Math 121B practice midterm 1.

Please make sure that your name is on everything you hand in. You are allowed calculators and 1 page of notes. All questions have about the same number of marks.

1. Express the integral
   \[ \int_0^\infty \frac{y^2 dy}{(1 + y)^6} \]
as a beta function, hence in terms of gamma functions, and use this to evaluate it explicitly. (Hint: put \( x = y/(1 + y) \) in the definition \( B(p, q) = \int_0^1 x^{p-1}(1 - x)^{q-1} dx = \Gamma(p)\Gamma(q)/\Gamma(p + q) \).)

2. Use Stirling’s formula \( n! \approx n^n e^{-n} \sqrt{2\pi n} \) to evaluate
   \[ \lim_{n \to \infty} \frac{(2n)!\sqrt{n}}{2^{2n}(n!)^2} \]

3. Find the general solution of
   \[ (x^2 + 1)y'' - 2xy' + 2y = 0 \]
   by writing \( y \) as a power series \( a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \) in \( x \).

4. Find the best (in the least squares sense) second-degree polynomial approximation \( a_0 + a_1 x + a_2 x^2 \) to the function \( x^4 \) for \( -1 \leq x \leq 1 \).
   (The first few Legendre polynomials are \( P_0(x) = 1, P_1(x) = x, P_2(x) = (3x^2 - 1)/2, P_3(x) = (5x^3 - 3x)/2, P_4(x) = (35x^4 - 30x^2 + 3)/8 \).)

5. Find \( P_6(0) \) from Rodrigues’ formula
   \[ P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l. \]