous section to the lattice $II_{1,25}$.

Note that orbits norm 4 vectors u of $II_{1,25}$ correspond naturally to 25 dimensional positive definite unimodular lattices, because $u^\perp$ is isomorphic to the lattice of even vectors in a 25 dimensional unimodular negative definite lattice. In particular we can classify the 665 positive definite 25 dimensional unimodular lattices, as in table 2; this is the main application of the algorithm of the previous section. Similarly norm 2 vectors of $II_{1,25}$ correspond to 25 dimensional even positive definite lattices of determinant 2. (Another interpretation of the vectors of $II_{25,1}$ of norm at least -2 is that they are the roots of the fake monster Lie algebra.)

First we have to show the existence of a vector w satisfying the property of section 1. This follows from Conway's theorem [C85] stating that the reflection group of $II_{1,25}$ has a Weyl vector w of norm 0, with the property that $(w,r)=1$ for all simple roots r of the reflection group. Conway's proof depends on the rather hard classification of the "deep holes" in the Leech lattice in [C-P-S]; there is a proof avoiding these long calculations in [B85]. It seems likely that 26 is the largest possible dimension of a lattice with a suitable vector w.

Next we have to classify the vectors u of D such that $u^\perp$ has roots but $u+r$ is not in D for any root $r \in u^\perp$. One obvious way this can happen is if u has norm 0, so we have to classify the norm 0 vectors in $II_{1,25}$. In any lattice $L = II_{8n+1,1}$ the orbits of primitive norm 0 vectors z correspond to the $8n$-dimensional even negative definite unimodular lattices $z^\perp/z$. So the orbits of primitive norm 0 vectors of $II_{1,25}$ correspond to the 24 Niemeier lattices ([C-S]). The non-primitive norm 0 vectors are of course either 0 or a positive integer multiple of a primitive norm 0 vector, so this gives the classification of all orbits of norm 0 vectors in $II_{1,25}$; see table 0.

Next suppose that u is a positive norm vector of D with $(u,u)=2n$ and r is a highest root in $u^\perp$ such that $u-r$ is not in D. Then $u-r$ is conjugate under the reflection group to some vector v such that $(v,u) < (u-r,u)$. But $(v,u)^2 \geq (u,u)(v,v) = 2n(2n-2)$ and $(v,u) < (u-r,u) = 2n$, so $(v,u) = 2n-1$. So if $z = u-v$ then $(z,u)=1$ and $z^2=0$. If we put $z' = u - nz$ then z and z' are norm 0 vectors with $(z,z')=1$ and $u = nz+z'$. So $II_{1,25} = B \oplus \langle z, z' \rangle$ for some Niemeier lattice B. If this Niemeier lattice has roots, then adding some of these roots to r gives a vector in D by the previous argument, so B must be the Leech lattice so we can assume that z is in the orbit of w. If $n>1$ then there are no roots in $u^\perp$, and if $n<1$ then $(u,u)\leq 0$, so we must have $n=1$. So the only possibility for u is that it is a norm 2 vector in the orbit of $w+w' = 2w+r$, where r is a simple root.

Putting everything together gives the following list of the vectors $u \in D$ such that $u^\perp$ has roots but $u-r$ is not in D for any root $r \in u^\perp$:

1. The zero vector.
2. The norm 0 vectors nz for $n \geq 1$ and z a primitive norm 0 vector of D corresponding to some Niemeier lattice other than the Leech lattice. The vectors for a given Niemeier lattice and a given value of n are all conjugate under $\mathrm{Aut}(D)$.
3. The norm 2 vectors of the form $2w+r$ for a simple root r of D. These form one orbit under $\mathrm{Aut}(D)$.

Lemma 2.1. *Suppose $u, v \in D$, $u^2 = 2n$, $v^2 = 2(n-1)$ and $(v,u) = 2n$. Then*

$$R_0(u) \subseteq R_0(v) \cup R_1(v) \cup R_2(v) = S(v)$$
$$R_i(u) \subseteq R_0(v) \cup R_1(v) \cup \cdots \cup R_i(v) \text{ for } i \geq 1.$$

Proof. The vector v is in D, so $v = u+r$ for some highest root r of $u^\perp$. The vector r has inner product 0, 1, or 2 with all simple roots of $u^\perp$, and $-r$ is a sum of roots of $R_0(u)$ with positive coefficients, so r has inner product ≥ 0 with all simple roots of D not in $R_0(u)$. The lemma follows from this and the fact that $(v,s) = (u,s)+(r,s)$ for any simple root s of D. This proves lemma 2.1.

We now start with a vector v of norm $2(n-1)$ and try to reconstruct u from it. The vector $u-v$ is a highest root of some component of $R_0(u)$, and $R_0(u)$ is contained in $S(v)$, so we should be able to find u from $S(v)$. By lemma 2.1 $S(u)$ is contained in $S(v)$, so we can repeat this process with u instead of v. The following theorem shows how to construct all possible vectors u as in lemma 2.1 from v and $S(v)$.

Theorem 2.2. *Suppose that v has norm $2(n-1)$ and is in D (so $n \geq 1$). Then there are bijections between*

(1) Norm $2n$ vectors u of D with $(u,v) = 2n$.

(2) Simple spherical Dynkin diagrams C contained in the Dynkin diagram Λ of D such that if r is the highest root of C and c in C satisfies $(c,r) = i$, then c is in $R_i(v)$.

(3) Dynkin diagrams C satisfying one of the following three conditions:

Either C is an a_1 and is contained in $R_2(v)$,

or C is an a_n ($n \geq 2$) and the two endpoints of C are in $R_1(v)$ while the other points of C are in $R_0(v)$,

or C is d_n ($n \geq 4$), e_6, e_7, or e_8 and the unique point of C that has inner product 1 with the highest root of C is in $R_1(v)$ while the other points of C are in $R_0(v)$.

Proof. Let u be as in (1) and put $r = u - v$. The vector r is orthogonal to u and has inner product ≤ 0 with all roots of R_0 (because $-v$ does) so it is a highest root of some component C of $R_0(u)$. The vector r therefore determines some simple spherical Dynkin diagram C contained in Λ. Any root c of C has $(c, v+r) = (c,u) = 0$, so c is in $R_i(v)$ where $i = (c,r)$. This gives a map from (1) to (2).

Conversely if we start with a Dynkin diagram C satisfying (2) and put $u = v + r$ (where r is the highest root of C) then $(c,u) = 0$ for all c in C, so $(r,u) = 0$ as r is a sum of the c's. This implies that $u^2 = 2n$ and $(u,v) = 2n$. We now have to show that u is in D. Let s be any simple root of D. If s is in C then $(s,r) = -(s,v)$ and if s is not in C then $(s,r) \geq 0$, so in any case $(s,u) = (s, v+r) \geq 0$ and hence u is in D. This gives a map from (2) to (1) and shows that (1) and (2) are equivalent.

Condition (3) is just the condition (2) written out explicitly for each possible C, so (2) and (3) are also equivalent. This proves theorem 2.2.

We define the **height** of a vector u in $II_{1,25}$ to be (u,w). We show how to calculate the heights of vectors of $II_{1,25}$ that have been found with the algorithm above.

Lemma 2.3. *Suppose u, v are vectors in D of norms $2n$, $2(n-1)$ with $(u,v) = 2n$ and suppose that $v = u - r$ for some root r of $u^\perp$ corresponds to the component C of $R_0(u)$. Then*

$$\text{height}(u) = \text{height}(v) + h - 1$$

where h is the Coxeter number of the component C.

Proof. We have $v = u - r$ where r is the highest root of C, so $\text{height}(u) = \text{height}(v) + (r,w)$. We have $r = \sum_i m_i c_i$ where the c_i are the simple roots of C with weights m_i and $\sum_i m_i = h - 1$. All the c_i have inner product 1 with w, so $(r,w) = h-1$. This proves lemma 2.3.

Lemma 2.4. *Let u be a primitive vector of D such that there is a norm 0 vector z with $(z,u) = 0$ or 1, and suppose that z corresponds to a Niemeier lattice B with Coxeter number h.*

(1) If u has norm 0 then its height is h. The Dynkin diagram of $u^\perp$ is the extended Dynkin diagram of B.

(2) If u has positive norm then $\text{height}(u) = 1 + (1 + u^2/2)h$. *The Dynkin diagram of $u^\perp$ is the Dynkin diagram of B if $u^2 > 2$ and the Dynkin diagram of B plus an a_1 if $u^2 = 2$.*

Proof.

(1) The Dynkin diagram of $u^\perp$ is a union of extended Dynkin diagrams. If this union is empty then u must be w and therefore has height $0 = h$. If not then let C be one of the components. We have $u = \sum_i m_i c_i$ where the c_i's are the simple roots of C with weights m_i. Also $\sum_i m_i = h$ because C is an extended Dynkin diagram and all the c_i's have height 1, so u has height h.

(2) As u has inner product 1 with a norm 0 vector z of D we can put $u = nz + z'$ with $u^2 = 2n$ and $z'^2 = 0$, $(z, z') = 1$. By part (1) z has height h. We have $z' = z + r$ where r is a simple root of D, so $\text{height}(z') = \text{height}(z) + \text{height}(r) = h + 1$. Hence $\text{height}(u) = nh + h + 1 = 1 + (1 + u^2/2)h$. The lattice $u^\perp$ is $B \oplus N$ where N is a one dimensional lattice of determinant $2n$, so the Dynkin diagram is that of B plus that of N, and the Dynkin diagram of (norm 2 roots of) N is empty unless $2n = 2$ in which case it is a_1. This proves that the Dynkin diagram of $u^\perp$ is what it is stated to be. This proves lemma 2.4.

Orbits of norm 2 vectors $u \in II_{1,25}$ correspond to even 25 dimensional positive definite lattices B of determinant 2, where $B(-1) \cong u^\perp$. One part of the algorithm for finding vectors of norm $2n$ consists of finding the vectors u such that there are no roots in $u^\perp$. For norm 2 vectors u the following lemma shows that there are no such vectors.

Lemma 2.5. *If $u \in II_{1,25}$ has norm 2 then $u^\perp$ contains roots. In other words every 25 dimensional even positive definite lattice of determinant 2 has a root.*

Proof. If $u^\perp$ contains no roots then, by the algorithm of section 1, $u = w + u_1$ for some u_1 in D. We have $u_1^2 = u^2 - 2\text{height}(u)$, so $u_1^2 = 0$ and u has height 1 because $u_1^2 \geq 0$, $u^2 = 2$ and the height of u is positive. Then $\text{height}(u_1) = \text{height}(u) = 1$, so u is a norm 0 vector in D that has inner product 1 with the norm 0 vector w of D, but this is impossible as $u - w$ would be a norm -2 vector separating the two vectors u and w of D. This proves lemma 2.5.

Theorem 2.6. *Suppose that $u \in D$ has norm 2. Then*

$$w = \rho + \text{height}(u)u/2$$

where ρ is the Weyl vector of the root system of $u^\perp$. Also $-2\rho^2 = \text{height}(u)^2$.

Proof. The vector w is fixed by any automorphism fixing D, so by lemma 1.4 the vector w must be in the space spanned by u and the roots of $u^\perp$. However w also has inner product 1 with all simple roots of $u^\perp$ and has inner product $\text{height}(u)$ with u, so w must be $\rho + \text{height}(u)u/2$. Taking norms of both sides of $w = \rho + \text{height}(u)u/2$, and using the facts that $w^2 = 0$, $(u, \rho) = 0$, and $(u, u) = 2$, shows that $-2\rho^2 = \text{height}(u)^2$. This proves theorem 2.6.

In particular we find the strange consequence that the norm of the Weyl vector of any 25 dimensional even positive definite lattice of determinant 2 must be a half a square.

Norm 4 vectors in the fundamental domain D of $II_{1,25}$ correspond to 25 dimensional unimodular lattices $A = A_1 \oplus I^n$, where $u^\perp$ is the lattice of even elements of $A(-1)$ and A_1 has no norm 1 vectors . The odd vectors of $A(-1)$ can be taken as the projections of the vectors y with $(y, u) = 2$ into $u^\perp$. A norm 4 vector u can behave in 4 different ways, depending on whether the unimodular lattice A_1 with no norm 1 vectors corresponding to u is at most 23 dimensional, or 24 dimensional and odd, or 24 dimensional and even, or 25 dimensional.

Theorem 2.7. *Norm 1 vectors of A correspond to norm 0 vectors z of $II_{1,25}$ with $(z, u) = 2$. Write $A = A_1 \oplus I^n$ where A_1 has no vectors of norm 1. Then u is in exactly one of the following four classes:*

(1) u has inner product 1 with a norm 0 vector. The lattice A_1 is a Niemeier lattice.

(2) A has at least 4 vectors of norm 1, so that A_1 is at most 23 dimensional (but may be even). There is a unique norm 0 vector z of D with $(z, u) = 2$ and this vector z is of the same type as either of the two even neighbors of $A_1 \oplus I^{n-1}$.

(3) A_1 is 24 dimensional and odd. There are exactly two norm 0 vectors that have inner product 2 with u, and they are both in D. They have the types of the two even neighbors of A_1.

(4) $A = A_1$ has no vectors of norm 1.

Proof. The vector z is a norm 0 vector with $(z, u) = 2$ if and only if $u/2 - z$ is a norm 1 vector of A. Most of 2.7 follows from this. The only non-trivial things to check are the statements about norm 0 vectors that are in D.

If u does not have inner product 1 with any norm 0 vector then a norm 0 vector z with $(z, u) = 2$ is in D if and only if it has inner product ≥ 0 with all simple roots of $u^\perp$, so there is one such vector in D for each orbit of such norm 0 vectors under the reflection group of $u^\perp$. If A has at least 4 vectors of norm 1 then they form a single orbit under the Weyl group of (the norm 2 vectors of) $u^\perp$, which proves (2), while if A has only two vectors of norm 1 then they are both orthogonal to all norm 2 vectors of A and so form two orbits under they Weyl group of $u^\perp$. This proves theorem 2.7.

Theorem 2.8. *Suppose that u is a norm 4 vector corresponding to a unimodular 25 dimensional lattice $A = A_1 \oplus I^{25-n}$ with $2n \geq 4$ vectors of norm 1. Let ρ be the Weyl vector of the root system of norm -2 roots of $u^\perp$ (which is the Weyl vector of the norm -2 vectors of $A(-1)$) and let let h be the Coxeter number of the even neighbors of the 24 dimensional unimodular lattice $A_1 \oplus I^{24-n}$. Then $\text{height}(u) = (w, u) = 2(h + n - 1)$, $w = \rho + \text{height}(u)u/4$, and $-\rho^2 = (h + n - 1)^2$.*

Proof. There is a unique norm 2 vector z of D with $(z, u) = 2$; we let i be its projection into $u^\perp$. The lattice A has at least 4 vectors of norm 1, so any vector of norm 1 and in particular i is in the vector space generated by vectors of norm -2 of $u^\perp$. Hence by lemma 1.5 and the same argument as in theorem 2.6 we have $w = \rho + \text{height}(u)u/4$. The norm -4

vector $2i$ of $u^\perp$ is the sum of $-2(n-1)$ simple roots of the d_n component of the Dynkin diagram of $u^\perp$, so $(2i, w) = (2i, \rho) = -2(n-1)$.

The vector i is the projection of z into $u^\perp$, so $i = z - u/2$, and hence

$$\begin{aligned}\text{height}(u) &= (w, u) \\ &= 2(w, z - i) \\ &= 2(\text{height}(z) + n - 1) \\ &= 2(h + n - 1).\end{aligned}$$

If we calculate the norms of both sides of $w = \rho + \text{height}(u)u/4$ we find that $-\rho^2 = (h + n - 1)^2$. This proves theorem 2.8.

Example 2.9. Suppose u corresponds to the lattice I^{25}. The number n is then 25 and the root system of the norm 2 vectors is D_{25}, so the Weyl vector ρ can be taken as $(0, 1, 2, \ldots, 24)$. The even neighbors of I^{24} are both D_{24} with Coxeter number $h = 46$, so we find that $0^2 + 1^2 + 2^2 + \cdots + 24^2 = \rho^2 = (h + n - 1)^2 = 70^2$. Watson [W] showed that the only solution of $0^2 + 1^2 + \cdots + k^2 = m^2$ with $k \geq 2$ is $k = 24$. See [C-S Chapter 26] for a construction of the Leech lattice using this equality.

Theorem 2.10. *Suppose that u is a norm 4 vector of D with exactly two norm 0 vectors z_1, z_2 that have inner product 2 with u, and suppose that there are no norm 0 vectors that have inner product 1 with u. Then z_1 and z_2 are both in D and have Coxeter numbers h_1, h_2 where $h_i = (z_i, w)$. Then*

$$w = \rho + (h_1 z_2 + h_2 z_1)/2$$

where ρ is the Weyl vector of the norm -2 vectors of $u^\perp$. Also $u = z_1 + z_2$, $\text{height}(u) = h_1 + h_2$, $-\rho^2 = h_1 h_2$, and $u^\perp$ has $8(h_1 + h_2 - 2)$ roots.

Proof. The vector $u - z_1$ is a norm 0 vector which has inner product 2 with u and so must be z_2. Hence $u = z_1 + z_2$ and $\text{height}(u) = \text{height}(z_1) + \text{height}(z_2) = h_1 + h_2$.

There is a norm 0 vector that has inner product 2 with u, and any automorphism of L fixing D also fixes w, so by lemma 1.5 w is a linear combination of z_1, z_2, and the roots of R in $u^\perp$. Using the facts that $(w, z_1) = h_1$, $(w, z_2) = h_2$, and $(w, r) = -r^2/2$ for any simple root r in $u^\perp$ shows that w must then be $\rho + (h_1 z_2 + h_2 z_1)/2$. Using the fact that $w^2 = 0$ this shows immediately that $-\rho^2 = h_1 h_2$. The number of roots follows from remark 2.12 below. This proves theorem 2.10.

Corollary 2.11. *If A_1 is an odd 24 dimensional positive definite unimodular lattice with no vectors of norm 1 and whose even neighbors have Coxeter numbers h_1 and h_2, then $\rho^2 = h_1 h_2$ where ρ is the Weyl vector of A_1.*

Proof. This follows immediately from theorem 2.10, using the fact that $A_1 \oplus I$ is the 25 dimensional unimodular lattice corresponding to u as in 2.10.

Remark. Let B_1, B_2 be the two even neighbors of A_1. Then it is not hard to show that $h_2 \leq 2h_1 + 2$, and there are several lattices A_1 for which equality holds.

Remark 2.12. Theorem 13.1 and corollary 13.2 of [B95] show that the height of a vector in the fundamental domain of $II_{1,25}$ can be written as an explicit linear combination of the theta functions of cosets of the lattice $u^\perp$. In particular we find that if u is a norm 2 vector then

$$12\text{height}(u) = 18 - 4z_1 + r$$

where r is the number of norm -2 vectors of $u^\perp$ and z_i is the number of norm 0 vectors having inner product i with u (so z_1 is 0 or 2 and is 2 if and only if the lattice $u^\perp$ is the sum of a one dimensional lattice and an even lattice). Similarly if u has norm 4 and corresponds to a 25 dimensional unimodular lattice A then

$$8t = 20 - 2z_2 - 8z_1 + r$$

where r is the number of norm 2 vectors of A, z_2 is the number of norm 1 vectors of A, and z_1 is 1 if A is the sum of a Niemeier lattice and a one dimensional lattice and is 0 otherwise. Note that these relations give congruences for the numbers of roots that immediately imply that 25 dimensional even lattices of determinant 2 and 25 dimensional unimodular lattices always have roots. There are similar relations and congruences for larger norm vectors of $II_{1,25}$.

There are several other genuses of lattices that can be classified using $II_{1,25}$. Most of these do not seem important enough to be worth publishing, but here is a summary of what is available just in case anyone finds a use for any of these. The 24 dimensional even positive definite lattices of determinant 5 are easy to classify as they turn out to correspond to pairs consisting of a norm 2 vector u of $II_{1,25}$ together with a norm -2 root r with $(r, u) = 1$, and these can easily be read off from the list of norm 2 vectors. The 25 dimensional positive definite even lattices of determinant 6 correspond to the norm 6 vectors in $II_{1,25}$ and can be classified from the norm 4 vectors using the algorithm; there are 2825 orbits if I have made no mistakes. A list of them is available from my home page. These can be used to classify the 26 dimensional even positive definite lattices of determinant 3, because the norm 2 roots of such lattices correspond to the norm 6 vectors of $II_{1,25}$. (There is a unique such lattice with no roots; see the next section.) There are between 677 and 681 such lattices, and a provisional list is available from my home page (there are a few small ambiguities that I have not yet got around to resolving). If such a lattice has no norm 6 roots then the number of norm 2 vectors is divisible by 6. With a lot more effort it should be possible to classify the 26 dimensional unimodular lattices by finding the (roughly 50000?) orbits of norm 10 vectors of $II_{1,25}$; see the next section.

3. Lattices with no roots.

In this section we show that there is a unique 26 dimensional positive definite unimodular lattice with no roots. Conway and Sloane use this result in their proof [C-S98] that there is a positive definite unimodular lattice with no roots in all dimensions greater than 25. We also show that the number of norm 2 vectors of a 26 dimensional unimodular lattice is divisible by 4, and sketch a construction of a 27 dimensional unimodular lattice with no roots.

Lemma 3.1. *A 26-dimensional unimodular lattice L with no vectors of norm 1 has a characteristic vector of norm 10.*

Proof. If L has a characteristic vector x of norm 2 then $x^\perp$ is a 25 dimensional even lattice of determinant 2 and therefore has a root r by theorem 2.6; $2r+x$ is a characteristic vector of norm 10. If the lemma is not true we can therefore assume that L has no vectors of norm 1 and no characteristic vectors of norm 2 or 10. Its theta function is determined by these conditions and turns out to be $1-156q^2+\cdots$ which is impossible as the coefficient of q^2 is negative. This proves lemma 3.1.

Lemma 3.2. *There is a bijection between isomorphism classes of*

(1) *Norm 10 characteristic vectors c in 26-dimensional positive definite unimodular lattices L, and*
(2) *Norm* 10 *vectors u in $II_{1,25}$*
given by $c^\perp(-1) \cong u^\perp$.
We have $\mathrm{Aut}(L,c) = \mathrm{Aut}(II_{1,25}, u)$.

Proof. Routine. Note that -1 is a square mod 10. This proves lemma 3.2.

Lemmas 3.1 and 3.2 give an algorithm for finding 26 dimensional unimodular lattices L. It is probably not hard to implement this on a computer if one is given a computer algorithm for deciding when 2 vectors of the Leech lattice are conjugate under its automorphism group; such an algorithm has been described by Allcock in [A]. The main remaining open problem is to find a use for these lattices! We now apply this algorithm to find the unique such lattice with no roots.

Lemma 3.3. *Take notation as in lemma 3.2. The lattice L has no roots if and only if $u^\perp$ has no roots and u does not have inner product 1, 2, 3, or 4 with any norm 0 vector.*

Proof. If $u^\perp$ has roots then obviously L has too. If there is a norm 0 vector z that has inner product 1, 2, 3, or 4 with u then the projection z_u of z into $u^\perp$ has norm $-1/10$, $-4/10$, $-9/10$, or $-16/10$. The lattice $L(-1)$ contains $u^\perp + c$, and the vector $z_u \pm 3c/10$, $z_u \pm 4c/10$, $z_u \pm c/10$, or $z_u \pm 2c/10$ is in L for some choice of sign and has norm -1, -2, -1, or -2. Hence if u has inner product 1, 2, 3, or 4 with some norm 0 vector then L has roots. Conversely if L has a root r then either r has norm 2 and inner product 0, ± 2, ± 4 with c or it has norm 1 and inner product ± 1, ± 3 with c, and each of these cases implies that $u^\perp$ has roots or that u has inner product 1, 2, 3, or 4 with some norm 0 vector by reversing the argument above. This proves lemma 3.3.

Now let L be a 26 dimensional unimodular lattice with no roots containing a characteristic vector c of norm 10, and let u be a norm 10 vector of D corresponding to it as in 3.2.

Lemma 3.4. *$u = z + w$, where z is a norm 0 vector of D corresponding to a Niemeier lattice with root system A_4^6, and w is the Weyl vector of D. In particular u is determined up to conjugacy under* $\mathrm{Aut}(D)$.

Proof. The lattice $u^\perp$ has no roots so $u = w + z$ for some vector z of D. By lemma 3.3 u does not have inner product 1, 2, 3, or 4 with any norm 0 vector, so $(z,w) = (u,w) \geq 5$. Hence

$$10 = u^2 = z^2 + 2(z,w) \geq 2(z,w) \geq 10$$

so $(z,w) = 5$ and $z^2 = 0$. The only norm 0 vectors z in D with $(z,w) = 5$ are the primitive ones corresponding to A_4^6 Niemeier lattices, which form one orbit under $\mathrm{Aut}(D)$. This proves lemma 3.4.

Lemma 3.5. *If $u = z + w$ is as in lemma 3.4 then the 26 dimensional unimodular lattice corresponding to u has no roots.*

Proof. The lattice $u^\perp$ obviously has no roots so by lemma 3.3 we have to check that there are no norm 0 vectors that have inner product 1, 2, 3, or 4 with u. Let x be any norm 0 vector in the positive cone. If x has type A_4^6 then $(x, u) \geq (x, w) \geq 5$; if x has Leech type then $(x, u) \geq (x, z) \geq 5$; if x has type A_1^{24} then $(x, u) = (x, w) + (x, z) \geq 2 + 3 = 5$ ((x, z) cannot be 2 as there are no pairs of norm 0 vectors of types A_1^{24} and A_4^6 that have inner product 2 by the classification of 24 dimensional unimodular lattices); and if x has any other type then $(x, u) = (x, w) + (x, z) \geq 3 + 2 = 5$. This proves lemma 3.5.

Theorem 3.6. *There is a unique 26 dimensional positive definite unimodular lattice L with no roots. Its automorphism group is isomorphic to the group $O_5(5) = 2.G.2$ of order $2^8.3^2.5^4.13$ and acts transitively on the 624 characteristic norm 10 vectors of L.*

Proof. By lemma 3.1 L has a characteristic vector of norm 10, so by lemmas 3.3 and 3.4 L is unique and its automorphism group acts transitively on the characteristic vectors of norm 10. By lemma 3.5 L exists. The theta function is determined by the conditions that L has no vectors of norm 1 or 2 and no characteristic vectors of norm 2, and it turns out that the number of characteristic vectors of norm 10 is 624. The stabilizer of such a vector is isomorphic to $\mathrm{Aut}(II_{1,25}, u)$, which is a group of the form $5^3.2.S_5$ where S_5 is the symmetric group on 5 letters. This determines the order of the automorphism group of the lattice. From this it is not difficult to determine it precisely; we omit the details. This proves theorem 3.6.

We now show that the number of norm 2 vectors of any 26 dimensional even positive definite unimodular lattice is divisible by 4. There are strictly 26 dimensional unimodular lattices with no roots or with 4 roots, so this is the best possible congruence. For unimodular lattices of dimension less than 26 there are congruences modulo higher powers of 2 for the number of roots.

Lemma 3.7. *If L is a 25-dimensional positive definite lattice of determinant 2 then the number of norm 2 roots of L is* $2 \bmod 4$.

Proof. The even vectors of L form a lattice isomorphic to the vectors that have even inner product with some vector b in an even 25-dimensional lattice B of determinant 2. (Note that b is not in $B' - B$.) The number of roots of B is $12t - 10$ or $12t - 18$ where t is the height of the norm 2 vector of D corresponding to B by remark 2.12, so it is sufficient to prove that the number of norm 2 vectors of B that have odd inner product with b is divisible by 4.

The vector b has zero inner product with u and integral inner product with w, so by theorem 2.6 b has integral inner product with ρ. Hence b has even inner product with the sum of the positive roots of B, so it has odd inner product with an even number of positive roots. This implies that the number of roots of B that have odd inner product with b is divisible by 4. This proves lemma 3.7.

Corollary 3.8. *If L is a 26 dimensional unimodular lattice then the number of norm 2 vectors of L is divisible by 4.*

Proof. The result is obvious if L has no norm 2 roots, so let r be a norm 2 vector of L. The lattice $r^\perp$ is a 25 dimensional even lattice of determinant so by remark 2.12 the number of roots of $r^\perp$ is 2 mod 4. The number of roots of L not in $r^\perp$ is $4h-6$ where h is the Coxeter number of the component of L containing r, so the number of norm 2 vectors of L is divisible by 4. This proves corollary 3.8.

Remark. A similar but more complicated argument can be used to show that there is a unique even 26 dimensional positive definite lattice of determinant 3 with no roots. Gluing on a one dimensional lattice to this gives a unique 27 dimensional unimodular lattice with no roots and a characteristic vector of norm 3. As a different proof of this has already been published in [E-Z] we will just give a brief sketch of the proof from [B]. (The preprint [B-V] shows that there are exactly three 27 dimensional positive definite unimodular lattices with no roots.) Let L be a 27 dimensional positive definite unimodular lattice with no roots and a characteristic vector c of norm 3. The theta function of L is determined by these conditions and this implies that L has vectors of norm 5; let v be such a vector. Then $\langle v, c\rangle^\perp$ is a 25 dimensional even lattice X of determinant 14 such that X'/X is generated by an element of norm $1/14 \bmod 2$. Such lattices X correspond to norm 14 vectors x in the fundamental domain D of $II_{1,25}$, and the condition that L has no vectors of norm 1 or 2 implies that there are exactly two possibilities for x: x is either the sum of w and a norm 0 vector of height 7 corresponding to A_6^4, or x is the sum of w and a norm 2 vector of height 6 corresponding to the 25 dimensional lattice of determinant 2 with root system a_2^9. Both of these x's turn out to give the same lattice L, which therefore has two orbits of norm 5 vectors and is the unique 27 dimensional positive definite unimodular lattice with no roots and with characteristic vectors of norm 3.

Table 0. The primitive norm 0 vectors of $II_{1,25}$.

We list the set of orbits of primitive norm 0 vectors z of $II_{1,25}$, which is of course more or less the same as the well known list of Niemeier lattices (see [C-S table 16.1]). The height is just (w, z) where w is the Weyl vector of a fundamental domain containing z. The letter after the height is just a name to distinguish vectors of the same height, and is the letter referred to in the column headed "Norm 0 vectors" of table 1. The column headed "Group" is the order of the subgroup of $\mathrm{Aut}(D)$ fixing the primitive norm 0 vector. However note that the group order is *not* (usually) the order of the quotient of the automorphism group of the Niemeier lattice by the reflection group; see [C-S chapter 16] for a description of the relation between these groups. For the vector w of height 0 the group is the infinite group of automorphisms of the affine Leech lattice and is an extension of a finite group of the order given by the group of translations of the Leech lattice Λ.

Height	Roots	Group
0x	None	$\Lambda\cdot 8315553613086720000$
2a	A_1^{24}	1002795171840
3a	A_2^{12}	138568320
4a	A_3^{8}	688128
5a	A_4^{6}	30000
6d	D_4^{6}	138240

6a	$A_5^4D_4$	3456
7a	A_6^4	1176
8a	$A_7^2D_5^2$	256
9a	A_8^3	324
10d	D_6^4	384
10a	$A_9^2D_6$	80
12e	E_6^4	432
12a	$A_{11}D_7E_6$	24
13a	A_{12}^2	52
14d	D_8^3	48
16a	$A_{15}D_9$	16
18d	$D_{10}E_7^2$	8
18a	$A_{17}E_7$	12
22d	D_{12}^2	8
25a	A_{24}	10
30e	E_8^3	6
30d	$D_{16}E_8$	2
46d	D_{24}	2

Table 1. The norm 2 vectors of $II_{1,25}$.

The following sets are in natural 1:1 correspondence:

(1) Orbits of norm 2 vectors in $II_{1,25}$ under $\mathrm{Aut}(II_{1,25})$.

(2) Orbits of norm 2 vectors u of D under $\mathrm{Aut}(D)$.

(3) 25 dimensional even bimodular lattices L.

The lattice $L(-1)$ is isomorphic to $u^\perp$. Table 1 lists the 121 elements of any of these three sets.

The *height* is the height of the norm 2 vector u of D, in other words (u, w) where w is the Weyl vector of D. The letter after the height is just a name to distinguish vectors of the same height, and is the letter referred to in the column headed "Norm 2" of table 2. An asterisk after the letter means that the vector u is of type 1, in other words the lattice L is the sum of a Niemeier lattice and a_1.

The column "Roots" gives the Dynkin diagram of the norm 2 vectors of L arranged into orbits under $\mathrm{Aut}(L)$. "Group" is the order of the subgroup of $\mathrm{Aut}(D)$ fixing u. The group $\mathrm{Aut}(L)$ is a split extension $R.G$ where R is the Weyl group of the Dynkin diagram and G is isomorphic to the subgroup of $\mathrm{Aut}(D)$ fixing u.

"S" is the maximal number of pairwise orthogonal roots of L.

The column headed "Norm 0 vectors" describes the norm 0 vectors z corresponding to each orbit of roots of $u^\perp$ where u is in D. A capital letter indicates that the corresponding norm 0 vector is twice a primitive vector, otherwise the norm 0 vector is primitive. x stands for a norm 0 vector of type the Leech lattice. Otherwise the letter a, d, or e is the first letter of the Dynkin diagram of the norm 0 vector, and its height is given by $\mathrm{height}(u) - h + 1$ where h is the Coxeter number of the component of the Dynkin diagram of u.

For example, the norm 2 vector of type $23a$ has 3 components in its root system, of Coxeter numbers 12, 12, and 6, and the letters are e, a, and d, so the corresponding norm

0 vectors have Coxeter numbers 12, 12, and 18 and hence are norm 0 vectors with Dynkin diagrams E_6^4, $A_{11}D_7E_6$, and $D_{10}E_7^2$.

For some remarks on the reliability of table 1 see the introduction to table 2.

Height	Roots	Group	S	Norm 0 vectors
1a*	a_1	8315553613086720000	1	X
2a	a_2	991533312000	1	x
3a	a_1^9	92897280	9	a
4a	$a_2a_1^{12}$	190080	13	aa
5a*	$a_1^{24}a_1$	244823040	25	aA
5b	$a_2^4a_1^9$	3456	13	aa
5c	$a_3a_1^{15}$	40320	17	aa
6a	a_2^9	3024	9	a
6b	$a_3a_2^5a_1^6$	240	13	aaa
7a*	$a_2^{12}a_1$	190080	13	aA
7b	$a_3^3a_2^4a_1^3$	48	13	aaa
7c	$a_3^4a_1^8a_1$	384	17	aad
7d	$a_4a_2^6a_1^5$	240	13	aaa
7e	$d_4a_1^{21}$	120960	25	ad
8a	$a_3^6a_2$	240	13	ad
8b	$a_4a_3^3a_2^3a_1^2$	12	13	aaaa
8c	$d_4a_2^9$	864	13	aa
9a*	$a_3^8a_1$	2688	17	aA
9b	$a_4^2a_3^4a_1$	16	13	aaa
9c	$a_4^3a_3a_2^2a_1^3$	12	13	aaaa
9d	$d_4a_3^4a_3a_1^3$	48	17	aada
9e	$a_5a_3^3a_2^4$	24	13	aaa
9f	$a_5a_3^4a_1^6$	48	17	aaa
10a	$d_4a_4^3a_2^3$	12	13	aaa
10b	$a_5a_4^2a_3^2a_2a_1$	4	13	aaaaa
11a*	$a_4^6a_1$	240	13	aA
11b	$d_4^4a_1^9$	432	25	dd
11c	$a_5d_4^2a_3^3$	24	17	daa
11d	$a_5a_5a_4^2a_3a_1$	4	13	aaaaa

11e	$a_5^2 d_4 a_3^2 a_1^2 a_1$	8	17	aaaad
11f	$a_5^3 a_2^4$	48	13	aa
11g	$d_5 a_3^6 a_1$	48	17	aad
11h	$a_6 a_4^2 a_3^2 a_2 a_1$	4	13	aaaaa
12a	$a_5^4 a_2$	24	13	ad
12b	$d_5 a_4^4 a_2$	8	13	aaa
12c	$a_6 d_4 a_4^3$	6	13	aaa
12d	$a_6 a_5^2 a_3 a_2^2$	4	13	aaaa
13a*	$a_5^4 d_4 a_1$	48	17	aaA
13b	$d_5 a_5^2 d_4 a_3 a_1$	4	17	aaada
13c	$d_5 a_5^3 a_1^3 a_1$	12	17	aaae
13d*	$d_4^6 a_1$	2160	25	dD
13e	$a_6^2 a_5 a_4 a_1^2$	4	13	aaaa
13f	$a_7 a_5 a_4^2 a_3$	4	13	aaaa
13g	$a_7 a_5 d_4 a_3^2 a_1^2$	4	17	aaaaa
14a	$a_6 a_6 d_5 a_4 a_2$	2	13	aaaaa
14b	$a_6^3 d_4$	12	13	aa
14c	$a_7 a_6 a_5 a_4 a_1$	2	13	aaaaa
15a*	$a_6^4 a_1$	24	13	aA
15b	$d_5^3 a_5 a_3$	12	17	ade
15c	$d_6 d_4^4 a_1^3$	24	25	ddd
15d	$d_6 a_5^2 a_5 a_3$	4	17	aada
15e	$a_7 d_5^2 a_3^2 a_1$	4	17	aaad
15f	$a_7^2 d_4^2 a_1$	8	17	aad
15g	$a_8 a_5^3$	6	13	aa
15h	$a_8 a_6 a_5 a_3 a_2$	2	13	aaaaa
16a	$a_7^3 a_2$	12	13	ad
16b	$a_8 a_6 d_5 a_4$	2	13	aaaa
17a*	$a_7^2 d_5^2 a_1$	8	17	aaA
17b	$e_6 a_5^3 d_4$	12	17	aae
17c	$a_7 d_6 d_5 a_5$	2	17	daaa
17d	$a_7^2 d_6 a_3 a_1$	4	17	aada
17e	$a_8 a_7^2 a_1$	4	13	aaa
17f	$a_9 d_5 a_5 d_4 a_1$	2	17	aaaaa
17g	$a_9 a_7 a_4^2$	4	13	aaa
18a	$e_6 a_6^3$	6	13	aa
18b	$a_9 a_8 a_5 a_2$	2	13	aaaa

19a*	$a_8^3a_1$	12	13	aA
19b	$d_6^3d_4a_1^3$	6	25	ddd
19c	$a_7e_6d_5^2a_1$	4	17	eaad
19d	$d_7a_7d_5a_5$	2	17	aaad
19e	$d_7a_7^2a_3a_1$	4	17	aaad
19f	$a_9a_7d_6a_1a_1$	2	17	aaaad
19g	$a_{10}a_7a_6a_1$	2	13	aaaa
20a	$a_8^2e_6a_2$	4	13	aaa
20b	$a_{10}a_8d_5$	2	13	aaa
21a*	$a_9^2d_6a_1$	4	17	aaA
21b	$a_{11}d_6a_5a_3$	2	17	aaaa
21c	$a_{11}a_8a_5$	2	13	aaa
21d*	$d_6^4a_1$	24	25	dD
21e	$a_9e_6d_6a_3$	2	17	aaad
23a	$d_7e_6^2a_5$	4	17	ead
23b	$d_8d_6^2d_4a_1$	2	25	dddd
23c	$a_9d_7^2$	4	17	da
23d	$a_9d_8a_7$	2	17	daa
23e	$a_{11}d_7d_5a_1$	2	17	aaad
24a	$a_{11}^2a_2$	4	13	ad
24b	$a_{12}e_6a_6$	2	13	aaa
25a*	$a_{11}d_7e_6a_1$	2	17	aaaA
25b	$a_{13}d_6d_5$	2	17	aaa
25e*	$e_6^4a_1$	48	17	eE
26a	$a_{13}a_{10}a_1$	2	13	aaa
27a*	$a_{12}^2a_1$	4	13	aA
27b	$e_7d_6^3$	3	25	dd
27c	$a_9a_9e_7$	2	17	ada
27d	$d_9a_9e_6$	2	17	ada
27e	$a_{11}d_9a_5$	2	17	aad
27f	$a_{14}a_9a_2$	2	13	aaa
29a	$a_{11}e_7e_6$	2	17	daa
29d*	$d_8^3a_1$	6	25	dD
31a	$d_8^2e_7a_1a_1$	2	25	ddde

31b	$d_{10}d_8d_6a_1$	1	25	dddd
31c	$a_{15}d_8a_1$	2	17	aad
33a*	$a_{15}d_9a_1$	2	17	aaA
33b	$a_{15}e_7a_3$	2	17	aad
33c	$a_{17}a_8$	2	13	aa
35a	$e_7^3d_4$	6	25	de
35b	$a_{13}d_{11}$	2	17	da
36a	$a_{18}e_6$	2	13	aa
37a*	$a_{17}e_7a_1$	2	17	aaA
37d*	$d_{10}e_7^2a_1$	2	25	ddD
39a	$d_{12}e_7d_6$	1	25	ddd
45d*	$d_{12}^2a_1$	2	25	dD
47a	$d_{10}e_8e_7$	1	25	edd
47b	$d_{14}d_{10}a_1$	1	25	ddd
47c	$a_{17}e_8$	2	17	da
48a	$a_{23}a_2$	2	13	ad
51a*	$a_{24}a_1$	2	13	aA
61d*	$d_{16}e_8a_1$	1	25	ddD
61e*	$e_8^3a_1$	6	25	eE
63a	$d_{18}e_7$	1	25	dd
93d*	$d_{24}a_1$	1	25	dD

Table 2. The norm 4 vectors of $II_{1,25}$.

There is a natural 1:1 correspondence between the elements of the following sets:

(1) Orbits of norm 4 vectors u in $II_{1,25}$ under $\mathrm{Aut}(II_{1,25})$.
(2) Orbits of norm 4 vectors in the fundamental domain D of $II_{1,25}$ under $\mathrm{Aut}(D)$.
(3) Orbits of norm 1 vectors v of $I_{1,25}$ under $\mathrm{Aut}(I_{1,25})$.
(4) 25 dimensional unimodular positive definite lattices L.
(5) Unimodular lattices L_1 of dimension at most 25 with no vectors of norm 1.
(6) 25 dimensional even lattices L_2 of determinant 4.

L_1 is the orthogonal complement of the norm 1 vectors of L, L_2 is the lattice of elements of L of even norm, $L_2(-1)$ is isomorphic to $u^\perp$, and $L(-1)$ is isomorphic to $v^\perp$. Table 2 lists the 665 elements of any of these sets.

The height is the height of the norm 4 vector u of D, in other words (u,w) where w is the Weyl vector of D. The things in table 2 are listed in increasing order of their height.

Dim is the dimension of the lattice L_1. A capital E after the dimension means that L_1 is even.

The column "roots" gives the Dynkin diagram of the norm 2 vectors of L_2 arranged into orbits under $\mathrm{Aut}(L_2)$.

"Group" gives the order of the subgroup of $\mathrm{Aut}(D)$ fixing u. The group $\mathrm{Aut}(L) \cong \mathrm{Aut}(L_2)$ is of the form $2 \times R.G$ where R is the group generated by the reflections of norm 2 vectors of L, G is the group described in the column "group", and 2 is the group of order 2 generated by -1. If $\dim(L_1) \leq 24$ then $\mathrm{Aut}(L_1)$ is of the form $R.G$ where R is the reflection group of L_1 and G is as above.

For any root r of $u^\perp$ the vector $v = u - r$ is a norm 2 vector of $II_{1,25}$. This vector v can be found as follows. Let X be the component of the Dynkin diagram of $u^\perp$ to which u belongs and let h be the Coxeter number of X. Then $u - r$ is conjugate to a norm 2 vector of $II_{1,25}$ in D of height $t - h + 1$ (or $t - h$ if the entry under "Dim" is $24E$) whose letter is the letter corresponding to X in the column headed "norm 2". For example let u be the vector of height 6 and root system $a_2^2 a_1^{10}$. Then the norm 2 vectors corresponding to roots from the components a_2 or a_1 have heights $6-3+1$ and $6-2+1$ and letters a and b, so they are the vectors $4a$ and $5b$ of table 1.

If $\dim(L_1) \leq 24$ then the column "neighbors" gives the two even neighbors of $L_1 + I^{24-\dim(L_1)}$. If $\dim(L_1) \leq 23$ then both neighbors are isomorphic so only one is listed, and if L_1 is a Niemeier lattice then the neighbor is preceded by 2 (to indicate that the corresponding norm 0 vector is twice a primitive vector). If the two neighbors are isomorphic then there is an automorphism of L exchanging them.

Tables 1 and 2 were originally calculated by hand. Most of the lattices were found several times, once for each orbit of roots, and this gave a large number of checks for most entries. I later ran a computer version of the algorithm of this paper, which turned up about 20 minor errors (mostly errors in column 5, and a few misprints in the group order and root systems which were due to copying errors). I also checked the Minkowski-Siegel mass formula. Any errors remaining are probably either copying errors (the tables are based on computer output, but have had some hand editing to turn them into nice looking TEX) or an error where a lattice should be split into 2 lattices each with twice the automorphism group. The second possibility cannot be detected by mass formulas but I think it unlikely that it occurs in these tables. (It becomes an irritating problem when classifying the 26-dimensional even lattices of determinant 3.)

Height	Dim	Roots	Group	norm 2	Neighbors	
1	24E	None	8315553613086720000		2Λ	
2	23	a_1^2	84610842624000	a	Λ	
2	24	None	1002795171840		Λ	A_1^{24}
3	25	a_1^2	88704000	a		
4	24	a_1^8	20643840	a	A_1^{24}	A_1^{24}

4	25	a_2^2	26127360	a		
4	25	a_1^6	138240	a		
5	24	a_1^{12}	190080	a	A_1^{24}	A_2^{12}
5	25	$a_2a_1^7$	5040	aa		
5	25	a_1^{10}	1920	a		
6	23	$a_1^{16}a_1^2$	645120	ca	A_1^{24}	
6	24	$a_2^2a_1^{10}$	5760	ab	A_2^{12}	A_2^{12}
6	24	a_1^{16}	43008	c	A_1^{24}	A_3^8
6	25	$a_3a_1^8$	21504	ac		
6	25	$a_2^2a_1^8$	128	ab		
6	25	$a_2a_1^{10}a_1$	120	abc		
6	25	$a_1^8a_1^6$	1152	bc		
7	24	$a_2^4a_1^8$	384	bb	A_2^{12}	A_3^8
7	24E	a_1^{24}	244823040	a	$2A_1^{24}$	
7	25	$a_2^5a_1^3$	720	bb		
7	25	$a_3a_2a_1^9$	72	acb		
7	25	$a_2^4a_1^4a_1^2$	24	bba		
7	25	$a_3a_1^{12}$	1440	ab		
7	25	$a_2^3a_1^6a_1^3$	12	bbb		
7	25	$a_2^2a_1^{12}$	144	cb		
8	22	$a_3a_1^{22}$	887040	ae	A_1^{24}	
8	23	$a_2^6a_1^6a_1^2$	1440	bda	A_2^{12}	
8	24	$a_3^2a_1^{12}$	768	cc	A_3^8	A_3^8
8	24	$a_3a_2^4a_1^6$	96	bbb	A_3^8	A_3^8
8	24	a_2^8	672	a	A_3^8	A_3^8
8	24	$a_2^6a_1^6$	240	bd	A_2^{12}	A_4^6
8	24	a_1^{24}	138240	e	A_1^{24}	D_4^6
8	25	$a_4a_1^{12}$	1440	ad		
8	25	$a_3a_2^4a_1^4$	16	bbb		
8	25	$a_3^2a_1^8a_1^2$	64	cbc		
8	25	$a_3a_2^3a_1^6a_1$	12	bbbc		
8	25	$a_3a_2^3a_1^3a_1^3a_1$	6	bbbbd		
8	25	$a_2^4a_2^2a_1^4$	8	bab		
8	25	$a_3a_2^2a_1^4a_1^4a_1^2$	16	bbcbd		
8	25	$a_2^4a_2a_1^4a_1^2a_1$	8	bbbdb		
8	25	$a_2^4a_1^8a_1^2$	48	bdc		
8	25	$a_3a_1^{15}a_1$	720	cce		
9	24	$a_3^2a_2^4a_1^4$	16	bbb	A_3^8	A_4^6
9	24	$a_2^8a_1^4$	384	dc	A_2^{12}	$A_5^4D_4$

9	25	$a_4a_2^3a_1^6a_1$	12	bdbb		
9	25	$a_3^2a_2^4a_1^2$	16	bba		
9	25	$a_3^2a_2^2a_2a_1^4a_1$	4	bbcba		
9	25	$a_3a_3a_2^2a_2a_1^2a_1^2a_1$	2	bbbbbbb		
9	25	$a_3a_2^6a_1^2$	6	abb		
9	25	$a_3^2a_2^2a_1^4a_1^4$	8	bcbb		
9	25	$a_3a_2^4a_2a_1^4a_1$	8	bbbbc		
9	25	$a_3a_2^2a_2^2a_2a_1^2a_1^2a_1$	2	bbbdbbb		
10	22	$a_3a_2^{10}$	2880	ac	A_2^{12}	
10	23	$a_3^4a_1^8a_1^2$	384	cfa	A_3^8	
10	23	$a_3^3a_2^4a_1^2a_1^2$	48	bbea	A_3^8	
10	24	$a_3^4a_2^2a_1^2$	32	bab	A_4^6	A_4^6
10	24	$a_4a_3a_2^4a_1^4$	16	bdbc	A_4^6	A_4^6
10	24	$a_3^2a_3a_2^4a_1^2$	16	bbbe	A_3^8	$A_5^4D_4$
10	24	$a_3^4a_1^4a_1^4$	48	cdf	A_3^8	$A_5^4D_4$
10	24	$a_3^4a_1^8$	384	cd	A_3^8	D_4^6
10	24E	a_2^{12}	190080	a	$2A_2^{12}$	
10	25	a_3^5	1920	c		
10	25	$d_4a_2^4a_1^6$	144	bcd		
10	25	$d_4a_3a_1^{12}$	576	ced		
10	25	$a_5a_1^{15}$	720	cf		
10	25	$a_4a_3a_2^4a_1^2$	8	bdbb		
10	25	$a_3^3a_3a_2a_1^3$	6	bcbb		
10	25	$a_4a_3a_2^2a_2a_1^2a_1^2a_1$	2	bdbbbce		
10	25	$a_3^3a_3a_1^4a_1^2$	24	bcce		
10	25	$a_3^2a_3^2a_1^4a_1^2$	16	cbbd		
10	25	$a_4a_2^6a_1^2$	6	abe		
10	25	$a_4a_3a_2^2a_1^4a_1^2a_1^2$	4	bdbccf		
10	25	$a_3^2a_3a_2^2a_2a_1^2a_1$	2	bbbbbc		
10	25	$a_3^2a_3a_2^2a_2a_1a_1a_1$	2	bbbadbc		
10	25	$a_4a_2^5a_1^5$	10	bbc		
10	25	$a_3^2a_2^6$	48	da		
10	25	$a_3^2a_2^4a_2^2$	8	bba		
10	25	$a_3^3a_2^2a_1^3a_1^2a_1$	12	bbdcf		
10	25	$a_3a_3a_3a_2^2a_1^2a_1a_1a_1a_1$	2	cbbbcebfd		
10	25	$a_3a_3a_2^2a_2^2a_2a_1^2a_1$	2	bbbbbee		
10	25	$a_3^2a_2^4a_1^4a_1^2$	8	dbcf		
10	25	$a_3^3a_1^{12}$	48	cf		
10	25	$a_3a_2^6a_2a_1^3$	12	dbce		
11	24	$a_4a_3^2a_3a_2^2a_1^2$	4	bbbbb	A_4^6	$A_5^4D_4$
11	24	$a_4^2a_2^4a_1^4$	16	dca	A_4^6	$A_5^4D_4$
11	24	a_3^6	240	a	A_4^6	D_4^6

11	24	$a_3^4a_2^4$	24	be	A_3^8	A_6^4
11	25	$a_5a_2^4a_2a_1^4$	8	befb		
11	25	$d_4a_3a_2^4a_1^4$	8	bcda		
11	25	$a_4a_3^2a_3a_2a_1^2a_1$	2	bbbcba		
11	25	$a_4^2a_2^2a_2a_1^4a_1$	4	dcbbb		
11	25	$a_4a_3^2a_2^4$	4	bbb		
11	25	$a_4a_3^3a_1^6$	6	cbb		
11	25	$a_4a_3^2a_2^2a_2a_1^2a_1$	2	bbbfab		
11	25	$a_4a_3a_3a_2a_2a_2a_1a_1a_1$	1	bbbbccbba		
11	25	$a_4a_3a_3a_2a_2a_2a_1a_1a_1$	1	bbbbecbbb		
11	25	$a_3^4a_3a_1^4$	8	bab		
11	25	$a_3^2a_3^2a_2^2a_2a_1$	2	bbbbb		
11	25	$a_3^2a_3a_3a_2^2a_2a_1$	2	bbabda		
11	25	$a_4a_3a_2^2a_2^2a_2a_1^2a_1$	2	bbeccba		
11	25	$a_3^2a_3^2a_2a_1^4$	4	bbdb		
12	22	$a_3^6a_3a_1^2$	96	dag	A_3^8	
12	23	$a_4^2a_3^2a_2^2a_1^2a_1^2$	8	bcbha	A_4^6	
12	23	$a_4a_3^5a_1^2$	40	aba	A_4^6	
12	24	$d_4a_4a_2^6$	24	dca	$A_5^4D_4$	$A_5^4D_4$
12	24	$d_4a_3^4a_1^4$	32	cde	$A_5^4D_4$	$A_5^4D_4$
12	24	$a_5a_3^3a_1^6a_1$	24	cfec	$A_5^4D_4$	$A_5^4D_4$
12	24	$a_4^2a_3^2a_3a_1^2$	8	bbbd	$A_5^4D_4$	$A_5^4D_4$
12	24	$a_5a_3^2a_2^4a_1$	16	bebf	$A_5^4D_4$	$A_5^4D_4$
12	24	$d_4a_3^4a_1^4$	48	cdc	D_4^6	$A_5^4D_4$
12	24	$d_4^2a_1^{16}$	1152	eb	D_4^6	D_4^6
12	24	$a_4^2a_3^2a_2^2a_1^2$	4	bcbh	A_4^6	A_6^4
12	24	$a_3^4a_3^2a_1^4$	32	fdg	A_3^8	$A_7^2D_5^2$
12	25	$a_5a_3^2a_3a_1^4a_1$	8	cefdc		
12	25	$a_5a_3^2a_2^2a_2a_1a_1$	2	bebbdh		
12	25	$d_4a_3^2a_3a_2^2a_1^2$	4	bddac		
12	25	$d_4a_4a_2^4a_1^4$	8	dcae		
12	25	$a_5a_3^2a_2^2a_1^2a_1^2a_1$	4	bfbdhf		
12	25	$a_5a_3a_3a_2^2a_1^2a_1a_1a_1$	2	bfebhdee		
12	25	$a_4^2a_3a_3a_2a_1^2a_1$	2	bbcade		
12	25	$a_4a_4a_3a_3a_2a_1a_1a_1$	1	bbccbddd		
12	25	$a_4^2a_3a_2^4$	4	bbb		
12	25	$a_4^2a_3^2a_1^4a_1^2$	4	bche		
12	25	$d_4a_3^2a_2^4a_1a_1$	8	bdaeg		
12	25	$d_4a_3^3a_1^6a_1a_1$	12	cdegb		
12	25	$a_4a_3^2a_3^2a_2a_1$	2	abcbc		
12	25	$a_4^2a_3a_2^2a_2a_1a_1a_1$	2	bcbbfdh		
12	25	$a_4a_4a_3a_2a_2a_2a_1a_1a_1$	1	bbcabbhhd		
12	25	$a_4a_3^4a_1^4$	8	ach		

12	25	$a_4a_3^2a_3a_3a_1^2a_1^2$	2	bbcfde		
12	25	$a_4a_3^3a_2^3a_1$	6	bbag		
12	25	$a_4a_3a_3a_3a_2a_2a_2a_1$	1	bbbebbah		
12	25	$a_3^4a_3a_3a_1^2$	8	bfdc		
12	25	$a_4a_3^2a_3a_2^2a_1^2a_1a_1$	2	bccbhge		
12	25	$a_4a_3^2a_3a_2^2a_1^2a_1^2$	2	bcfbhh		
12	25	$a_3^2a_3^2a_3a_2^2a_1^2$	4	edbbh		
13	24	$a_4^4a_1^4$	24	cc	$A_5^4D_4$	A_6^4
13	24	$a_5a_4a_3a_3a_2a_2a_1$	2	bebbddd	$A_5^4D_4$	A_6^4
13	24	$a_4^2a_3^4$	16	bb	A_4^6	$A_7^2D_5^2$
13	24	$a_4^2a_4a_3a_2^2a_1^2$	4	ccahb	A_4^6	$A_7^2D_5^2$
13	24E	a_3^8	2688	a	$2A_3^8$	
13	25	$a_6a_2^6a_1^3$	12	dhd		
13	25	$a_4^4a_1^2$	24	ca		
13	25	$a_5a_4a_3^2a_2a_1^2$	2	bfbdc		
13	25	$d_4a_4a_3^3a_1^2$	6	bdac		
13	25	$d_4^2a_2^6$	72	cc		
13	25	$a_5a_4a_3a_2a_2a_2a_1a_1$	1	bebhdeda		
13	25	$a_5a_4a_3a_2a_2a_2a_1a_1$	1	bebhdddd		
13	25	$d_4a_4a_3a_3a_2a_2a_1a_1$	1	bdaaeccb		
13	25	$a_4^2a_4a_3a_2^2$	2	bcbd		
13	25	$a_4a_4a_4a_3a_2a_1a_1a_1$	1	bccbdbcd		
13	25	$a_5a_3^3a_2^3$	6	bbd		
13	25	$a_5a_3^2a_3a_2^2a_2$	2	abbhc		
13	25	$a_5a_4a_2^2a_2^2a_2a_1^2$	2	behdfd		
13	25	$a_5a_3^2a_3a_2^2a_1^2a_1$	2	bbbedd		
13	25	$a_4a_4a_3a_3a_3a_2a_1$	1	bcbbbdc		
13	25	$a_4a_4a_3a_3a_3a_2a_1$	1	bbbabdb		
13	25	$a_4^3a_2^3a_1^3$	6	cdb		
13	25	$d_4a_3^3a_2^3a_2$	6	bacg		
13	25	$a_4^2a_3a_3a_2^2a_2a_1$	2	ebaddd		
13	25	$a_5a_2^9$	72	cf		
14	21	$d_4a_3^7$	336	ag	A_3^8	
14	22	$a_4^4a_3a_2^2$	16	aab	A_4^6	
14	23	$a_5a_4^2a_3a_3a_1^2a_1$	4	bbddaf	$A_5^4D_4$	
14	23	$d_4a_4^3a_2^2a_1^2$	12	caca	$A_5^4D_4$	
14	23	$a_5d_4a_3^3a_1^3a_1^2$	12	dfega	$A_5^4D_4$	
14	23	$a_5^2a_3a_2^4a_1^2$	16	efda	$A_5^4D_4$	
14	23	$d_4^2a_3^4a_1^2$	96	dcd	D_4^6	
14	24	$a_5a_4^3a_1^3$	12	cbe	A_6^4	A_6^4
14	24	$a_5^2a_3^2a_2^2$	8	eda	A_6^4	A_6^4
14	24	$a_6a_3^3a_2^3$	12	bhd	A_6^4	A_6^4

14	24	$d_4a_4^2a_4a_2^2$	4	caac	$A_5^4D_4$	$A_7^2D_5^2$
14	24	$a_5d_4a_3^2a_3a_1^2a_1$	4	dfecbg	$A_5^4D_4$	$A_7^2D_5^2$
14	24	$a_5^2a_3^2a_1^2a_1^2a_1^2$	8	febcg	$A_5^4D_4$	$A_7^2D_5^2$
14	24	$a_5a_4^2a_3a_3a_1$	4	bbddf	$A_5^4D_4$	$A_7^2D_5^2$
14	24	$d_4^2a_3^4$	32	dc	D_4^6	$A_7^2D_5^2$
14	24	$a_4^3a_3^3$	12	bh	A_4^6	A_8^3
14	24	a_3^8	384	g	A_3^8	D_6^4
14	25	$d_5a_3^2a_2^4a_1^2$	8	bgbb		
14	25	$d_5a_3^3a_1^8$	48	cgc		
14	25	$a_6a_3^2a_3a_2^2a_1$	2	bhhcf		
14	25	$a_5^2a_3a_3a_1^4$	8	efee		
14	25	$a_6a_3^2a_3a_2a_1^2a_1^2$	2	bhhdeg		
14	25	$d_4a_4^3a_1^3a_1$	6	cabc		
14	25	$a_5d_4a_3a_3a_2^2a_1$	2	deeccb		
14	25	$a_5a_5a_3a_2^2a_1^2a_1a_1$	2	efddefg		
14	25	$a_5a_4^2a_3a_2a_1^2$	2	bbfdf		
14	25	$a_5a_4a_4a_3a_2a_1a_1$	1	cbbdabe		
14	25	$d_4a_4^2a_3^2a_1^2$	4	baeb		
14	25	$d_4^2a_3^2a_3a_1^4$	8	dcbb		
14	25	$a_5a_4^2a_3a_1^2a_1^2a_1$	2	cbhcee		
14	25	$a_5d_4a_3^2a_1^4a_1^2a_1$	4	dfegcg		
14	25	$a_5a_4a_3a_3a_3a_2$	1	bbhddc		
14	25	$a_5a_4a_3a_3a_3a_2$	1	bbeddc		
14	25	$a_5a_4^2a_2^2a_2a_1^2$	2	cbddf		
14	25	$a_4^4a_2^2$	8	ba		
14	25	$d_4a_4a_4a_3a_2a_2a_1a_1$	1	caaecbbg		
14	25	$a_5a_4a_3a_3a_3a_1a_1a_1$	1	bbhedbfg		
14	25	$a_5a_3^2a_3^2a_3a_1$	4	dddcg		
14	25	$a_5a_3^2a_3a_3a_3a_1$	2	bheddf		
14	25	$a_5a_4a_3^2a_2^2a_1^2a_1$	2	cbhdge		
14	25	$a_4^2a_4a_3^2a_2a_1$	2	bbdbf		
14	25	$a_4a_4a_4a_3a_3a_2a_1$	1	abbdhcf		
14	25	$a_4^2a_4a_3a_2^2a_2a_1$	2	bahbdf		
14	25	$d_4a_3^4a_3a_1^4$	8	fegg		
15	24	$a_5^2a_4^2a_1^2$	4	bdc	A_6^4	$A_7^2D_5^2$
15	24	$a_6a_4a_4a_3a_2a_1a_1$	2	chhceaa	A_6^4	$A_7^2D_5^2$
15	24	$a_5^2a_4a_3a_2^2$	4	bfdf	$A_5^4D_4$	A_8^3
15	24	$a_4^4a_3^2$	16	hb	A_4^6	$A_9^2D_6$
15	25	$d_5a_3^5$	20	ab		
15	25	$d_5a_4a_3^2a_2^2a_1^2$	2	bgbba		
15	25	$a_6a_4^2a_3a_1^2a_1$	2	chdcb		
15	25	$a_6a_4a_4a_2a_2a_1a_1a_1$	1	chhefacc		
15	25	$a_5d_4a_4^2a_1^2a_1$	2	abeab		

15	25	$a_5a_5a_4a_3a_2a_1$	1	bbddea		
15	25	$a_6a_4a_3a_3a_2a_2a_1$	1	bhdcfga		
15	25	$a_6a_4a_3^2a_2^2a_1$	2	bhdfc		
15	25	$d_4^2a_4^2a_2^2$	4	acb		
15	25	$a_5^2a_4a_3a_1^2a_1^2$	2	bedcc		
15	25	$a_5d_4a_4a_3a_2a_2a_1$	1	abecbba		
15	25	$a_5^2a_3^2a_3a_1^2$	2	bdac		
15	25	$a_5a_4^2a_4a_2a_1^2$	2	adfec		
15	25	$a_5a_4^2a_4a_2a_1^2$	2	addca		
15	25	$a_5a_4a_4a_4a_2a_1a_1$	1	bhddeca		
15	25	a_4^5	5	d		
15	25	$a_5a_4a_4a_3a_3a_2$	1	bhdcab		
15	25	$d_4a_4^2a_3^2a_3$	2	bccb		
16	21	$d_4a_4^5$	40	ab	A_4^6	
16	22	$a_5^2d_4a_3^2a_3$	8	ceba	$A_5^4D_4$	
16	22	$a_5^3a_3a_3a_1^3$	12	ecad	$A_5^4D_4$	
16	22	$d_4^4a_3a_1^6$	144	bdc	D_4^6	
16	23	$a_6a_5a_4a_3a_2a_1^2a_1$	2	bhdecah	A_6^4	
16	23	$a_5^3a_4a_1^2a_1$	6	daag	A_6^4	
16	24	$d_5d_4a_3^4$	16	dgb	$A_7^2D_5^2$	$A_7^2D_5^2$
16	24	$d_5a_5a_3^2a_3a_1^2a_1$	4	fgbceb	$A_7^2D_5^2$	$A_7^2D_5^2$
16	24	$a_5^2d_4^2a_1^2$	16	cef	$A_7^2D_5^2$	$A_7^2D_5^2$
16	24	$a_7a_3^4a_1^4$	16	fge	$A_7^2D_5^2$	$A_7^2D_5^2$
16	24	$d_5a_4a_4^2a_2^2$	4	cbba	$A_7^2D_5^2$	$A_7^2D_5^2$
16	24	$a_6d_4a_4^2a_2$	4	ahcb	$A_7^2D_5^2$	$A_7^2D_5^2$
16	24	$a_6a_5a_4a_3a_2a_1$	2	bhdech	A_6^4	A_8^3
16	24	$a_5^2d_4a_3^2a_1^2$	4	eegd	$A_5^4D_4$	$A_9^2D_6$
16	24	$a_5a_5a_4^2a_3$	4	dddf	$A_5^4D_4$	$A_9^2D_6$
16	24	$a_5^2d_4a_3^2a_1^2$	8	eebd	$A_5^4D_4$	D_6^4
16	24	$d_4^4a_1^8$	48	bc	D_4^6	D_6^4
16	24E	a_4^6	240	a	$2A_4^6$	
16	25	$a_7a_3^4a_1^2$	8	fff		
16	25	$a_7a_3a_3a_3a_2^2a_1^2$	2	efggch		
16	25	$a_5d_4^3a_1^3$	36	bcb		
16	25	$d_5a_4^2a_3a_3a_1a_1$	2	bbcbdb		
16	25	$d_5d_4a_3^3a_1^3a_1$	6	dgbec		
16	25	$d_5a_5a_3a_2^4a_1$	4	egcad		
16	25	$a_6a_5a_4a_2a_2a_1a_1$	1	bhdcagh		
16	25	$d_5a_4a_3^4$	4	bbb		
16	25	$d_5a_4^2a_3a_2^2a_1^2$	2	cbcae		
16	25	$a_6d_4a_4a_3a_2a_1a_1$	1	ahcgafe		
16	25	$a_5^2a_5a_3a_2$	2	deeb		
16	25	$a_6a_5a_3a_3a_2a_2$	1	bhfebc		

16	25	$a_6a_5a_3a_3a_2a_1a_1a_1$	1	bhegchhh		
16	25	$a_6a_4^2a_3^2a_1$	2	bcee		
16	25	$a_5a_5a_5a_2^2a_1^2a_1$	2	defchd		
16	25	$a_5^2d_4a_3a_2^2$	4	cfcb		
16	25	$a_5a_5d_4a_3a_2^2$	2	ccdga		
16	25	$a_6d_4a_3^2a_2^2a_2$	2	ahgab		
16	25	$a_5a_5a_4a_4a_2a_1$	1	ddddcg		
16	25	$a_5a_5a_4a_4a_2a_1$	1	ddadad		
16	25	$a_6a_4a_4a_3a_2a_2a_1$	1	bddecah		
16	25	$a_5d_4a_4^2a_3a_1$	2	edccb		
16	25	$a_5d_4^2a_3^2a_1a_1a_1$	2	cebecf		
16	25	$a_5^2a_4a_3^2a_1^2$	2	haeh		
16	25	$a_5^2a_4a_3a_3a_1^2$	2	ddebd		
16	25	$a_5a_5a_4a_3a_3a_1a_1$	1	eddefdg		
16	25	$a_5^2a_3^4$	8	cf		
16	25	$a_5^2a_4a_3a_2^2a_1^2$	2	hdcch		
16	25	$d_4a_4^2a_4a_3^2$	2	hcbb		
17	24	$a_7a_4^2a_3^2$	4	bfc	$A_7^2D_5^2$	A_8^3
17	24	$a_6^2a_4a_3a_1^2$	4	hebb	$A_7^2D_5^2$	A_8^3
17	24	$a_6a_5a_5a_3a_2$	2	dddcg	A_6^4	$A_9^2D_6$
17	24	$a_6^2a_3^2a_2^2$	4	hah	A_6^4	$A_9^2D_6$
17	24	a_5^4	24	a	A_6^4	D_6^4
17	25	$a_7a_4a_4a_3a_2a_1$	1	bfgcgb		
17	25	$a_6^2a_3^2a_2a_1$	2	hcfa		
17	25	$a_7a_4^2a_2^2a_2a_1$	2	bfhea		
17	25	$d_5d_4a_4^2a_2^2$	2	abbb		
17	25	$d_5a_5a_4a_3a_3a_1$	1	bbbaab		
17	25	$d_5a_5a_4a_3a_2a_2a_1$	1	bbbaedb		
17	25	$a_6a_5d_4a_3a_2a_1$	1	ecdafb		
17	25	$a_6a_5a_4^2a_1^2$	2	ddeb		
17	25	$a_6a_5a_4a_3a_3$	1	dcgbc		
17	25	$a_6a_5a_4a_3a_3$	1	dcfcc		
17	25	$a_6a_5a_4a_3a_3$	1	dceac		
17	25	$a_5^2d_4a_4a_3$	2	cdbb		
17	25	$a_6a_4a_4a_4a_3a_1$	1	deffab		
18	21	$a_5^3d_4d_4a_1$	12	bcab	$A_5^4D_4$	
18	22	$a_6^2a_4^2a_3$	4	caa	A_6^4	
18	23	$a_6d_5a_4a_4a_2a_1^2$	2	bhaaba	$A_7^2D_5^2$	
18	23	$a_6^2d_4a_4a_1^2$	4	ceba	$A_7^2D_5^2$	
18	23	$d_5a_5^2d_4a_1^2a_1^2$	4	ebcfa	$A_7^2D_5^2$	
18	23	$a_7a_5a_4^2a_1^2a_1$	4	dfcag	$A_7^2D_5^2$	
18	23	$a_7d_4^2a_3^2a_1^2$	8	cgfa	$A_7^2D_5^2$	

18	23	$d_5^2a_3^4a_1^2$	16	gea	$A_7^2D_5^2$	
18	24	$a_7a_5^2a_2^2$	8	ffa	A_8^3	A_8^3
18	24	$a_7a_5a_4^2a_1$	4	dfcg	$A_7^2D_5^2$	$A_9^2D_6$
18	24	$a_7a_5d_4a_3a_1a_1a_1$	2	eggfdfc	$A_7^2D_5^2$	$A_9^2D_6$
18	24	$a_6d_5a_4a_4a_2$	2	bhaab	$A_7^2D_5^2$	$A_9^2D_6$
18	24	$d_5^2a_3^4$	16	gb	$A_7^2D_5^2$	D_6^4
18	24	$d_5a_5^2d_4a_1^2$	4	ebbc	$A_7^2D_5^2$	D_6^4
18	24	$a_5^2a_5d_4a_3a_1$	4	gbcdb	$A_5^4D_4$	$A_{11}D_7E_6$
18	24	$a_5^4a_1^4$	48	cb	$A_5^4D_4$	E_6^4
18	25	$d_6a_3^4a_3a_1^2$	8	ddcd		
18	25	$a_7a_5^2a_1^4$	8	fge		
18	25	$a_7a_5d_4a_2^2a_1$	2	egfbc		
18	25	$a_7d_4^2a_3a_1^4$	4	cgef		
18	25	$a_7a_5a_4a_3a_2$	1	dfchb		
18	25	$a_6^2a_5a_2a_1a_1$	2	deaed		
18	25	$a_7a_5a_4a_3a_1a_1a_1$	1	dgchfeg		
18	25	$a_6d_5a_4a_3a_2a_1a_1$	1	bhaebfd		
18	25	$d_5a_5a_5a_4a_1a_1$	1	dbcacc		
18	25	$a_7a_5a_3^2a_3a_1$	2	dggfd		
18	25	$a_7a_5a_3^2a_3a_1$	2	dfhfg		
18	25	$a_6a_6a_4a_3a_2a_1$	1	cdchbg		
18	25	$a_6a_5a_5a_4a_1$	1	aeecc		
18	25	$d_5a_5d_4a_3^2a_1^2a_1$	2	ebcefb		
18	25	$d_5d_4^2a_3^2a_3$	4	cbbc		
18	25	$d_5a_5a_4^2a_3a_1$	2	dbadf		
18	25	$d_5a_5a_4^2a_3a_1$	2	dbabb		
18	25	$a_6a_5d_4a_4a_2a_1$	1	cgeabf		
18	25	$a_6a_5a_4a_4a_3$	1	cfacg		
18	25	$a_5^2d_4^2a_3a_1^2$	2	bgff		
18	25	$a_6a_4^2a_4^2a_1$	2	bcag		
19	24	$a_8a_4^2a_3^2$	4	hhb	A_8^3	$A_9^2D_6$
19	24	$a_7a_6a_5a_2a_1$	2	dfceb	A_8^3	$A_9^2D_6$
19	24	$a_6a_6a_5a_4a_1$	2	eeaha	A_6^4	$A_{11}D_7E_6$
19	24E	d_4^6	2160	d	$2D_4^6$	
19	24E	$a_5^4d_4$	48	aa	$2A_5^4D_4$	
19	25	$d_6a_4^2a_4a_2^2$	2	addc		
19	25	$a_8a_4a_4a_3a_2a_1$	1	hghbfb		
19	25	$a_8a_4^2a_2^2a_2a_1$	2	hhgeb		
19	25	$a_7a_6a_4a_2a_2a_1$	1	dfheeb		
19	25	$d_5^2a_4a_4a_2^2$	2	bbec		
19	25	$a_6d_5d_4a_4a_2$	1	bcaec		
19	25	$a_6d_5a_5a_3a_2a_1$	1	bdabca		
19	25	$a_7a_5^2a_3a_1^2$	2	dcab		

19	25	$d_5a_5^3a_1$	3	aaa		
19	25	$a_7a_5^2a_2^2a_1^2$	2	dcdb		
19	25	$a_7a_5a_4a_4a_2$	1	achgc		
19	25	$a_7d_4a_4a_4a_3$	1	ccefb		
19	25	$a_6^2a_5a_3a_2$	2	fbad		
19	25	$a_6a_5^3$	6	cb		
19	25	$a_6a_5^2a_5$	2	ecb		
19	25	$a_6a_5a_5d_4a_2$	1	babcc		
19	25	$d_5a_5^2a_4a_2^2$	2	dadc		
20	20	$d_5a_5^4$	16	ad	$A_5^4D_4$	
20	20	$d_5d_4^5$	120	dc	D_4^6	
20	21	$a_6^3d_4a_2$	6	aaa	A_6^4	
20	22	$a_7d_5a_5a_3a_3a_1$	2	bgecad	$A_7^2D_5^2$	
20	22	$d_5^2a_5^2a_3$	8	bba	$A_7^2D_5^2$	
20	22	$a_7^2a_3^2a_3a_1^2$	8	gdae	$A_7^2D_5^2$	
20	23	$a_7a_6^2a_1^2a_1^2$	4	ecga	A_8^3	
20	23	$a_7^2a_4a_3a_1^2$	4	faea	A_8^3	
20	23	$a_8a_5^2a_2^2a_1^2$	4	dhba	A_8^3	
20	24	$a_7^2d_4a_1^4$	8	gff	$A_9^2D_6$	$A_9^2D_6$
20	24	$d_6a_5^2a_3^2$	8	edd	$A_9^2D_6$	$A_9^2D_6$
20	24	$a_8a_5^2a_3$	4	dge	$A_9^2D_6$	$A_9^2D_6$
20	24	$d_6a_5^2a_3^2$	4	edc	D_6^4	$A_9^2D_6$
20	24	$d_6d_4^3a_1^6$	12	bcb	D_6^4	D_6^4
20	24	$a_7d_5a_5a_3a_1a_1a_1$	2	cgefecd	$A_7^2D_5^2$	$A_{11}D_7E_6$
20	24	$d_5d_5a_5^2a_1^2$	4	bced	$A_7^2D_5^2$	$A_{11}D_7E_6$
20	24	$a_7d_5d_4a_3a_3$	2	bgecf	$A_7^2D_5^2$	$A_{11}D_7E_6$
20	24	$a_6a_6d_5a_4$	2	aaeb	$A_7^2D_5^2$	$A_{11}D_7E_6$
20	24	$d_5^2a_5^2a_1^2$	8	cbc	$A_7^2D_5^2$	E_6^4
20	24	$a_6^2a_5^2$	4	ch	A_6^4	A_{12}^2
20	24	d_4^6	48	c	D_4^6	D_8^3
20	25	$a_8a_5^2a_1^2a_1^2$	2	dhfg		
20	25	$d_6a_5^2a_3a_1^2a_1a_1$	2	eddfeb		
20	25	$d_6a_5d_4a_3^2a_1$	2	ccdcd		
20	25	$a_7a_7a_3a_2^2a_1^2$	2	fggbg		
20	25	$a_8d_4a_4a_3a_3$	1	chbff		
20	25	$d_5^2a_5d_4a_1^2a_1$	2	bbecb		
20	25	$a_7d_5a_4^2a_1a_1$	2	cfbcf		
20	25	$a_7a_6a_5a_3$	1	ecfe		
20	25	$a_6^2d_5a_3a_1^2$	2	aedd		
20	25	$a_7d_5a_4a_3a_3$	1	bfbfc		
20	25	$a_7a_6a_5a_2a_1a_1a_1$	1	echbgfe		
20	25	$a_7a_6a_4a_4a_1$	1	ecbad		
20	25	$a_6^2a_6a_3a_1$	2	abee		

20	25	$a_6a_6a_6a_3a_1$	1	caceg		
20	25	$a_6d_5a_5a_4a_2a_1$	1	aeebad		
20	25	$a_6a_6a_5d_4a_1$	1	abfhd		
20	25	$a_7a_5^2a_4a_1^2$	2	ehag		
20	25	$d_5a_5a_5a_4^2$	2	fbdb		
21	24	$a_8a_6a_5a_2a_1$	2	ehbga	A_8^3	$A_{11}D_7E_6$
21	24	$a_7^2a_4^2$	4	cg	$A_7^2D_5^2$	A_{12}^2
21	25	$d_6a_6a_4a_4a_2$	1	cdcdd		
21	25	$a_8a_6a_4a_3a_1$	1	eggbb		
21	25	$a_7a_6d_5a_2a_1a_1$	1	aecfba		
21	25	$a_6d_5d_5a_4a_2$	1	baacc		
21	25	$a_7d_5a_5a_4a_1$	1	acbdb		
21	25	$a_7a_6a_5a_4$	1	cgbe		
22	21	$a_7d_5^2d_4a_3$	4	beac	$A_7^2D_5^2$	
22	22	$a_8a_6^2a_3$	4	bba	A_8^3	
22	23	$a_8a_7a_5a_1^2a_1$	2	cgeac	$A_9^2D_6$	
22	23	$a_8a_6d_5a_2a_1^2$	2	abhba	$A_9^2D_6$	
22	23	$a_7d_6a_5a_3a_1^2a_1$	2	dgdfab	$A_9^2D_6$	
22	23	$a_9a_5a_4^2a_1^2$	4	fgba	$A_9^2D_6$	
22	23	$d_6d_5a_5^2a_1^2$	4	bdcd	D_6^4	
22	23	$d_5^4a_1^2$	48	bd	D_6^4	
22	24	$a_9a_5d_4a_3a_1a_1$	2	gfffeb	$A_9^2D_6$	$A_{11}D_7E_6$
22	24	$a_7d_6a_5a_3a_1$	2	dgcfe	$A_9^2D_6$	$A_{11}D_7E_6$
22	24	$a_8a_6d_5a_2$	2	abhb	$A_9^2D_6$	$A_{11}D_7E_6$
22	24	$d_6d_5a_5^2$	4	bdc	D_6^4	$A_{11}D_7E_6$
22	24	d_5^4	48	b	D_6^4	E_6^4
22	24	$a_8a_7a_4a_3$	2	chbg	A_8^3	A_{12}^2
22	24	$a_7d_5^2a_3^2$	4	eed	$A_7^2D_5^2$	D_8^3
22	24	$a_7^2d_4^2$	8	fd	$A_7^2D_5^2$	D_8^3
22	24E	a_6^4	24	a	$2A_6^4$	
22	25	$e_6d_4^2a_3^3$	12	cbc		
22	25	$e_6a_5a_4^2a_3a_1$	2	dbace		
22	25	$e_6a_5^2a_2^4$	8	fba		
22	25	$d_7a_3^6$	24	ge		
22	25	$a_9a_5d_4a_2^2$	2	gfgb		
22	25	$a_9a_5a_4a_3a_1a_1$	1	ffbgbc		
22	25	$a_9a_5a_3^2a_3$	2	fggf		
22	25	$a_8a_7a_4a_2a_1$	1	chbbc		
22	25	$d_6d_5a_5a_3a_3a_1$	1	bdcdbe		
22	25	$a_7d_6a_3^2a_3a_1^2$	2	dgfeb		
22	25	$a_8d_5a_5a_3a_1$	1	agffe		
22	25	$a_8d_5a_5a_2a_2a_1$	1	ahfbab		

22	25	$a_8a_6a_5a_3$	1	bbgg		
22	25	$a_8a_6a_5a_3$	1	cbef		
22	25	$d_6a_5^2d_4a_3$	2	bcdb		
22	25	$d_6a_5^3a_1^3$	6	cdb		
22	25	$a_8a_5^2d_4$	2	bfe		
22	25	$a_8a_6a_4a_4a_1$	1	cbbbc		
22	25	$a_7a_7a_5a_3a_1$	1	ghefc		
22	25	$d_5^3a_3^3$	6	ec		
22	25	$a_7d_5d_4^2a_3$	2	bffc		
23	24	$a_8^2a_3^2$	4	hb	$A_9^2D_6$	A_{12}^2
23	24	$a_9a_6a_5a_2$	2	cgbc	$A_9^2D_6$	A_{12}^2
23	24	a_7^3	12	a	A_8^3	D_8^3
23	25	$d_7a_4^4$	4	bd		
23	25	$a_9a_6a_4a_3$	1	cfgb		
23	25	$a_8d_5^2a_2^2$	2	ebe		
23	25	$d_6a_6a_6a_4$	1	accf		
23	25	$d_6a_6^2a_4$	2	bce		
23	25	$a_8a_7d_4a_3$	1	fbbb		
23	25	$a_7^2d_5a_3$	2	baa		
23	25	$a_6^2d_5^2$	2	cb		
24	20	$a_7^2d_5d_5$	4	cda	$A_7^2D_5^2$	
24	21	$a_8^2d_4a_4$	4	baa	A_8^3	
24	22	$a_9a_7d_4a_3a_1$	2	fffad	$A_9^2D_6$	
24	22	$a_7^2d_6a_3$	4	cfa	$A_9^2D_6$	
24	22	$d_6^2d_4^2a_3a_1^2$	4	cbdb	D_6^4	
24	24	$d_7a_5^2a_5a_1$	4	cdea	$A_{11}D_7E_6$	$A_{11}D_7E_6$
24	24	$a_9d_5^2a_1^2a_1$	4	efec	$A_{11}D_7E_6$	$A_{11}D_7E_6$
24	24	$a_7e_6d_4a_3^2$	4	bgce	$A_{11}D_7E_6$	$A_{11}D_7E_6$
24	24	$e_6a_6^2a_4$	4	eaa	$A_{11}D_7E_6$	$A_{11}D_7E_6$
24	24	$e_6d_5a_5^2a_1^2$	4	cbca	E_6^4	$A_{11}D_7E_6$
24	24	$d_6^2d_4^2a_1^4$	4	cbb	D_6^4	D_8^3
24	24	$a_7^2d_6a_1^2$	4	dfd	$A_9^2D_6$	D_8^3
24	24	$a_7^2d_5d_4$	4	fde	$A_7^2D_5^2$	$A_{15}D_9$
24	25	$d_7a_5^2d_4a_1a_1$	2	bdeeb		
24	25	$a_9a_7a_3^2a_1$	2	fgcd		
24	25	$d_6d_5^2a_5a_1$	2	bcba		
24	25	$a_7d_6d_5a_3a_1a_1$	1	cedeeb		
24	25	$a_8a_7a_6a_1$	1	aebd		
24	25	$a_7d_6a_5d_4a_1$	1	cfdfb		
24	25	$a_7^2a_7a_1^2$	2	edd		
24	25	$a_8a_7d_4a_4$	1	bfgb		
24	25	$a_7a_7d_5a_4$	1	ccgb		

24	25	$a_7^2d_5a_4$	2	cea		
25	24	$a_{10}a_6a_5a_1$	2	hgbb	$A_{11}D_7E_6$	A_{12}^2
25	24	$a_8a_7^2$	4	eb	A_8^3	$A_{15}D_9$
25	24E	$a_7^2d_5^2$	8	aa	$2A_7^2D_5^2$	
25	25	$d_7a_6a_6a_2a_2$	1	adedc		
25	25	$a_{10}a_6a_4a_2a_1$	1	hgcea		
25	25	$e_6a_6d_5a_4a_2$	1	acaea		
25	25	$e_6a_6^2d_4$	2	bca		
25	25	$a_7e_6a_5a_4a_1$	1	acaeb		
25	25	$a_{10}a_5a_4a_4$	1	gbcb		
25	25	$a_8d_6a_6a_2$	1	dbfc		
25	25	$a_9a_6d_5a_2a_1$	1	bfbeb		
25	25	$a_9a_6^2a_2$	2	agc		
25	25	$a_9d_5a_5a_4$	1	bbbe		
26	19	$a_7^2d_6d_5$	4	dae	$A_7^2D_5^2$	
26	21	$a_9d_6a_5d_4$	2	cfea	$A_9^2D_6$	
26	23	$a_9d_6d_5a_1^2a_1$	2	cffab	$A_{11}D_7E_6$	
26	23	$a_{10}a_6d_5a_1^2$	2	bbga	$A_{11}D_7E_6$	
26	23	$a_8e_6a_6a_2a_1^2$	2	ahaba	$A_{11}D_7E_6$	
26	23	$d_7a_7d_5a_3a_1^2$	2	edeea	$A_{11}D_7E_6$	
26	23	$e_6d_6a_5^2a_1^2$	4	dbea	$A_{11}D_7E_6$	
26	23	$e_6d_5^3a_1^2$	12	bce	E_6^4	
26	24	$a_9^2a_2^2$	8	ga	A_{12}^2	A_{12}^2
26	24	$d_7a_7d_5a_3$	2	eddc	$A_{11}D_7E_6$	D_8^3
26	24	$a_9d_6a_5a_3$	2	dfbd	$A_9^2D_6$	$A_{15}D_9$
26	24	$a_9a_8a_5$	2	ebc	$A_9^2D_6$	$A_{15}D_9$
26	24	$a_7^2d_5^2$	8	ce	$A_7^2D_5^2$	$D_{10}E_7^2$
26	25	$d_7d_5^2a_3a_3$	2	bdba		
26	25	$a_{10}d_5a_5a_2a_1$	1	bgbbb		
26	25	$a_7d_6^2a_3$	2	bcc		
26	25	$a_9d_6d_4a_3a_1$	1	cfbeb		
26	25	$a_9a_7a_6$	1	egb		
26	25	$a_9a_7a_6$	1	efb		
26	25	$a_7d_6d_5a_5a_1$	1	dffeb		
27	24	$a_8^2a_7$	4	ga	A_8^3	$A_{17}E_7$
27	25	$a_8d_7a_4a_4$	1	dbde		
27	25	$a_9^2a_3a_1^2$	2	baa		
27	25	$a_{10}a_6^2a_1$	2	eca		
28	20	$a_9^2d_5a_1^2$	4	fac	$A_9^2D_6$	
28	20	$d_6^3d_5a_1^2$	6	bdb	D_6^4	

28	22	$a_9e_6d_5a_3a_1$	2	cfead	$A_{11}D_7E_6$	
28	22	$a_9d_7a_5a_3$	2	dfca	$A_{11}D_7E_6$	
28	22	$a_{11}d_5a_5a_3a_1$	2	fbeae	$A_{11}D_7E_6$	
28	22	$e_6^2a_5^2a_3$	8	bae	E_6^4	
28	23	$a_{10}a_9a_2a_1^2a_1$	2	bgaaf	A_{12}^2	
28	23	$a_{11}a_7a_4a_1^2$	2	gcaa	A_{12}^2	
28	24	$d_8d_4^4$	8	cb	D_8^3	D_8^3
28	24	$a_9d_7a_5a_1a_1$	2	efede	$A_{11}D_7E_6$	$A_{15}D_9$
28	24	$a_{11}d_5d_4a_3$	2	fbeb	$A_{11}D_7E_6$	$A_{15}D_9$
28	24	$a_9a_7d_6a_1$	2	fefc	$A_9^2D_6$	$A_{17}E_7$
28	24	$a_9a_7d_6a_1$	2	fbfc	$A_9^2D_6$	$D_{10}E_7^2$
28	24	$d_6^2d_6d_4a_1^2$	2	bbbb	D_6^4	$D_{10}E_7^2$
28	24E	a_8^3	12	a	$2A_8^3$	
28	25	$d_8a_5^2a_5a_1$	2	ddbe		
28	25	$a_{11}a_7a_2^2a_1^2$	2	gbaf		
28	25	$a_{11}d_5a_4a_3$	1	fcbb		
28	25	$e_6d_6d_5a_5a_1$	1	cceab		
28	25	$d_7d_6a_5a_5$	1	cdcb		
28	25	$a_8a_7e_6a_1a_1$	1	aeedc		
28	25	$a_9e_6a_4^2a_1$	2	cgbc		
28	25	$a_{10}a_8a_4a_1$	1	bbae		
28	25	$a_9^2a_4a_1^2$	2	gaf		
28	25	$a_9a_8d_5a_1$	1	fbcd		
29	24	$a_{11}a_8a_3$	2	bca	A_{12}^2	$A_{15}D_9$
29	25	$a_{10}d_6a_6$	1	fbe		
30	21	$d_7a_7e_6d_4$	2	cada	$A_{11}D_7E_6$	
30	22	$a_{10}^2a_3$	4	ba	A_{12}^2	
30	23	$d_7^2a_7a_1^2$	4	dcd	D_8^3	
30	23	$d_8a_7^2a_1^2$	4	ddd	D_8^3	
30	24	$d_8a_7^2$	4	dd	D_8^3	$A_{15}D_9$
30	24	$a_{11}d_6a_5a_1$	2	fbba	$A_{11}D_7E_6$	$A_{17}E_7$
30	24	$a_{10}e_6a_6$	2	agb	$A_{11}D_7E_6$	$A_{17}E_7$
30	24	$d_7a_7e_6a_3$	2	caed	$A_{11}D_7E_6$	$D_{10}E_7^2$
30	24	$a_9e_6d_6a_1$	2	efea	$A_{11}D_7E_6$	$D_{10}E_7^2$
30	24	$e_6^2d_5^2$	8	ca	E_6^4	$D_{10}E_7^2$
30	25	$d_8a_7d_5a_3$	1	cbdd		
30	25	$a_{10}a_9a_4$	1	bca		
30	25	$d_7a_7d_5^2$	2	cae		
31	24	$a_{12}a_7a_4$	2	gbf	A_{12}^2	$A_{17}E_7$
31	24E	d_6^4	24	d	$2D_6^4$	
31	24E	$a_9^2d_6$	4	aa	$2A_9^2D_6$	

31	25	$a_{12}a_6a_5$	1	gba		
31	25	$a_8e_6^2a_2^2$	2	aaa		
31	25	$a_{10}e_6d_5a_2$	1	ebba		
31	25	$a_{11}a_8d_4$	1	bea		
31	25	$a_{11}a_7d_5$	1	bba		
31	25	$a_8a_8d_7$	1	dcb		
31	25	$a_8^2d_7$	2	ca		
32	18	$a_9^2d_7$	4	da	$A_9^2D_6$	
32	18	$d_7d_6^3$	6	db	D_6^4	
32	20	$a_{11}e_6d_5a_3$	2	ebaa	$A_{11}D_7E_6$	
32	21	$a_{12}a_8d_4$	2	bba	A_{12}^2	
32	22	$d_8d_6^2a_3a_1^2$	2	bbdb	D_8^3	
32	24	$d_8d_6^2a_1^2a_1^2$	2	bbba	D_8^3	$D_{10}E_7^2$
32	24	d_6^4	8	b	D_6^4	D_{12}^2
32	25	$a_9d_8a_5a_1a_1$	1	dfecb		
32	25	$a_{12}a_7d_4$	1	bbf		
32	25	$a_9d_7d_6a_1$	1	ceeb		
34	19	$a_{11}d_7d_6$	2	cea	$A_{11}D_7E_6$	
34	19	$e_6^3d_6$	12	ae	E_6^4	
34	23	$a_{11}d_8a_3a_1^2$	2	dbca	$A_{15}D_9$	
34	23	$a_{13}a_8a_1^2a_1$	2	caac	$A_{15}D_9$	
34	23	$d_9a_7^2a_1^2$	4	eea	$A_{15}D_9$	
34	24	$a_{13}d_6a_3a_1$	2	bbcb	$A_{15}D_9$	$A_{17}E_7$
34	24	$d_9a_7^2$	4	ed	$A_{15}D_9$	$D_{10}E_7^2$
34	24	$a_{11}d_7d_5$	2	eee	$A_{11}D_7E_6$	D_{12}^2
34	25	$e_7a_7d_5a_5$	1	cbca		
34	25	$e_7a_7^2a_3a_1$	2	dcab		
34	25	$d_9a_7d_5a_3$	1	ddeb		
34	25	$a_{13}a_7a_3a_1$	1	cfcc		
34	25	$d_7^2e_6a_3$	2	aca		
34	25	$d_8a_7e_6a_3$	1	edda		
34	25	$a_{11}d_6^2$	2	cb		
35	24	a_{11}^2	4	a	A_{12}^2	D_{12}^2
35	25	$a_{12}d_7a_4$	1	ebc		
36	20	$d_8^2d_5d_4$	2	bda	D_8^3	
36	22	$a_{13}d_7a_3a_1$	2	ebab	$A_{15}D_9$	
36	24	$a_9e_7a_7$	2	cfa	$D_{10}E_7^2$	$A_{17}E_7$
36	24	$e_7d_6d_6d_4a_1$	2	bbbaa	$D_{10}E_7^2$	$D_{10}E_7^2$
36	24	$d_8^2d_4^2$	2	bb	D_8^3	D_{12}^2
36	25	$a_{12}a_{10}a_1$	1	aab		

37	24E	e_6^4	48	e	$2E_6^4$	
37	24E	$a_{11}d_7e_6$	2	aaa	$2A_{11}D_7E_6$	
37	25	$a_{13}e_6a_4a_1$	1	baba		
38	17	$a_{11}d_8e_6$	2	dae	$A_{11}D_7E_6$	
38	21	$a_{11}d_9d_4$	2	dea	$A_{15}D_9$	
38	23	$d_9e_6^2a_1^2$	4	add	$D_{10}E_7^2$	
38	23	$a_9e_7e_6a_1^2$	2	aecd	$D_{10}E_7^2$	
38	23	$a_{14}e_6a_2a_1^2$	2	bfaa	$A_{17}E_7$	
38	23	$a_{11}e_7a_5a_1^2$	2	cbba	$A_{17}E_7$	
38	24	$a_{11}d_9a_3$	2	eeb	$A_{15}D_9$	D_{12}^2
38	24	$a_{12}a_{11}$	2	af	A_{12}^2	A_{24}
38	25	$d_9d_7a_7$	1	cdb		
38	25	$a_{11}d_8d_5$	1	dbc		
40	20	$a_{15}d_5d_5$	2	bba	$A_{15}D_9$	
40	22	$d_8e_7d_6a_3a_1$	1	bbada	$D_{10}E_7^2$	
40	22	$d_{10}d_6^2a_3$	2	bbd	$D_{10}E_7^2$	
40	22	$a_{15}d_6a_3$	2	bca	$A_{17}E_7$	
40	24	$d_{10}d_6^2a_1^2$	2	bba	$D_{10}E_7^2$	D_{12}^2
40	24E	a_{12}^2	4	a	$2A_{12}^2$	
40	25	$d_{10}a_9a_5$	1	dbb		
40	25	$a_{15}d_5a_4$	1	bca		
40	25	$e_7e_6^2a_5$	2	aaa		
40	25	$a_9e_7d_7$	1	aca		
40	25	$a_{11}e_7d_5a_1$	1	aeba		
40	25	$a_{14}a_9$	1	ac		
41	24	$a_{15}a_8$	2	ac	$A_{15}D_9$	A_{24}
42	21	$a_{13}e_7d_4$	2	aba	$A_{17}E_7$	
43	24	$a_{16}a_7$	2	fa	$A_{17}E_7$	A_{24}
43	24E	d_8^3	6	d	$2D_8^3$	
44	16	$d_9d_8^2$	2	db	D_8^3	
44	20	$e_7^2d_6d_5$	2	bad	$D_{10}E_7^2$	
44	24	$d_8^2d_8$	2	ba	D_8^3	$D_{16}E_8$
44	24	d_8^3	6	a	D_8^3	E_8^3
46	23	$d_{11}a_{11}a_1^2$	2	ebd	D_{12}^2	
46	24	$a_{15}d_8$	2	cb	$A_{15}D_9$	$D_{16}E_8$
46	25	$d_{11}a_7e_6$	1	dab		

47	25	$a_{16}d_7$	1	ca		
48	18	$d_{10}e_7d_7a_1$	1	abda	$D_{10}E_7^2$	
48	18	$a_{17}d_7a_1$	2	cac	$A_{17}E_7$	
48	22	$d_{10}^2a_3a_1^2$	2	bdb	D_{12}^2	
48	24	$a_{15}e_7a_1$	2	bcc	$A_{17}E_7$	$D_{16}E_8$
48	24	$d_{10}e_7d_6a_1$	1	abaa	$D_{10}E_7^2$	$D_{16}E_8$
48	24	$d_8e_7^2a_1^2$	2	aaa	$D_{10}E_7^2$	$D_{16}E_8$
48	25	$a_{13}d_{10}a_1$	1	bcb		
49	24E	$a_{15}d_9$	2	aa	$2A_{15}D_9$	
50	15	$a_{15}d_{10}$	2	ba	$A_{15}D_9$	
52	20	$d_{12}d_8d_5$	1	bad	D_{12}^2	
52	23	$a_{19}a_4a_1^2$	2	caa	A_{24}	
52	24	$d_{12}d_8d_4$	1	bab	D_{12}^2	$D_{16}E_8$
55	24E	$d_{10}e_7^2$	2	dd	$2D_{10}E_7^2$	
55	24E	$a_{17}e_7$	2	aa	$2A_{17}E_7$	
55	25	$a_{19}d_5$	1	aa		
56	14	$d_{11}e_7^2$	2	da	$D_{10}E_7^2$	
56	21	$a_{20}d_4$	2	aa	A_{24}	
58	25	$e_8a_{11}e_6$	1	aac		
58	25	$a_{11}d_{13}$	1	bb		
60	24	$e_8d_8^2$	2	aa	E_8^3	$D_{16}E_8$
62	23	$a_{15}e_8a_1^2$	2	cbd	$D_{16}E_8$	
64	22	$e_8e_7^2a_3$	2	aae	E_8^3	
64	22	$d_{14}e_7a_3a_1$	1	abda	$D_{16}E_8$	
67	24E	d_{12}^2	2	d	$2D_{12}^2$	
68	12	$d_{13}d_{12}$	1	db	D_{12}^2	
68	20	$d_{12}e_8d_5$	1	aad	$D_{16}E_8$	
68	24	d_{12}^2	2	b	D_{12}^2	D_{24}
71	24	a_{23}	2	a	A_{24}	D_{24}

76	16E	$d_{16}d_9$	1	bd	$D_{16}E_8$	
76	16E	$d_9e_8^2$	2	ea	E_8^3	
76	24	$d_{16}d_8$	1	ba	$D_{16}E_8$	D_{24}
76	24E	a_{24}	2	a	$2A_{24}$	
91	24E	e_8^3	6	e	$2E_8^3$	
91	24E	$d_{16}e_8$	1	dd	$2D_{16}E_8$	
92	8E	$d_{17}e_8$	1	da	$D_{16}E_8$	
100	20	$d_{20}d_5$	1	ad	D_{24}	
139	24E	d_{24}	1	d	$2D_{24}$	
140	0E	d_{25}	1	d	D_{24}	

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