37  Simple real Lie algebras

37.1  Real forms

(The stuff about E8 is duplicate and needs to be removed)

In general, suppose \( L \) is a Lie algebra with complexification \( L \otimes \mathbb{C} \). How can we find another Lie algebra \( M \) with the same complexification? \( L \otimes \mathbb{C} \) has an anti-linear involution \( \omega_L : l \otimes z \mapsto l \otimes \bar{z} \). Similarly, it has an anti-linear involution \( \omega_M \). Notice that \( \omega_L \omega_M \) is a linear involution of \( L \otimes \mathbb{C} \). Conversely, if we know this involution, we can reconstruct \( M \) from it. Given an involution \( \omega \) of \( L \otimes \mathbb{C} \), we can get \( M \) as the fixed points of the map \( a \mapsto \omega_L \omega(a) = \omega(a) \).

Another way is to put \( L = L^+ \oplus L^- \), which are the +1 and −1 eigenspaces, then \( M = L^+ \oplus iL^- \).

Thus, to find other real forms, we have to study the involutions of the complexification of \( L \). The exact relation is subtle, but this is a good way to go.

Example 393  Let \( L = sl_2(\mathbb{R}) \). It has an involution \( \omega(m) = -m^T \). \( su_2(\mathbb{R}) \) is the set of fixed points of the involution \( \omega \) times complex conjugation on \( sl_2(\mathbb{C}) \), by definition.

So to construct real forms of \( E_8 \), we want some involutions of the Lie algebra \( E_8 \) which we constructed. What involutions do we know about? There are two obvious ways to construct involutions:

1. Lift \(-1\) on \( L \) to \( \hat{e}^\alpha \mapsto (-1)^{\alpha^2/2}(\hat{e}^\alpha)^{-1} \), which induces an involution on the Lie algebra.

2. Take \( \beta \in L/2L \), and look at the involution \( \hat{e}^\alpha \mapsto (-1)^{(\alpha,\beta)}\hat{e}^\alpha \).

(2) gives nothing new: we get the Lie algebra we started with. (1) only gives one real form. To get all real forms, we multiply these two kinds of involutions together.

Recall that \( L/2L \) has 3 orbits under the action of the Weyl group, of size 1, 120, and 135. These will correspond to the three real forms of \( E_8 \). How do we distinguish different real forms? The answer was found by Cartan: look at the signature of an invariant quadratic form on the Lie algebra.

A bilinear form \( (, \) \) on a Lie algebra is called invariant if \( ([a, b], c) + (b[a, c]) = 0 \) for all \( a, b, c \). This is called invariant because it corresponds to the form being invariant under the corresponding group action. Now we can construct an invariant bilinear form on \( E_8 \) as follows:

1. \( (\alpha, \beta) \) in the Lie algebra = \( (\alpha, \beta) \) in the lattice

2. \( (\hat{e}^\alpha, (\hat{e}^\alpha)^{-1}) = 1 \)

3. \( (a, b) = 0 \) if \( a \) and \( b \) are in root spaces \( \alpha \) and \( \beta \) with \( \alpha + \beta \neq 0 \).

This gives an invariant inner product on \( E_8 \), which we prove by case-by-case check

Exercise 394  do these checks
We constructed a Lie algebra of type $E_8$, which was $L \oplus \bigoplus \hat{e}^\alpha$, where $L$ is the root lattice and $\alpha^2 = 2$. This gives a double cover of the root lattice:

$$1 \to \pm 1 \to \hat{e}^L \to e^L \to 1.$$ 

We had a lift for $\omega(\alpha) = -\alpha$, given by $\omega(\hat{e}^\alpha) = (-1)^{\alpha^2/2}(\hat{e}^\alpha)^{-1}$. So $\omega$ becomes an automorphism of order 2 on the Lie algebra. $e^\alpha \mapsto (-1)^{\alpha,\beta}e^\alpha$ is also an automorphism of the Lie algebra.

Suppose $\sigma$ is an automorphism of order 2 of the real Lie algebra $L = L^+ + L^-$. Thus, we have a map from conjugacy classes of automorphisms with $\sigma^2 = 1$ to real forms of $L$. This is not in general in isomorphism.

$E_8$ has an invariant symmetric bilinear form $(e^\alpha, (e^\alpha)^{-1}) = 1$, $(\alpha, \beta) = (\beta, \alpha)$. The form is unique up to multiplication by a constant since $E_8$ is an irreducible representation of $E_8$. So the absolute value of the signature is an invariant of the Lie algebra.

For the split form of $E_8$, what is the signature of the invariant bilinear form (the split form is the one we just constructed)? On the Cartan subalgebra $L$, $(\ , \ )$ is positive definite, so we get $+8$ contribution to the signature. On $\{e^\alpha, (e^\alpha)^{-1}\}$, the form is $(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$, so it has signature $0 - 120$. Thus, the signature is 8. So if we find any real form with a different signature, we will have found a new Lie algebra.

We first try involutions $e^\alpha \mapsto (-1)^{\alpha,\beta}e^\alpha$. But this does not change the signature. $L$ is still positive definite, and we still have $(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$ or $(-1)^{\alpha^2/2}(\hat{e}^\alpha)^{-1}$ on the other parts. These Lie algebras actually turn out to be isomorphic to what we started with (though we have not shown that they are isomorphic).

Now try $\omega: e^\alpha \mapsto (-1)^{\alpha^2/2}(\hat{e}^\alpha)^{-1}$, $\alpha \mapsto -\alpha$. What is the signature of the form? We write down the + and - eigenspaces of $\omega$. The + eigenspace will be spanned by $e^\alpha = e^{-\alpha}$, and these vectors have norm 2 and are orthogonal. The - eigenspace will be spanned by $e^\alpha + e^{-\alpha}$ and $L$, which have norm 2 and are orthogonal, and $L$ is positive definite. What is the Lie algebra corresponding to the involution $\omega$? It will be spanned by $e^\alpha - e^{-\alpha}$ where $\alpha^2 = 2$ (norm -2), and $i(e^\alpha + e^{-\alpha})$ (norm -2), and $iL$ (which is now negative definite). So the bilinear form is negative definite, with signature $-248(\neq \pm 8)$.

With some more work, we can actually show that this is the Lie algebra of the compact form of $E_8$. This is because the automorphism group of $E_8$ preserves the invariant bilinear form, so it is contained in $O_{0,248}(\mathbb{R})$, which is compact.

Now we look at involutions of the form $e^\alpha \mapsto (-1)^{\alpha,\beta}\omega(e^\alpha)$. Notice that $\omega$ commutes with $e^\alpha \mapsto (-1)^{\alpha,\beta}e^\alpha$. The $\beta$'s in $(\alpha, \beta)$ correspond to $L/2L$ modulo the action of the Weyl group $W(E_8)$. Remember this has three orbits, with 1 norm 0 vector, 120 norm 2 vectors, and 135 norm 4 vectors. The norm 0 vector gives us the compact form. Let's look at the other cases and see what we get.

Suppose $V$ has a negative definite symmetric inner product $(\ , \ )$, and suppose $\sigma$ is an involution of $V = V_+ \oplus V_-$(eigenspaces of $\sigma$). What is the signature of the invariant inner product on $V_+ \oplus iV_- $? On $V_+$, it is negative definite, and on $iV_-$ it is positive definite. Thus, the signature is $\dim V_- - \dim V_+ = -\operatorname{tr}(\sigma)$. So we want to work out the traces of these involutions.

Given some $\beta \in L/2L$, what is $\operatorname{tr}(e^\alpha \mapsto (-1)^{\alpha,\beta}e^\alpha)$? If $\beta = 0$, the traces is obviously 248 because we just have the identity map. If $\beta^2 = 2$, we need to
figure how many roots have a given inner product with $\beta$. Recall that this was determined before:

<table>
<thead>
<tr>
<th>$(\alpha, \beta)$</th>
<th># of roots $\alpha$ with given inner product</th>
<th>eigenvalue</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>56</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>126</td>
<td>1</td>
</tr>
<tr>
<td>-1</td>
<td>56</td>
<td>-1</td>
</tr>
<tr>
<td>-2</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Thus, the trace is $1 - 56 + 126 - 56 + 1 + 8 = 24$ (the 8 is from the Cartan subalgebra). So the signature of the corresponding form on the Lie algebra is $-24$. We’ve found a third Lie algebra.

If we also look at the case when $\beta^2 = 4$, what happens? How many $\alpha$ with $\alpha^2 = 2$ and with given $(\alpha, \beta)$ are there? In this case, we have:

<table>
<thead>
<tr>
<th>$(\alpha, \beta)$</th>
<th># of roots $\alpha$ with given inner product</th>
<th>eigenvalue</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>14</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>64</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>84</td>
<td>1</td>
</tr>
<tr>
<td>-1</td>
<td>64</td>
<td>-1</td>
</tr>
<tr>
<td>-2</td>
<td>14</td>
<td>1</td>
</tr>
</tbody>
</table>

The trace will be $14 - 64 + 84 - 64 + 14 + 8 = -8$. This is just the split form again.

Summary: We’ve found 3 forms of $E_8$, corresponding to 3 classes in $L/2L$, with signatures $8$, $-24$, $-248$, corresponding to involutions $e^\alpha \rightarrow (-1)^{(\alpha,\beta)} e^{-\alpha}$ of the compact form. If $L$ is the compact form of a simple Lie algebra, then Cartan showed that the other forms correspond exactly to the conjugacy classes of involutions in the automorphism group of $L$ (this doesn’t work if we don’t start with the compact form — so always start with the compact form).

In fact, these three are the only forms of $E_8$, but we won’t prove that.

**Example 395** Let’s go back to various forms of $E_8$ and figure out (guess) the fundamental groups. We need to know the maximal compact subgroups.

1. One of them is easy: the compact form is its own maximal compact subgroup. What is the fundamental group? Remember or quote the fact that for compact simple groups, $\pi_1 \cong \frac{\text{weight lattice}}{\text{root lattice}}$, which is 1. So this form is simply connected.

2. $\beta^2 = 2$ case (signature $-24$). Recall that there were 1, 56, 126, 56, and 1 roots $\alpha$ with $(\alpha, \beta) = 2, 1, 0, -1,$ and -2 respectively, and there are another 8 dimensions for the Cartan subalgebra. On the 1, 126, 1, 8 parts, the form is negative definite. The sum of these root spaces gives a Lie algebra of type $E_7A_1$ with a negative definite bilinear form (the 126 gives you the roots of an $E_7$, and the 1s are the roots of an $A_1$). So it a reasonable guess that the maximal compact subgroup has something to do with $E_7A_1$. $E_7$ and $A_1$ are not simply connected: the compact form of $E_7$ has $\pi_1 = \mathbb{Z}/2$ and the compact form of $A_1$ also has $\pi_1 = \mathbb{Z}/2$. So the universal cover of $E_7A_1$ has center $(\mathbb{Z}/2)^2$. Which part of this acts trivially on $E_8$? We look at the $E_8$ Lie algebra as a representation of $E_7 \times A_1$. You can read off how
it splits form the picture above: \( E_8 \cong E_7 \oplus A_1 \oplus 56 \otimes 2 \), where 56 and 2 are irreducible, and the centers of \( E_7 \) and \( A_1 \) both act as \(-1\) on them. So the maximal compact subgroup of this form of \( E_8 \) is the simply connected compact form of \( E_7 \times A_1/(-1, -1) \). This means that \( \pi_1(E_8) \) is the same as \( \pi_1 \) of the compact subgroup, which is \((\mathbb{Z}/2)^2/(-1, -1) \cong \mathbb{Z}/2\). So this simple group has a nontrivial double cover (which is non-algebraic).

3. For the other (split) form of \( E_8 \) with signature 8, the maximal compact subgroup is \( \text{Spin}_{16}(\mathbb{R})/(\mathbb{Z}/2) \), and \( \pi_1(E_8) \) is \( \mathbb{Z}/2 \).

You can compute any other homotopy invariants with this method.

Let’s look at the 56 dimensional representation of \( E_7 \) in more detail. We had the picture

\[
\begin{array}{c|c}
(\alpha, \beta) & \# \text{ of } \alpha's \\
\hline
2 & 1 \\
1 & 56 \\
0 & 126 \\
-1 & 56 \\
-2 & 1 \\
\end{array}
\]

The Lie algebra \( E_7 \) fixes these 5 spaces of \( E_8 \) of dimensions 1, 56, 126 + 8, 56, 1.

From this we can get some representations of \( E_7 \). The 126 + 8 splits as \( 1 + (126 + 7) \). But we also get a 56 dimensional representation of \( E_7 \). Let’s show that this is actually an irreducible representation. Recall that in calculating \( W(E_8) \), we showed that \( W(E_7) \) acts transitively on this set of 56 roots of \( E_8 \), which can be considered as weights of \( E_7 \).

An irreducible representation is called minuscule if the Weyl group acts transitively on the weights. This kind of representation is particularly easy to work with. It is really easy to work out the character for example: just translate the 1 at the highest weight around, so every weight has multiplicity 1.

So the 56 dimensional representation of \( E_7 \) must actually be the irreducible representation with whatever highest weight corresponds to one of the vectors.

37.2 Every possible simple Lie group

We will construct them as follows: Take an involution \( \sigma \) of the compact form \( L = L^+ + L^- \) of the Lie algebra, and form \( L^+ + iL^- \). The way we constructed these was to first construct \( A_n, D_n, E_6, \) and \( E_7 \) as for \( E_8 \). Then construct the involution \( \omega: e^\alpha \mapsto -e^{-\alpha} \). We get \( B_n, C_n, F_4, \) and \( G_2 \) as fixed points of the involution \( \omega \).

Kac classified all automorphisms of finite order of any compact simple Lie group. The method we’ll use to classify involutions is extracted from his method.

We can construct lots of involutions as follows:

1. Take any Dynkin diagram, say \( E_8 \), and select some of its verticals, corresponding to simple roots. Get an involution by taking \( e^\alpha \mapsto \pm e^\alpha \) where the sign depends on whether \( \alpha \) is one of the simple roots we’ve selected. However, this is not a great method. For one thing, we get a lot of repeats (recall that there are only 3, and we’ve found \( 2^8 \) this way).

2. Take any diagram automorphism of order 2, such as

This gives you more involutions.

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Next time, we’ll see how to cut down this set of involutions. Split form of Lie algebra (we did this for $A_n$, $D_n$, $E_6$, $E_7$, $E_8$): $A = \bigoplus \mathbb{C} \epsilon^\alpha \oplus L$. Compact form $A^+ + iA^-$, where $A^\pm$ eigenspaces of $\omega: \epsilon^\alpha \mapsto (-1)^{\alpha^2/2} \epsilon^{-\alpha}$.

We talked about other involutions of the compact form. You get all the other forms this way.

The idea now is to find ALL real simple Lie algebras by listing all involutions of the compact form. We will construct all of them, but we won’t prove that we have all of them.

We’ll use Kac’s method for classifying all automorphisms of order $N$ of a compact Lie algebra (and we’ll only use the case $N = 2$). First let’s look at inner automorphisms. Write down the AFFINE Dynkin diagram

Choose $n_i$ with $\sum n_i m_i = N$ where the $m_i$ are the numbers on the diagram.

We have an automorphism $e^\epsilon_j \mapsto e^{2\pi m_j/N} e^\epsilon_j$ induces an automorphism of order dividing $N$. This is obvious. The point of Kac’s theorem is that all inner automorphisms of order dividing $N$ are obtained this way and are conjugate if and only if they are conjugate by an automorphism of the Dynkin diagram. We won’t actually prove Kac’s theorem because we just want to get a bunch of examples.

Example 396 Real forms of $E_8$. We’ve already found three, and it took us a long time. We can now do it fast. We need to solve $\sum n_i m_i = 2$ where $n_i \geq 0$; there are only a few possibilities:

The points NOT crossed off form the Dynkin diagram of the maximal compact subgroup. Thus, by just looking at the diagram, we can see what all the real forms are!

Example 397 Let’s do $E_7$. Write down the affine diagram:

We get the possibilities

(* The number of ways is counted up to automorphisms of the diagram.

(**) In the split real form, the maximal compact subgroup has dimension equal to half the number of roots. The roots of $A_7$ look like $\epsilon_i - \epsilon_j$ for $i, j \leq 8$ and $i \neq j$, so the dimension is $8 \cdot 7 + 7 = 56 = \frac{112}{2}$.

(*** The maximal compact subgroup is $E_6 \oplus \mathbb{R}$ because the fixed subalgebra contains the whole Cartan subalgebra, and the $E_6$ only accounts for 6 of the 7 dimensions. You can use this to construct some interesting representations of $E_6$ (the minuscule ones). How does the algebra $E_7$ decompose as a representation of the algebra $E_6 \oplus \mathbb{R}$?

We can decompose it according to the eigenvalues of $\mathbb{R}$. The $E_6 \oplus \mathbb{R}$ is the zero eigenvalue of $\mathbb{R}$ [why?], and the rest is 54 dimensional. The easy way to see the decomposition is to look at the roots. Remember when we computed the Weyl group we looked for vectors like

The 27 possibilities (for each) form the weights of a 27 dimensional representation of $E_6$. The orthogonal complement of the two nodes is an $E_6$ root system whose Weyl group acts transitively on these 27 vectors (we showed that these form a single orbit, remember?). Vectors of the $E_7$ root system are the vectors of the $E_6$ root system plus these 27 vectors plus the other 27 vectors. This splits up the $E_7$ explicitly. The two 27s form single orbits, so they are irreducible. Thus, $E_7 \cong E_6 \oplus \mathbb{R} \oplus 27 \oplus 27$, and the 27s are minuscule.
Let $K$ be a maximal compact subgroup, with Lie algebra $\mathbb{R} + E_6$. The factor of $\mathbb{R}$ means that $K$ has an $S^1$ in its center. Now look at the space $G/K$, where $G$ is the Lie group of type $E_7$, and $K$ is the maximal compact subgroup. It is a Hermitian symmetric space. Symmetric space means that it is a (simply connected) Riemannian manifold $M$ such that for each point $p \in M$, there is an automorphism fixing $p$ and acting as $-1$ on the tangent space. This looks weird, but it turns out that all kinds of nice objects you know about are symmetric spaces. Typical examples you may have seen: spheres $S^n$, hyperbolic space $\mathbb{H}^n$, and Euclidean space $\mathbb{R}^n$. Roughly speaking, symmetric spaces have nice properties of these spaces. Cartan classified all symmetric spaces: they are non-compact simple Lie groups modulo the maximal compact subgroup (more or less ... depending on simply connectedness hypotheses 'n such). Historically, Cartan classified simple Lie groups, and then later classified symmetric spaces, and was surprised to find the same result. Hermitian symmetric spaces are just symmetric spaces with a complex structure. A standard example of this is the upper half plane $\{x + iy | y > 0\}$. It is acted on by $SL_2(\mathbb{R})$, which acts by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$ 

Let's go back to this $G/K$ and try to explain why we get a Hermitian symmetric space from it. We'll be rather sketchy here. First of all, to make it a symmetric space, we have to find a nice invariant Riemannian metric on it. It is sufficient to find a positive definite bilinear form on the tangent space at $p$ which is invariant under $K$ ... then we can translate it around. We can do this as $K$ is compact (so you have the averaging trick). Why is it Hermitian? We'll show that there is an almost complex structure. We have $S^1$ acting on the tangent space of each point because we have an $S^1$ in the center of the stabilizer of any given point. Identify this $S^1$ with complex numbers of absolute value 1. This gives an invariant almost complex structure on $G/K$. That is, each tangent space is a complex vector space. Almost complex structures don’t always come from complex structures, but this one does (it is integrable). Notice that it is a little unexpected that $G/K$ has a complex structure ($G$ and $K$ are odd dimensional in the case of $G = E_7$, $K = E_6 \oplus \mathbb{R}$, so they have no hope of having a complex structure).

**Example 398** Let’s look at $E_6$, with affine Dynkin diagram

We get the possibilities

In the last one, the maximal compact subalgebra is $D_5 \oplus \mathbb{R}$. Just as before, we get a Hermitian symmetric space. Let’s compute its dimension (over $\mathbb{C}$). The dimension will be the dimension of $E_6$ minus the dimension of $D_5 \oplus \mathbb{R}$, all divided by 2 (because we want complex dimension). Notice that it is a little unexpected that $G/K$ has a complex structure ($G$ and $K$ are odd dimensional in the case of $G = E_7$, $K = E_6 \oplus \mathbb{R}$, so they have no hope of having a complex structure).

There are also some OUTER automorphisms of $E_6$ coming from the diagram automorphism

The fixed point subalgebra has Dynkin diagram obtained by folding the $E_6$ on itself. This is the $F_4$ Dynkin diagram. The fixed points of $E_6$ under the diagram automorphism is an $F_4$ Lie algebra. So we get a real form of $E_6$ with maximal compact subgroup $F_4$. This is probably the easiest way to construct $F_4$, by the way. Moreover, we can decompose $E_6$ as a representation of $F_4$. $dim E_6 = 78$ and $dim F_4 = 52$, so $E_6 = F_4 \oplus 26$, where 26 turns out
to be irreducible (the smallest non-trivial representation of $F_4$ ... the only one anybody actually works with). The roots of $F_4$ look like $(\ldots, \pm 1, \pm 1, \ldots)$ (24 of these) and $(\pm \frac{1}{2}, \ldots, \pm \frac{1}{2})$ (16 of these), and $(\ldots, \pm 1, \ldots)$ (8 of them) ... the last two types are in the same orbit of the Weyl group.

The 26 dimensional representation has the following character: it has all norm 1 roots with multiplicity 1 and 0 with multiplicity 2 (note that this is not minuscule).

There is one other real form of $E_6$. To get at it, we have to talk about Kac’s description of non-inner automorphisms of order $N$. The non-inner automorphisms all turn out to be related to diagram automorphisms. Choose a diagram automorphism of order $r$, which divides $N$. Let’s take the standard thing on $E_6$.

Fold the diagram (take the fixed points), and form a TWISTED affine Dynkin diagram (note that the arrow goes the wrong way from the affine $F_4$)

There are also numbers on the twisted diagram, but never mind them. Find $n_i$ so that $r \sum n_i m_i = N$. This is Kac’s general rule. We’ll only use the case $N = 2$.

If $r > 1$, the only possibility is $r = 2$ and one $n_1$ is 1 and the corresponding $m_i$ is 1. So we just have to find points of weight 1 in the twisted affine Dynkin diagram. There are just two ways of doing this in the case of $E_6$.

one of these gives us $F_4$, and the other has maximal compact subalgebra $C_4$, which is the split form since dim $C_4 = \#$roots of $F_4/2 = 24$.

Example 399 $F_4$. The affine Dynkin is

We can cross out one node of weight 1, giving the compact form (split form), or a node of weight 2 (in two ways), giving maximal compacts $A_1C_3$ or $B_4$. This gives us three real forms.

Example 400 $G_2$. We can actually draw this root system ... UCB won’t supply me with a four dimensional board. The construction is to take the $D_4$ algebra and look at the fixed points of:

We want to find the fixed point subalgebra.

Fixed points on Cartan subalgebra: $\rho$ fixes a two dimensional space, and has 1 dimensional eigenspaces corresponding to $\omega$ and $\bar{\omega}$, where $\omega^3 = 1$. The 2 dimensional space will be the Cartan subalgebra of $G_2$.

Positive roots of $D_4$ as linear combinations of simple roots (not fundamental weights):

There are six orbits under $\rho$, grouped above. It obviously acts on the negative roots in exactly the same way. What we have is a root system with six roots of norm 2 and six roots of norm 2/3. Thus, the root system is $G_2$:

One of the only root systems to appear on a country’s national flag. Now let’s work out the real forms. Look at the affine:

we can delete the node of weight 1, giving the compact form:

. We can delete the node of weight 2, giving $A_1A_1$ as the compact subalgebra:

... this must be the split form because there is nothing else the split form can be.

Let’s say some more about the split form. What does the Lie algebra of $G_2$ look like as a representation of the maximal compact subalgebra $A_1 \times A_1$? In this case, it is small enough that we can just draw a picture:

We have two orthogonal $A_1$s, and we have leftover the stuff on the right. This thing on the right is a tensor product of the 4 dimensional irreducible
representation of the horizontal and the 2 dimensional of the vertical. Thus, \( G_2 = 3 \times 1 + 1 \otimes 3 + 4 \otimes 2 \) as irreducible representations of \( A_1^{(\text{horizontal})} \otimes A_1^{(\text{vertical})} \).

Let’s use this to determine exactly what the maximal compact subgroup is. It is a quotient of the simply connected compact group \( SU(2) \times SU(2) \), with Lie algebra \( A_1 \times A_1 \). Just as for \( E_8 \), we need to identify which elements of the center act trivially on \( G_2 \). The center is \( \mathbb{Z}/2 \times \mathbb{Z}/2 \). Since we’ve decomposed \( G_2 \), we can compute this easily. A non-trivial element of the center of \( SU(2) \) acts as 1 (on odd dimensional representations) or \(-1\) (on even dimensional representations). So the element \( z \times z \in SU(2) \times SU(2) \) acts trivially on \( 3 \otimes 1 + 1 \otimes 3 + 4 \times 2 \). Thus the maximal compact subgroup of the non-compact simple \( G_2 \) is \( SU(2) \times SU(2)/(z \times z) \cong SO_4(\mathbb{R}) \), where \( z \) is the non-trivial element of \( \mathbb{Z}/2 \).

So we have constructed \( 3 + 4 + 5 + 3 + 2 \) (from \( E_8 \), \( E_7 \), \( E_6 \), \( F_4 \), \( G_2 \)) real forms of exceptional simple Lie groups.

There are another 5 exceptional real Lie groups: Take COMPLEX groups \( E_8(\mathbb{C}) \), \( E_7(\mathbb{C}) \), \( E_6(\mathbb{C}) \), \( F_4(\mathbb{C}) \), and \( G_2(\mathbb{C}) \), and consider them as REAL. These give simple real Lie groups of dimensions \( 248 \times 2 \), \( 133 \times 2 \), \( 78 \times 2 \), \( 52 \times 2 \), and \( 14 \times 2 \).