A Spectral Approach to Polytope Diameter

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Results Summary

1. Worst Case Theorem: Given an integer polytope $\{Ax \leq b\}$ with $A \in \mathbb{Z}^{m \times d}$ having bounded entries, its diameter is at most $O(d \log m / {}_0)$ where ${}_0$ is the minimum angle in between adjacent facets.

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2. Smoothed Case Theorem: given a well-rounded unit polytope $\{\langle a_i, x\rangle \quad 1\}$ for $\|a_i\| \quad 1$, the random polytope $\{\langle a_i + g_i, x\rangle \quad 1\}$ contains a subgraph with $1 \quad 1/\text{poly}(m)$ of the vertices (with respect to the mean curvature measure) and diameter $\text{poly}(md)/\ ^2 r^3$.

Roadmap

Formal Hessian and Spectral Gap

Bound from Chebyshev Polynomials

Bound from Continuous Time Markov Chain

- Smoothed Setting
- Denominator lower bound
- Numerator upper bound - large angles
- Numerator upper bound - small perimeter

Formal Hessian and Spectral Gap

'Formal Hessian' of a polytope K is

$$H_{ij} = \begin{cases} |F_{ij}| \csc(\theta_{ij}) & i \neq j \\ \sum_k |F_{ik}| \cot(\theta_{ij}) & i = j \end{cases}$$

where F_{ij} is the intersection between the ith and jth facets, and θ_{ij} is the angle between their normals.

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- Also holds when K is not simple.

Formal Hessian and Spectral Gap

Let $D = H + L$ where D is diagonal and L is Laplacian.

$$L_{ij} = \begin{cases} |F_{ij}| \csc(\ _{ij}) & i \neq j \\ \sum_k |F_{ik}| \csc(\ _{ij}) & i = j \end{cases}.$$

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Then

$$D^{\ 1/2} H D^{\ 1/2} = \quad D^{\ 1/2} L D^{\ 1/2} + I.$$

The left-hand side has exactly one positive eigenvalue, the right-hand side has maximum eigenvalue exactly at 1.

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$$\sum_{j} j(\;\; {}^{1}H)_{ij} j \quad \frac{2 \sum_{i \; j} jF_{ij}j \csc(\;\; _{ij})}{\sum_{i \;\; k} jF_{ik}j \tan(\;\; _{ij}=2)} \quad \sup_{i \;\; j} \frac{2\csc(\;\; _{ij})}{\tan(\;\; _{ij}=2)} = \sup_{i \;\; j} \csc^{2}(\;\; _{ij}=2) \quad \frac{10}{\frac{2}{0}}:$$

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$$v_{\min}^{2} = \frac{\min_i D_{ii}}{k1^{T} D^{1=2} k^{2}} \quad \frac{\min_i D_{ii}}{N \max_i D_{ii}} \quad \frac{1}{N} \frac{\min_i \sum_k jF_{ik}j \tan(\;\; _{ik}=2)}{\max_i \sum_k jF_{ik}j \tan(\;\; _{ik}=2)} \quad \frac{0 \;\; 1}{4N^{2}} \frac{\min_{i;j} jF_{ij}j}{\max_{i;j} jF_{ij}j};$$

$$0 \;\; 1 \quad 2^{\;\; O(d \log d + dL)}; \frac{\min_{i;j} F_{ij}}{\max_{i;j} F_{ij}} \quad 2^{\;\; O(dL + d \log d)}; N \quad m^{d}.$$

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$$v_{\min}^2 = \frac{\min_i D_{ii}}{\|1^T D^{1=2}\|^2} \quad \frac{\min_i D_{ii}}{N \max_i D_{ii}} \quad \frac{1}{N}\frac{\min_i \sum_k |F_{ik}|\tan(\ _{ik}=2)}{\max_i \sum_k |F_{ik}|\tan(\ _{ik}=2)} \quad \frac{0\ \ 1}{4N^2}\frac{\min_{i;j}|F_{ij}|}{\max_{i;j}|F_{ij}|};$$

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This gives $O((dL + d\log m)=\ _0)$.

Roadmap

Bound from Continuous Time Markov Chain

Continuous time Markov chain with generator Q and transition matrix $P(t) = \exp(tQ)$.

- If at state i at time t_0, then ith row of $P(t \quad t_0)$ is distribution on state at time t.
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Construct a continuous time Markov chain with two properties:

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Pick $Q := -D^{-1}L$. Top eigenvalue of $\exp(tQ)$ is 1 with eigenvector $1^T D$. Other eigenvalues are in $(0, e^{-t}]$.

Average Transition Rate

Define two measures:

$$(S) := \sum_{T \in F} |F_{ST}| \tan(\ _{ST}/2). \qquad \text{(diagonal entries of } D\text{)}$$

$$(S) := \sum_{T \in F} |F_{ST}| \csc(\ _{ST}). \qquad \text{(diagonal entries of } L\text{)}$$

See that $(S)/\ (S) = Q(S, S)$ are the transition rates. Average transition rate is

$$\mathbb{E}_S \quad [Q(S, S)] = \frac{\sum_S \ (S) Q(S, S)}{\sum_S \ (S)} = \frac{\sum_S \ (S)}{\sum_S \ (S)} =: J_{ave}.$$

Roadmap

Smoothed Setting

Our result holds for small perturbations of well-rounded unit polytopes. Let

$$P_0 = \{x \mid \langle a_i, x\rangle \leq 1\} \qquad K_0 := P_0^{o} = \mathrm{conv}\{a_i\}_{i \in [m]}$$

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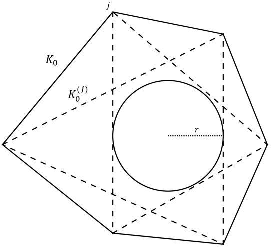
$$rB_2^d \subseteq \bigcap_{j\in[m]} K_0^{(j)} \subseteq K_0 \subseteq B_2^d.$$



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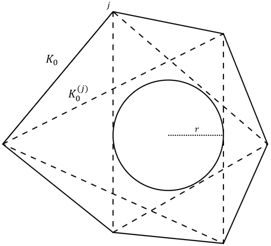

Perturb the constraints as $v_i = a_i + g_i$ for g_i i.i.d. from $N(0, \sigma^2 I_d)$, where $\sigma := 6\sqrt{d \log m} < r/d^2$. See that $\sup \|g_i\|$ w.h.p.

$$P = \{x \mid \langle v_i, x\rangle \le 1\} \qquad K := P^o = \operatorname{conv}\{v_i\}_{i\in[m]}$$

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Denominator lower bound

$$(S) = \sum_{T \in F} |F_{ST}| \tan(\ _{ST}/2) \quad \sum_{T \in F} |F_{ST}| \ _{ST} =: \ _2(S)$$

We have the quermassintegral formulas:

$$d \ \ V(K[d \quad 1]; B_2^d[1]) = \sum_{S \in F} |F_S| = |\partial K|;$$

$$\binom{d}{2} V(K[d \quad 2]; B_2^d[2]) = \sum_{S;T \in F} |F_{ST}| \ _{ST}.$$

By Alexandrov–Fenchel,

$$\sum_{S \in F} \ _2(S) \quad \binom{d}{2} V(K[d \quad 2]; B_2^d[2])$$

$$\binom{d}{2} V(K[d \quad 1]; B_2^d[1])^{\frac{d \ \ 2}{d \ \ 1}} V(B_2^d[d])^{\frac{1}{d \ \ 1}}$$

$$d|\partial K|.$$

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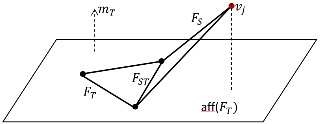
$$\mathbb{E} \sum_S \ (S) = \mathbb{E}\left[\sum_{S,T \in \binom{[m]}{d}} |F_{ST}| \csc \ _{ST} K_S K_T K_{ST} \right]$$

$$O(d \log m / \ ) \mathbb{E}\left[\sum_{S,T \in \binom{[m]}{d}} |F_{ST}| K_{ST}^{(S \setminus T)} \right]$$

Large angles proof

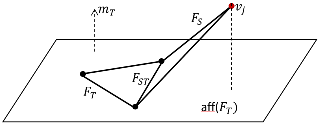
If $K_S K_T K_{ST} = 1$ then $SnT = fjg$ is a singleton. Fix v_i for $i \neq j$ and consider the randomness of $v_j = a_j + g_j$.



$jF_{ST}jK_{ST}$ $\quad$ $jF_{ST}jK^{(j)}_{ST}$

Large angles proof

If $|K_S|, |K_T|, |K_{ST}| = 1$ then $S \setminus T = \{j\}$ is a singleton. Fix v_i for $i \neq j$ and consider the randomness of $v_j = a_j + g_j$.



$$|F_{ST}|\,|K_{ST}| \qquad |F_{ST}|\,|K_{ST}^{(j)}|, \qquad \csc\ _{ST} = \frac{\mathrm{dist}(v_j, \mathrm{aff}(F_{ST}))}{\mathrm{dist}(v_j, \mathrm{aff}(F_T))} \qquad \frac{3}{\mathrm{dist}(v_j, \mathrm{aff}(F_T))}$$

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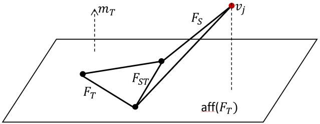

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where $h_j = \langle g_j, m_T \rangle$ and $x_T = \operatorname{dist}(0, \operatorname{aff}(F_T) \quad a_j) \quad 4.$

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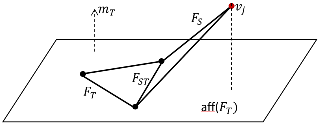

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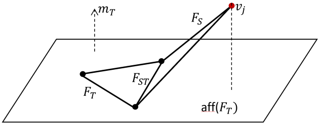

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$$\text{E}[|h_j \quad X_T|^{\;\;1}] = \int_{m^{\;\;5d}}^{5} \frac{1}{t} dt \quad (\log(m^{5d}) + \log 4)/ \qquad O(d \log m / \quad)$$

Large angles - Recap

$$\mathbb{E}\left[\sum_{S} \ (S)\right] = \mathbb{E}\left[\sum_{S,T\in\binom{[m]}{d}} |F_{ST}| \csc \ _{ST} K_S K_T K_{ST}\right]$$

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$$(\text{codim-2 perimeter}) = \mathbb{E}\left[\sum_{S,T\in\binom{[m]}{d}} |F_{ST}| K_{ST}^{(j)}\right]$$

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Small Perimeter

We can relate the codimension 2 perimeter to the surface area:

$$\text{(codim-2 perimeter)} \quad \text{poly}(d; \log m) = r \quad jK_0^{(j)}j$$

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Quadrature by 2d planes / cross-sections:

Quadrature by planes

Proof is almost entirely local: for every small patch D contained in two codim-2 perimeter we want

$$|K_0| \cdot \Pr[W \cap D \neq \emptyset] \geq \frac{r}{\mathrm{poly}(d)\sqrt{\log m}} |D|.$$