We will need to use duals of representations. These can get rather confusing because there are in fact 8 different natural ways of constructing a new representation from an old one, many of which have been called the dual. The three main ways are as follows:

- The complex conjugate of a representation $V$: keep the same $G$-action, but change the action of $i$ to $-i$. If we represent the elements of $G$ by complex matrices, this corresponds to taking the complex conjugate of a matrix.

- Change the left action on $V$ to a right action. If we define $vg = gv$ with does not work (why?) but we get a right action by putting $vg = g^{-1}v$. (This is really using the antipode of the group ring of $G$ thought of as a Hopf algebra: any left module over a Hopf algebra can be turned into a right module in a similar way.) This corresponds to taking inverses of a matrix.

- The usual vector space dual of $V$ is a representation, but we have to be careful how we define the $G$-action. Putting $(fg)(v) = f(gv)$ fails. However we can make the dual into a right $G$-module by putting $(fg)(v) = f(gv)$. This operation corresponds to taking transposes of a matrix.

By combining these three operations in various ways we can construct other representations. For example if we want the dual to be a left module, we first construct the dual as a right module, then change it to a left module, so we use the transpose inverse of matrices. If we want the Hermitian dual as a right module we take the complex conjugate of the dual; if we like we can turn this into a left module by taking inverses as well. (Physicists like to leave it as a right module in their bra-ket notation, while mathematician like to make it into a left module.)

For finite (and compact) groups all irreducible complex representations are unitary (or rather can be made unitary in a way that is unique up to multiplication by non-zero scalars). If we work with unitary representations then taking conjugate inverse transpose leaves everything fixed there are only 4 things we can do, and if we stick to left modules this leaves just two things: $V$ and its dual, which can be given by taking complex conjugates. However if we work with non-unitary representations then there are still 4 left modules we can construct from $V$ and taking ordinary or Hermitian duals is no longer the same as taking complex conjugates.

**Example 254** The $G$ to be the circle group $\mathbb{R}/2\pi \mathbb{Z}$. Then its irreducible representations are 1-dimensional and are given by $x \mapsto e^{inx}$ for integers $n$. There functions form an orthonormal base for the $L^2$ functions on $G$ (using Lebesgue measure divided by $2\pi$), and in particular every $L^2$ function can be written as a linear combination of them: this is just its Fourier series expansion. In this case the dual of a representation is given by the complex conjugate, as we expect for compact groups.

**Exercise 255** Show that the representations of a finite abelian group give an orthonormal basis for the functions on the group in a similar way.
We would like to generalize this to all finite (and compact) groups: in other words find an orthonormal basis for functions on \( G \) related to the irreducible representations.

**Lemma 256** (Schur’s lemma). Suppose \( V \) and \( W \) are irreducible representations of a group \( G \) over some field \( k \). Then the algebra of linear transformations of \( V \) that commute with \( G \) is a division algebra over \( k \). The space of linear transformations commuting with \( G \) from \( V \) to \( W \) is 0 if \( V \) is not isomorphic to \( W \).

**Proof** This is almost trivial: suppose \( T \) is any endomorphism commuting with \( G \). Then the image and kernel of \( T \) are invariant subspaces, so must be 0 on the whole space. So either \( T \) is 0, or it has zero kernel so is an isomorphism and has an inverse. \( \square \)

For complex representations the only finite dimensional division algebra over \( \mathbb{C} \) is \( \mathbb{C} \), so the space of linear maps from \( V \) to itself is 1-dimensional. Over other fields more interesting things can happen:

**Example 257** If \( G \) is the group of order 4 acting on the real plane by rotations, then the algebra of endomorphisms commuting with \( G \) is the algebra of complex numbers. This also gives an example of a representation that is irreducible but not absolutely irreducible: it becomes reducible over an algebraic closure.

**Example 258** If \( G \) is the quaternion group of order 8 acting by left multiplication on the quaternions (thought of as a 4-dimensional real vector space) then the algebra commuting with it is the algebra of quaternions (acting by right multiplication in itself).

If \( G \) is a finite group we can construct its group ring \( \mathbb{C}[G] \): this is the complex algebra with basis \( G \) and multiplication given by the product of \( G \). Alternatively we can think of it as functions on \( G \) with the product given by convolution: this definition generalizes better to Lie groups. The regular representation of \( G \) is the action of \( G \) on its group algebra by left multiplication.

The functions \( \langle gv, w \rangle \) are called matrix coefficients of the representation: if we choose a basis for \( V \) and the dual basis for \( W \) they the endomorphisms of \( V \) are given by matrices, and the representation of \( G \) is given by matrix-valued functions of \( G \). We will now show that these matrix coefficients are mostly orthogonal to each other under the obvious inner product on \( \mathbb{C}[G] \) where the elements of \( G \) form an orthonormal base.

**Lemma 259** Suppose the representation \( V \) does not contain the trivial representation. Then \( \sum_g \langle gv, w \rangle = 0 \) for any \( v \in V, w \in V^* \).

**Proof** The vector \( \sum_g gv \) is fixed by \( G \) and is therefore 0, so has bracket 0 with any \( w \). \( \square \)

**Lemma 260** If the irreducible representations \( V \) and \( W \) are not dual, then matrix coefficients of \( V \) are orthogonal to matrix coefficients of \( W \) under the symmetric inner product on \( \mathbb{C}[G] \).
**Proof** By assumption $V \otimes W$ does not contain the trivial representation (using Schur’s lemma) so

$$\sum_{g \in G} \langle g(a), c \rangle \langle g(b), d \rangle = \sum_{g \in G} \langle g(a \times b), c \times d \rangle = 0$$

□

The character of the dual of a unitary representation is given by taking the complex conjugate, so we get:

**Lemma 261** If $V$ and $W$ are irreducible and not isomorphic, then the matrix coefficients of $V$ and $W$ are orthogonal under the hermitian inner product of $\mathbb{C}[G]$.  

**Exercise 262** If $V$ is an irreducible complex representation with some basis show that the sum of matrix coefficients $\sum_{g \in G} g_{ij} g_{kl}^{-1}$ is $|G|/\dim(V)$ if $i = l$, $j = k$ and 0 otherwise. (This is similar to the proof when we have two different representations, except that now there is a non-trivial map from $V$ to $V$ that makes some of the inner products of matrix coefficients non-zero.)

We give the group ring the Hermitian scalar product such that the elements of $G$ are orthogonal and have norm $1/|G|$. To summarize: if we take a representative of each irreducible representation of $G$ and take an orthonormal base of each representation, then the matrix coefficients we get form an orthogonal set in the group ring of $G$. The norms are given by $1/\dim$ (if we normalize the measure on $G$ so that $G$ has measure 1: this generalizes to compact groups).

**Definition 263** The character of a representation is the function from $G$ to $\mathbb{C}$ given by the trace.

The character is just the sum of the diagonal entries of a matrix, so by the orthogonality for matrix coefficients we see that the characters of irreducible representations form an orthonormal set of irreducible functions on $G$. Characters are rather spacial functions on $G$ because they are class functions: this means they only depend on the conjugacy class of an element of $G$ (which follows from the fact that matrices $g^{-1}hg$ and $h$ have the same trace).

**Exercise 264** If $V$ and $W$ are representations, show that $V \otimes W$ is a representation whose character is the product of the characters of $V$ and $W$.

**Exercise 265** If $G$ acts on a set $S$, form a representation of $G$ on the vector space with basis $S$. Show that the character of this representation is given by taking the number of fixed points of an element of $G$. Show that this representation always contains the trivial 1-dimensional representation as a subrepresentation. How many times does the trivial 1-dimensional representation occur?

**Exercise 266** Show that the character of the symmetric square of a representation with character $\chi$ is given by $(\chi(g)^2 + \chi(g^2))/2$ and find a formula for the character of the alternating square. (If $g$ has eigenvalues $\lambda_i$, then the eigenvalues on the symmetric square or alternating square are $\lambda_i \lambda_j$ for $i \leq j$ or $i < j$.)