The symplectic group is obviously not compact. For orthogonal groups it was easy to find a compact form: just take the orthogonal group of a positive definite quadratic form. For symplectic groups we do not have this option as there is essentially only one symplectic form. Instead we can use the following method of constructing compact forms: take the intersection of the corresponding complex group with the compact subgroup of unitary matrices. For example, if we do this with the complex orthogonal group of matrices with $AA^t = I$ and intersect it with the unitary matrices $A\overline{A}^t = I$ we get the usual real orthogonal group. In this case the intersection with the unitary group just happens to consist of real matrices, but this does not happen in general. For the symplectic group we get the compact group $Sp_{2n}(\mathbb{C}) \cap U_{2n}(\mathbb{C})$. We should check that this is a real form of the symplectic group, which means roughly that the Lie algebras of both groups have the same complexification $sp_{2n}(\mathbb{C})$, and in particular has the right dimension. We show that if V is any †-invariant complex subspace of $M_k(\mathbb{C})$ then the Hermitian matrices of V are a real form of V. This follows because any matrix $v \in V$ can be written as a sum of hermitian and skew hermitian matrices $v = (v + v^\dagger)/2 + (v - v^\dagger)/2$, in the same way that a complex number can be written as a sum of real and imaginary parts. (Of course this argument really has nothing to do with matrices: it works for any antilinear involution † of any complex vector space.) Now we need to check that the Lie algebra of the complex symplectic group is †-invariant. This Lie algebra consists of matrices a auch that $aJ + Ja^T = 0$; this is obviously closed under complex conjugation, and is closed under taking transposes if we choose J so tthat $J^2 = 1$.

For orthogonal groups we can find an index 2 subgroup because the determinant can be positive or negative, and one might guess that one can do something similar for symplectic groups. However for symplectic groups the determinant is always 1:

Lemma 221 *if B is a symplectic matrix preserving the non-degenerate alternating form of A, so that $BAB^t = A$, then* $\det(B) = 1$.

Proof A non-degenerate alternating form on a $2n$-dimensional vector space V gives a 2-form ω in $\Lambda^2(V)$ and $\omega \wedge \cdots \wedge \omega$ is a non-degenerate $2n$-form preserved by B, So B has determinant 1, as the determinant is the amount by which a matrix multiplies a nondegenerate $2n$-form. □

For example, Riemannian manifolds, where the structure group is reduced to the orthogonal group, can be non-orientable, but symplectic manifolds, where the structure group is reduced to the symplectic group, are always orientable.

Definition 222 *An alternating form can be represented by an alternating matrix A. Since this is equivalent to the standard form J with diagonal blocks $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ we can write the matrix as TJT^t for some T. Any two matrices T differ by a symplectic matrix that necessarily has determinant 1, so the determinant of T depends only on A and is called the Pfaffian of the alternating matrix A. (More generally, the Pfaffian is really a function of two alternating forms on a vector space, given by the determinant of a map taking one to the other.)*

The Pfaffian can also be given as follows: the highest degree part of

$$\exp(\sum A_{ij}\omega_i \wedge \omega_j) = \mathrm{Pf}(A)\omega_1 \wedge \cdots \wedge \omega_{2n}$$

.

Exercise 223 If A is alternating and B is any matrix show that

$$\mathrm{Pf}(BAB^t) = \det(B)\mathrm{Pf}(A)$$

$$\det(A) = \mathrm{Pf}(A)^2$$

We can write the determinant as the square of an explicit polynomial in the entries of A.

Lemma 224 *If A is a skew symmetric matrix over a field of characteristic 0, the Pfaffian of A is given by*

$$\omega^n = 2^n n! Pf(A) e_1 \wedge \cdots \wedge e_n$$

where $\omega = \sum_{i,j} a_{ij} e_i \wedge e_j$.

We can write this as

$$Pf(A) = \frac{1}{2^n n!} \sum_{w \in S_{2n}} \epsilon(w) a_{w(1)w(2)} a_{w(3)w(4)} \cdots$$

and since each term on the right occurs $2^n n!$ times, we get a definition of the Pfaffian over any commutative ring by just summing over the permutations with $w(1) < w(3) < w(5) \cdots$, $w(1) < w(2)$, $w(3) < w(4), \cdots$.

Exercise 225 Find the Pfaffian of

$$\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}$$

Example 226 The orthogonal group acts on its Lie algebra, which is the vector space of skew symmetric real matrices. We can ask about the invariants of $O_n(\mathbb{R})$ and its subgroup $SO_n(\mathbb{R})$ for this representation, in other words the polynomials in the entries of A that are invariant under changing A to BAB^{-1} for B an orthogonal or special orthogonal matrix (so $B^{-1} = B^t$). By the formula $\mathrm{Pf}(BAB^t) = \det(B)\mathrm{Pf}(A)$, we see that the Pfaffian is an invariant of the special orthogonal group, but changes sign under reflections. In fact the invariants of the special orthogonal group form a 2-dimensional module over the invariants of the orthogonal group, with a basis given by 1 and the Pfaffian.

16.1 Perfect matchings and domino tilings

The Pfaffian turns up in several problems of statistical mechanics, where it can sometimes be used to give exact solutions in 2 dimensions. As an example we will use it to count the number of perfect matchings of a bipartite planar graph. (Bipartite means that the vertices can be colored black and white such that the two endpoints of any edge have different colors, or equivalently that all cycles have even length.) The idea is to write this number as the Pfaffian of a matrix, and then evaluate the determinant of the matrix by diagonalizing it.

We can form the adjacency matrix of this graph with $a_{ij} = 0, 1$ counting the number of edges from vertex ii to vertex j, and can see that the Pfaffian of this adjacency matrix would b the number of bipartite matchings if it were not for the signs in the Pfaffian. The idea is to cleverly change the signs of some of the entries in the adjacency matrix to nullify the signs in the Pfaffian.

Order the vertices of the graph, and call a cycle odd or even depending on whether as we go around the cycle we have an odd or even number of edges where we go to a larger vertex. (This does not depend on which way we transverse the cycle as the cycle has even length.)

Suppose we label each edge of the graph with a sign. Now we take the adjacency matrix of the graph, and change signs of its entries as follows.

- First change the sign of a_{ij} so that it has the same sign as the corresponding edge of the graph.
- Then make it antisymmetric by changing the sign of a_{ij} if $i < j$.

Lemma 227 *Suppose that for every cycle the product of signs in the cycle of length a is* $(-1)^{a/2+1}$ *if the cycle is even and minus this if the cycle is odd. Then the Pfaffian of the matrix above is (up to sign) the number of perfect matchings of the graph.*

Proof The non-zero terms of the expansion of the Pfaffian correspond to perfect matchings. The problem is to check that any two terms have the same sign.

Suppose that we have two perfect matchings. Color the edges of one red, and the edges of the other blue. Then we get a collection of even length cycles, whose edges alternate red and blue. (These are allowed to have one double edge colored both red and blue.) We examine a single cycle $v_1v_2\cdots v_{2k}$ and check that the sign of the term of the Pfaffian does not change if we switch from the red to the blue edges. The sign of the red edges comes from:

- The sign of the permutation $v_1v_2\cdots v_{2k}$.
- The number of pairs $v_{2i-1}v_{2i}$ that are decreasing (using the order of the vertices)
- The number of pairs $v_{2i-1}v_{2i}$ whose edge has sign -1.

while the sign of the blue permutation comes from

- The sign of the permutation $v_2\cdots v_{2k}v_1$.
- The number of pairs $v_{2i}v_{2i+1}$ that are decreasing (using the order of the vertices)
- The number of pairs $v_{2i}v_{2i+1}$ whose edge has sign -1.

So we pick up a factor of -1 as the sign of a cycle of length $2n$, and a factor of -1 each time the vertices decrease as we go around the cycle, and a factor of -1 for each edge of the cycle whose edge has sign -1. By assumption the signs of the edges are chosen so that these signs cancel out over every cycle. □

Exercise 228 If we have a planar bipartite graph then we can assign signs to each edge so that the product of signs in a cycle of length a is $(-1)^{a/2+1}$ if the cycle is even and minus this if the cycle is odd. (By induction on the size of the graph we can arrange that this is true for the cycles bounding faces of the planar graph (remove an outer edge, add signs to the remainder of the graph, then add a sign to the removed edge so that its face has the correct number of signs). Then check that all cycles, not just those bounding faces, have the correct parity of signs by induction on the number of faces inside the cycle.)

Example 229 We use this to count the number of domino tilings of a chessboard, or more generally an $m \times n$ rectangle. The number of domino tilings of a chessboard is the number of bipartite matchings of the graph formed by joining all centers of squares to the centers of adjacent squares. We have to choose an ordering of the vertices and signs for the edges satisfying the condition above. We choose lexicographic ordering of vertices. Then every 1 by 1 square has 2 increases as we go around it, so needs an odd number of signs on its edges. We can achieve this by putting signs on the horizontal edges of rows $2, 4, 6, \ldots$. We let Q_n be the $n \times n$ matrix with 1s just above the diagonal, -1 just below it, and 0s elsewhere. We let I_n be the $n \times n$ identity matrix. We let F_n be the diagonal matrix whose entries alternate 1 and -1. Then the matrix whose Pfaffian we want to evaluate is

$$Q_n \otimes I_m + F_n \otimes Q_m$$

Here Q_n comes from the horizontal edges, Q_m from the vertical edges, and F_n comes from the fact that we twiddle the signs of the edges in even rows.

Now we find the determinant of this matrix by (almost) diagonalizing it. First we diagonalize Q_n by finding its eigenvectors. If $(a_1, a_2, \cdots)$ is an eigenvector with eigenvalue λ, then $-a_{k-1} + a_{k+1} = \lambda a_k$ (with $a_0 = a_{n+1} = 0$). This is a difference equation for a_k with solution $a_k = z_1^k - z_2^k$ where $z_1^{n+1} - z_2^{-n-1} = a_{n+1} = 0$, $z_1 z_2 = -1$, $z_1 + z_2 = \lambda$ so $z_1 = e^{(2j+n+1)\pi i/2(n+1)}$ for integers j. The eigenvalue λ is $z_1 + z_2 = 2i\sin((2j+n+1)\pi/2(n+1)) = 2i\cos(j/(n+1))$.

If we replaced F_n by I_n we would be finished, because the vectors $v_i \otimes v_j$ would be a set of eigenvectors for out matrix, where v_i and v_j run through eigenvectors of the matrices Q. We now seem to run into a problem, because the matrices F_n and Q_n do not commute, so we cannot simultaneously diagonalize them. However they are not too far from commuting: in fact $F_n Q_n = -Q_n F_n$. This means that F_n switches the eigenspaces of Q with eigenvalues λ and $-\lambda$, so (at least when λ is nonzero) we can find a basis of eigenvectors of Q_n in which F_n can be written with 2 by 2 blocks $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ down the diagonal.

What is really going on is that we have a representation of the algebra generated by Q_n, Q_m, and F_n on a space of dimension mn, and the key point is that this representation breaks up as a sum of small representations of dimension at most 2.

So the mn by mn matrix can be written with $mn/2$ diagonal 2 by 2 blocks, each of the form

$$\begin{pmatrix} 2i\cos(j/(n+1)) & 2i\cos(k/(m+1)) \\ 2i\cos(k/(m+1)) & 2i\cos(j/(n+1)) \end{pmatrix}$$

down the diagonal. We could diagonalize this, but there is not much point as all we need is its determinant which is easy to evaluate. The determinant is the product of the determinants of all these 2 by 2 blocks, which is the square root of

$$\prod_{j=1}^{n}\prod_{k=1}^{m}(2i\cos(j/(n+1)))^2+(2i\cos(k/(m+1)))^2$$

So the number of domino tilings is (up to sign) the Pfaffian, in other words the 4th root of the absolute value of this product. For example, there are 12988816 domino tilings of a chessboard.