5.1a This is the tacnode. The singular points are the points with \( x^2 = x^4 + y^4, 2x = 4x^3, \) and \( 4y^3 = 0, \) so (at least in characteristic 0) the only singular point is \((0,0)\).

5.1b This is the node; singular point is \((0,0)\).

5.1c This is the cusp; singular point is \((0,0)\).

5.1d This is the triple point; singular point is \((0,0)\).

5.2 The singular points of \( f(x,y,z) = 0 \) are given by \( f = 0, \frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \) and \( \frac{\partial f}{\partial z} = 0. \)

5.2a This is the pinch point; singular points are where \( xy = z^2, y^2 = 0, 2xy = 0, \) and \( 2z = 0, \) which is the line \( y = z = 0. \)

5.2b This is the conical double point; singular points are where \( x^2 + y^2 = z^2, 2x = 0, 2y = 0, \) and \( 2z = 0, \) which is the point \((0,0,0)\).

5.2c This is the double line; singular points are where \( xy + x^3 + y^3 = 0, y + 3x^2 = 0, x + 3y^2 = 0, \) and \( 0 = 0, \) which is the line \( x = y = 0. \)

5.3a If \( P \) is a point on \( Y \) then \( P \) is a nonsingular point of \( Y \) is equivalent to saying that one of \( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \) are nonzero at \( P, \) which is equivalent to saying that \( f \) has a term of degree 1 in \( x \) and \( y, \) which is equivalent to saying that \( \mu_P(Y) = 1. \)

5.3b The singularities in 1a, 1b, and 1c have multiplicity 2, and 1d has multiplicity 3.

5.4a \( f \) and \( g \) both vanish at only a finite number of points, so we can find a polynomial \( h(y) \) which vanishes whenever \( f \) and \( g \) both vanish, so \( h^n \in (f,g) \) for some \( n, \) so we can assume \( n = 1. \) The submodules of \( O_P/(f,g) \) correspond to ideals of \( O_P \) containing \( f \) and \( g, \) so it is sufficient to show that \( k[x,y]/(f,g) \) is finite dimensional (as its dimension is at least the length of \( O_P/(f,g) \)). But if we have polynomials \( h_1(x) \) and \( h_2(y) \) of degrees \( m \) and \( n \) in \( (f,g) \) then \( k[x,y]/(f,g) \) has dimension at most that of \( k[x,y]/(h_1,h_2) \) which is \( mn \) which is finite.

5.4b Put \( P = (0,0) \) and take any line \( L \) not in the tangent cone of \( Y. \) We can assume that \( L \) is the line \( y = 0, \) so the terms of lowest degree in \( f \) contain \( x^m \) (where \( m \) is the multiplicity of \( Y \) at \( P. \) Then \( O_P/(f,g) = O_P/(y,x^m + \cdots) = O_Q/(x^m + \cdots) \) which has length \( m \) (where \( O_Q \) is the local ring of \( Q = 0 \) in \( A^1). \)

5.4c We can assume that \( L \) is \( y = 0. \) If \( z \neq 0, \) the equation of the curve \( Y \) is \( f(x) + y(*) = 0 \) where \( f \) if a polynomial in \( x \) of some degree \( n. \) Then if \( x \) is a root of \( f \) of multiplicity \( m, \) we have \( (L.Y)(x,0) = m, \) so the sums of the intersection multiplicities along the \( x \) axis is the number of roots of \( f \) which is \( n. \) On the other hand, at the point \((0:1:0)\) the intersection multiplicity is \( d - n \) as the equation for \( f \) is locally \( z^{d-n} + \cdots + x(*) = 0. \) So the sum of all intersection multiplicities is \( n + d - n = d. \)

5.5 If the characteristic \( p \) does not divide \( d \) we can use \( x^d + y^d + z^d = 0 \) Otherwise we can use \( xy^{d-1} + yz^{d-1} + zx^{d-1} = 0. \)