What is a vertex algebra?.

Arbeitstagung talk 1997
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The answer to the question in the title is that a vertex algebra is really a sort of commutative ring. I will try to explain this in the rest of the talk, and show how to use this to generalize the idea of a vertex algebra to higher dimensions. The picture to keep in mind is that a commutative ring should be thought of as somehow related to quantum field theories in 0 dimensions, and vertex algebras are related in the same way to 1 dimensional quantum field theories, and we want to find out what corresponds to higher dimensional field theories. This talk is an exposition of the paper q-alg/9706008, which contains (some of) the missing details. There is also some overlap with unpublished notes of Soibelman, which have just appeared on the q-alg preprint server as q-alg/9709030.

The relation of vertex algebras to commutative rings is obscured by the rather bad notation generally used for vertex algebras. Recall that for any element v of a vertex algebra V we have a vertex operator denoted by $V(v,z)$ taking V to the Laurent power series in V. I am going to change notation and write $V(v,z)u$ as $v^z u$. Let us see what several standard formulas look like in this new notation:

Old notation	New notation
$V(u,z)v$	$u^z v$
$V(V(a,x)b,y)c = V(a,x+y)V(b,y)c$	$(a^x b)^y c = a^{xy}(b^y c)$
$V(a,x)V(b,y)c = V(b,y)V(a,x)c$	$a^x b^y c = b^y a^x c$
$V(a,x)b = e^{xL_{-1}}(V(b,-x)a)$	$a^x b = (b^{x^{-1}} a)^x$
$V(1,x)b = b$	$1^x b = b$

The formulas on the right hand side are all easy to recognize: they are just standard formulas for a commutative ring acted on by a group G, where a, b, c are in the ring V, x, y, z are elements of the group G, and the action of $x \in G$ on $a \in V$ is denoted by a^x. This suggests that we should try to set things up so that vertex algebras are exactly the commutative rings objects over some sort of mysterious group-like thing G.

For simplicity we will work over a field of characteristic 0. This is not an important assumption; it just saves us from some minor technicalities about divided powers of derivations.

We will first look at the special case of vertex algebras such that all the vertex operators $V(a,x)$ are holomorphic. We show that such vertex algebras are the same as commutative algebras with a derivation D. The correspondence is given as follows. First suppose that V is a commutative algebra with derivation D. We define the vertex operator $V(a,x)$ by $V(a,x)b = \sum_{i\geq 0}(D^i a)bx^i/i!$. Conversely if V is a vertex algebra we define the product

* Supported by a Royal Society professorship.

by $ab = V(a,0)b$ and the derivation by $Da =$ coefficient of x^1 in $V(a,x)b$. (We cannot really check that this turns commutative algebras into holomorphic vertex algebras and vice versa because we have not yet said exactly what the axioms for a vertex algebra are.)

In the new notation for vertex algebras above we would put $a^x = \sum_i x^i D^i a/i!$, $a^x b = \sum_i x^i D^i ab/i!$. Here we think of x as being an "element" of the one dimensional formal group $\hat{G}_a$. This formal group has as its formal group ring H the algebra of polynomials $k[D]$ and its coordinate ring is the ring of formal power series $k[[x]]$. (In characteristic 0 it does not matter whether we use Lie algebras or formal groups which are essentially equivalent, but in other characteristics formal groups are better than Lie algebras.) The (tensor) category of modules with a derivation is the same as the category of modules over the formal group ring H, so holomorphic vertex algebras are the same as the commutative ring objects in this category.

What is the difference between a commutative algebra over $\hat{G}_a$ and a (non holomorphic) vertex algebra? The only difference is that expressions like $a^x b^y c$ are no longer holomorphic in x, y but can have singularities; more precisely $a^x b^y c$ lies in $V[[x,y]][x^{-1}, y^{-1}, (x-y)^{-1}]$. In other words we can provisionally define a vertex algebra to be a module V such that we are given functions $a^x b^y c^z \cdots$ for each $a, b, c, \ldots \in V$ which behave just like the corresponding functions for commutative rings over $\hat{G}_a$, except that they are allowed to have certain sorts of singularities. Notice that we can no longer reconstruct a commutative ring structure on V by defining $ab = a^x b$ at $x = 1$, because $a^x b$ may have a singularity at $x = 1$.

The definition above is too vague to be useful, so we try to make it more precise. What we really want to do is to define some sort of category, whose multilinear maps are somehow allowed to have the sort of singularities above, and whose commutative ring objects are just vertex algebras. We first ask in what sort of categories we can define commutative ring objects. The obvious answer is tensor categories, such as the category of modules over $\hat{G}_a$ (or over any cocommutative Hopf algebra) but this turns out to be too restrictive. (We will implicitly assume that all categories are additive and have some sort of "symmetric" structure.) A tensor category requires that multilinear maps should be representable, but this is sometimes not the case for the categories we are interested in, and in any case this assumption is unnecessary. It is sufficient to assume that for each collection of objects $A_1, \ldots, A_n, B$ of the category we are given the space of multilinear maps $Multi(A_1, \ldots, A_n, B)$, and that these satisfy a large number of fairly obvious properties which I cannot be bothered to write down.

Unfortunately multilinear categories are not really the right objects either. The problem is the following. An expression like $a^x b^y c$ should live in a space like $V[[x,y]][x^{-1}, y^{-1}, (x-y)^{-1}]$. However the expression $a^x(b^y c)$ does not naturally live in this space, but in the larger space $V[[y]][y^{-1}][[x]][x^{-1}]$, and $(a^{xy^{-1}} b)^y c$ lives in a different larger space. This makes it hard to compare these expressions in a clean way. The easiest way to solve this problem is to define it out of existence by using "relaxed multilinear categories". The key idea is that instead of just once space of multilinear maps $Multi(A_1, \ldots, A_n, B)$ we are given many different spaces $Multi_p(A_1, \ldots, A_n, B)$ of multilinear maps, parameterized by trees p with a root (corresponding to B) and n leaves, corresponding to $A_1, \ldots, A_n$. We should also have some extra structures, consisting of maps between different spaces of multilinear maps corresponding to collapsing maps between trees, and a composition of multilinear maps taking multilinear maps of types $p_1, \dots, p_n, p$ to a multilinear map of type $p(p_1, \dots, p_n)$. (Here $p(p_1, \dots, p_n)$ is the tree obtained by attaching $p_1, \dots, p_n$ to the leaves of p.) For details see my paper or Soibelman's notes, or better still work them out for yourself.

The main point is that in a relaxed multilinear category it is still possible to define commutative associative algebras. Joyal has pointed out that the definition of associative algebras in a relaxed multilinear category is strikingly similar to the definition of an A_∞ algebra; for example, the cells of the complexes used to define A_∞ algebras are parameterized by rooted trees with n leaves, and the boundary maps correspond to the collapsing maps between trees.

One way of constructing relaxed multilinear categories is as the representations of "vertex groups". A vertex group can be thought of informally as a group together with certain sorts of allowed singularities of functions on the group. More precisely a vertex group is given by a cocommutative Hopf algebra H, which we thing of as its group ring, together with an algebra of "singular functions" K over the "coordinate ring" H^* of H. The axioms for a vertex group say that K behaves as if it were the ring of meromorphic functions over the "group" G; for example, the ring of meromorphic functions is acted on by left and right translations, so K should have good left and right actions of H. A typical example of a vertex group is to take $H = k[D]$, $H^* = k[[x]]$ (so that H is the formal group ring of $\hat{G}_a$), and to take K to be the quotient field $k[[x]][x^{-1}]$ of H^*, which we can think of as the field of rational functions on the formal group $\hat{G}_a$.

We can construct a relaxed multilinear category from a vertex group roughly as follows. The underlying category is the same as that of the Hopf algebra of the vertex group. However the spaces of multilinear maps are different. Rather than define these in general, which is a bit complicated, we will just look at one example. We take G to be the vertex group above (whose commutative rings are vertex algebras), and take 3 G-modules A, B, and C. Then the space of bilinear maps from A, B to C is defined to be the ordinary space of bilinear maps from $A \times B$ to $C[[x, y]][(x - y)^{-1}]$ which are invariant under an action of G^3. (The easiest way to work out what the action of G^3 should be is to see what it has to be for the invariant bilinear maps from $A \times B$ to $C[[x, y]]$ taking $a \times b$ to $\sum_{i,j} f(D^i a, D^j b) x^i y^j / i! j!$ to be the same as invariant maps f from $A \times B$ to C.)

We summarize what we have done so far:

1 We have introduced "vertex groups"'.

2 The modules over a vertex group form a "relaxed multilinear category".

3 The commutative ring objects over the simplest nontrivial vertex group are exactly vertex algebras.

Now that we have set up this machinery, it is easy to find higher dimensional analogues of vertex algebras: all we have to do is look at commutative algebras over higher dimensional vertex groups G; we will call these vertex G algebras. As an example we will construct vertex algebras related to free quantum field theories in higher dimensions. (Can one construct vertex algebras corresponding to nontrivial quantum field theories in higher dimensions? At the moment this is just a daydream, as it is too vague to be called a conjecture.)

We first need to construct a suitable vertex group G. We take its underlying Hopf algebra H to be the polynomial algebra $\mathbf{R}[D_0, \ldots, D_n]$ where $D_i = \partial/\partial x_i$, which we think of as the universal enveloping algebra of the Lie algebra of translations of spacetime (with $x_0 = t$). The dual H^* is then $\mathbf{R}[[x_0, \ldots, x_n]]$, which we think of as the algebra of functions on spacetime. We define K to be $H^*[(x_0^2 - x_1^2 - \cdots x_n^2)^{-1}]$, which we think of as the algebra of functions on spacetime which are allowed to have singularities (poles) on the light cone.

Now we define the vertex G algebra V. The underlying space of V is the universal commutative H-algebra generated by an element ϕ, so $V = \mathbf{R}[\phi, D_0\phi, D_1\phi, \ldots, D_0^2\phi, \ldots]$ is a ring of polynomials in an infinite number of variables. We think of V as the ring of classical fields generated by ϕ, and it is a (holomorphic) vertex G algebra as it is a commutative ring acted on by H. We will turn it into a a nontrivial vertex G algebra by "deforming" this trivial vertex G algebra structure. (In general, for vertex G-algebras, quantization means deforming the structure on some commutative ring to turn it into a vertex G algebra.)

To do this we recall the following method of constructing commutative rings: if V is a space acted on by commuting operators v_n, and if V is generated by an element $1 \in V$ by the action of these operators, then V has a unique commutative ring structure such that 1 is the identity and the actions of all the operators are given by multiplication by elements of V. (Proof: easy exercise.) A similar theorem holds for vertex algebras (as was proved by Frenkel, Kac, Rado, and Wang). We will make V into a vertex G algebra by finding a vertex operator $\phi(x) = \phi(x_0 \ldots, x_n)$ acting on V such that $\phi(x)\phi(y) = \phi(y)\phi(x)$ and applying the construction above.

To construct $\phi(x)$, we first put

$$\phi^+(x) = \sum_i D^i \phi x^i / i!.$$

The vertex algebra structure on V defined by this vertex operator is just the commutative ring structure on V, so we need to deform ϕ^+. We define $\phi^-(x)$ to be the unique G-invariant derivation from V to $V[x][(x_0^2 - x_1^2 \cdots)^{-1}]$ taking ϕ to some even function $\Delta(x)$ (called the propagator). This is uniquely defined by the universal property of V. Finally we put

$$\phi(x) = \phi^+(x) + \phi^-(x).$$

It is easy to check that $\phi(x)$ and $\phi(y)$ commute, as $\phi^+(x)$ and $\phi^+(y)$ commute, $\phi^-(x)$ and $\phi^-(y)$ commute, and $[\phi^-(x), \phi^+(y)] = -[\phi^+(x), \phi^-(y)] = \Delta(x - y)$. Therefore we can make V into a commutative vertex G-algebra.

Notice that in quantum field theory, $\phi(x)$ means the value of some operator valued distribution ϕ at some point x of a manifold. On the other hand, for vertex G algebras, $\phi(x)$ should be thought of as the action of a "group" element x on an element ϕ of a vertex G algebra. Several other concepts in quantum field theory can also be translated into vertex algebra theory; for example, the correlation functions are $Tr(\phi(x)\phi(y)\phi(z)\cdots)$, where Tr is some G invariant linear function on V. Some concepts are not so easy to extend; for example, vertex G algebras do not seem to be able to cope with arbitrary curved spacetimes other than Lie groups, and it can be difficult to reconstruct a Hilbert space (this includes as a very special case the problem of deciding which representations of the Virasoro algebra are unitary).

Finally we will briefly describe a special case of a theorem about vertex G-algebras generalizing the usual identity for vertex algebras. This theorem says (roughly) that under fairly general conditions, the integral of a vertex operator over an n-cycle is a vertex differential operator of order n. We will illustrate this for ordinary vertex algebras, when it says that the integral of $a(x)$ around the origin is a vertex differential operator of order 1. This means

$$\int a(x)dxb(y) - b(y)\int a(x)dx$$

has order 0 and is therefore of the form $t(y)$ for some t. Applying both sides to 1 and taking $y = 0$ shows that $t = \int a(x)dxb$. Therefore

$$\int a(x)dxb(y)c - b(y)\int a(x)dxc = (\int a(x)dxb)(y)c.$$

Finally integrating both sides around $y = 0$ shows that

$$\int a(x)dx\int b(y)dyc - \int b(y)dy\int a(x)dxc = \int(\int a(x)dxb)(y)dyc.$$

which is a special case of the vertex algebra identity. (The other cases can be deduced in a similar way.)