2 Lie algebras

Lie groups such as $GL_n(\mathbb{R})$ are quite complicated nonlinear objects. A Lie algebra is a way of linearizing a Lie group, which is often easier to handle. Roughly speaking, the addition and Lie bracket of the Lie algebra are given by the lowest order terms in the product and commutator of the Lie group. By a minor miracle (the Campbell-Baker-Hausdorff formula) we do not need any higher order terms: the Lie algebra is enough to determine the group product locally. We first recall some background about vector fields and differential operators on a manifold. We will then define the Lie algebra of a Lie group to be the left invariant vector fields on the group.

For any algebra over a ring we define the Lie bracket $[a, b]$ to be $ab - ba$. It satisfies the identities

- $[a, b]$ is bilinear
- $[a, b] = -[b, a]$
- $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$ (Jacobi identity)

Definition 19 *A Lie algebra over a ring is a module with a bracket satisfying the conditions above, in other words it is bilinear, skey symmetric, and satisfies the Jacobi idenity.*

These conditions make sense in any additive tensor category, so for example we can define Lie algebras of sheaves, or graded Lie algebras. An intersting variation is Lie superalgebras, where we use the tensor category of supermodules over a ring or field. Some authors add the non-linear condition that $[a, a] = 0$.

Example 20 The basic example of a Lie algebra is given by taking V to be an associative algebra and defining $[a, b]$ to be $[ab - ba]$.

The Lie algebra of a Lie group can be defined as its tangent space at the identity, with the Lie bracket given by the lowest order part of the commutator. The lowest-order terms of the group law are just given by addition on the Lie algebra, as can be seen in $GL_n($
$R)$: the product of $1 + \epsilon A$ and $1 + \epsilon B$ is $1 + \epsilon(A + B)$ to first order. However defining the Lie bracket in terms of the commutator is a little messy, and it is technically more convenient to define the Lie algebra as the left invariant vector fields on the manifold.

There are several different ways to think of vector fields:

- Informally, a vector field is a little tangent vector at each point.
- A vector field is informally an infinitesimal diffeomorphism, where we get an infinitesimal diffeomorphism from a vector field by pushing each point slightly in the direction of the vector field.
- More formally, a vector field is a section of the tangent bundle or sheaf.
- A vector field is a normalized differential operator of order at most 1
- A vector field is a derivation of the ring of smooth functions.

The last two seem less intuitive but turn out to be the easiest definitions to work with.

Suppose we have a manifold M, with its ring R of smooth functions. A differential operator on M should be something that in local coordinates looks like a partial differential operator times a smooth function. It is easier to forget about local coordinates, and just use the following key property of differential operators: the commutator of an nth order operator with a smooth function is a differential operator of smaller order. This is really just a form of Leibniz's rule for differentiating a product. We will use this to DEFINE differential operators as follows.

Definition 21 *A differential operator of order less than 0 is 0. A differential operator of order at most $n \geq 0$ is an operator on R whose commutator with elements of R is a differential operator of order at most $n-1$.*

Differential operators on R form a filtered ring $D^0 \subset D^1 \subset D^2 \cdots$, where D^n is the differential operators of order at most n. The differential operators of order at most 0 can be identified with the ring R (look at their action on 1), and any differential operator can be normalized by adding a function so that it kills 1. So a differential operator can be written canonically as a function (order 0 operator) plus a normalized differential operator.

The product of differential operators of orders at most m, n has order at most $m+n$. Differential operators do not quite commute with each other; however the commutator or Lie bracket $[D_1, D_2]$ of operators of orders at most m, n has order at most $m+n-1$; in other words differential operators commute "up to lower order terms". This means that the associated graded ring $D^0 \oplus D^1/D^0 \oplus D^2/D^1 \oplus \cdots$ is a commutative graded ring (whose elements are sometimes called symbols).

We will call a differential operator normalized if it kills the function 1. Differential operators of order at most 1 can be written canonically as the sum of an order 0 differential operator and a normalized differential operator. (However there is no canonical way to write an operator of order $n > 1$ as an operator of order less than n and something "homogeneous" of order n.) A vector field on a manifold is the same as a normalized differential operator of order at most 1. Vector fields are closed under the Lie bracket, and in particular form a Lie algebra. It is useful to think of a vector field as a sort of infinitesimal diffeomorphism of the manifold: each point is moved an infinitesimal distance in the direction of the vector at that point. Since the Lie algebra of a group can be thought of as the "infinitesimal" elements of the group, this means that the vector fields on a manifold are more or less the Lie algebra of the group of diffeomorphisms.

The Lie algebra of vector fields is an infinite dimensional Lie algebra, which is too big for this course, so we cut it down.

Definition 22 *The Lie algebra of a Lie group is the Lie algebra of left-invariant vector fields on the group.*

We explain what this means. The group is a manifold, so we have the Lie algebra of all vector fields on it forming an infinite dimensional lie algebra. The group acts on itself by left translation, and so acts on everything constructed from the manifold, such as vector fields. We just take the vector fields fixed by this action of left translation. It is automatically a subalgebra of the lie algebra of all vector fields, as the group action preserves the Lie bracket.

We can also identify the Lie algebra of the group with the tangent space at the origin. The reason is that if we pick a tangent vector at the origin, there is a unique vector field on G given by left translating this vector everywhere. We could have defined the Lie algebra to be the tangent space at the origin, but then it would not have been so clear how (or why) we can define the Lie bracket.

Now we will calculate the left invariant vector fields on the group $GL_n(R)$ and find the Lie bracket. We will then be able to find the Lie algebras of other groups by mapping them to $GL_n(R)$. There are obvious coordinates x_{ij} for $GL_n(R) \subset R^{n\times n}$, and corresponding vector fields $\partial/\partial x_{ij}$. Of course there are not left invariant under $GL_n(R)$: they are left invariant vector fields on the abelian group R^{n^2}, and have zero Lie bracket.

We let $x = (x_{ij})$ be the matrix whose entries are the coordinate functions. We can think of x as the identity function from R^{n^2} to itself, so might guess that G acts trivially on it, but this is wrong: the point is that the two copies of R^{n^2} are not really the same as the domain is acted on by G by translations, while the range is acted on trivially by G. This is very confusing. The action of an element g on x is given by right multiplying it by g^{-1}. Next if $a = (a_{ij})$ is a matrix we can consider the matrix of differential operators with entries a_{ij} We consider the matrix D of differential operators $x_{ik}\partial/\partial x_{jk}$ (using the Einstein summation convention). This acts on x as left multiplication by the matrix e_{ij} (one entry in position i, j, other entries 0). Since left multiplication by a matrix commutes with right multiplication we see that these differential operators all commute with left translation on the entries of x, and therefore are left invariant differential operators.

So we get a natural correspondence between n by n matrices and these left invariant differential operators. Finally we can work out the Lie bracket of two such differential operators

$$[x_{ik}\frac{\partial}{\partial x_{jk}}, x_{i\prime k\prime}\frac{\partial}{\partial x_{j\prime k\prime}}] = [j = i\prime]x_{ij}\frac{\partial}{\partial x_{j\prime k}} - [i = j\prime]x_{i\prime j}\frac{\partial}{\partial x_{j\prime k}}$$

, and we see that it just corresponds to the Lie bracket

$$[e_{ij}, e_{i\prime j\prime}] = [j = i\prime]e_{ij\prime} - [i = j\prime]e_{i\prime j}$$

of n by n matrices e_{ij} that have a one in position (i, j) and are zero elsewhwere.

To summarize, the Lie algebra of $GL_n(R)$ is just $M_n(R)$, with the Lie bracket given by $[A, B] = AB - BA$.

To find the Lie algebras of subgroups of general linear groups, which covers most practical cases, we just have to find the tangent space at the identity. The easy way to do this to to find the matrices A such that $1 + \epsilon A$ satisfies the equations defining the Lie group, where $\epsilon^2 = 0$.

Example 23 The orthogonal group consists of matrices g such that $gg^T = I$. So its Lie algebra conssits of matrices a such that $(1 + \epsilon a)(1 + \epsilon A^T) = 1$ to first order in ϵ, in other words $a + a^T = 0$, so that a s skew-symmetric.

Example 24 The special linear group consists of matrices g such that $\det g = 1$. So its Lie algebra consits of matrices a such that $\det(1 + \epsilon a) = 1$ to first order in ϵ. Since $\det(1 + a) = 1 + \mathrm{Trace}(A)\epsilon$ to first order in epsilon, the Lie algebra consists of the matrices of trace 0.

Exercise 25 Show that the Lie algebra of the unitary group consists of skew Hermitian matrices, which are Hermitian matrices multiplied by i.

Identifying skew hermitean matrices with hermitean matrices by multiplication by i shows that defining $[a, b] - i(ab - ba)$ makes Hermitean matrices into a Lie algebra. This Lie bracket is not the only interesting algebraic structure one can put on Hermitian matrices.

Exercise 26 Show that if a and b are Hermitian then so is their Jordan product $a \circ b = (ab + ba)/2$. Show that this is a commutative but non-associative product, satisfying the Jordan identity $(x \circ y) \circ (x \circ x) = x \circ (y \circ (x \circ x))$. Algebras with these properties are called Jordan algebras.

In the early days of quantum mechanics it was hoped that a suitable Jordan algebra would explain the universe, but this hope was abandoned when the simple finite dimensional Jordan algebras were classified: they are mostly algebras of Hermitean matrices, and none of them explain the known elementary particles.

Exercise 27 Find the Lie algebra of the group $Sp_{2n}(\mathbb{R})$ of symplectic matrices.