3/18 Quantum Error correcting codes

Classical Error correcting code

C   k-dim subspace $\mathbb{F}_2^n$

rate: k/n Hamming ↓ $x \neq 0$

distance: $\min\{wt(x) : x \in C\}$

$wt(x-y)$ $x, y \in C$ $(0 \in C)$

Goal: Encode length k strings, add redundancy.

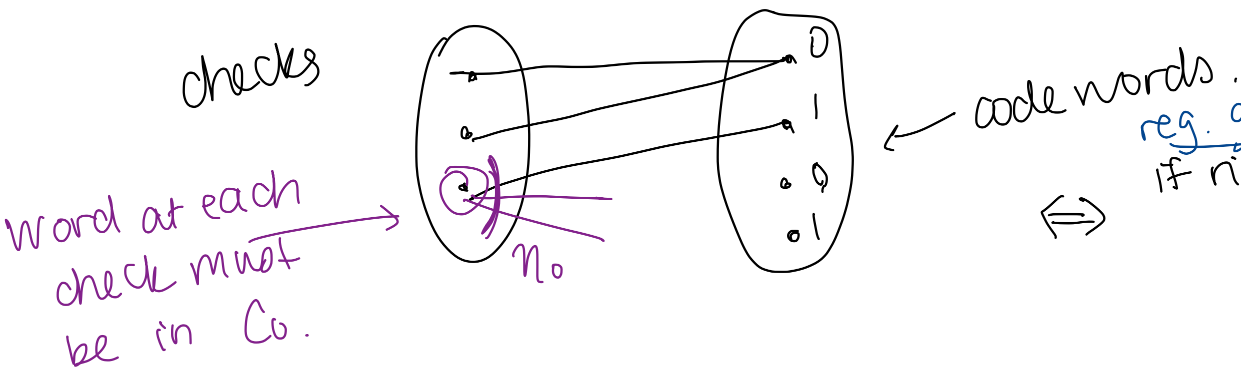
Ex: Repetition code $C = \{\underbrace{0\ldots0}_{n}, \underbrace{1 \quad 1}_{n}\}$

Rate: $1/n$ Dist: n

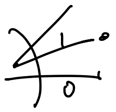
Ex: Parity code $C = \{x \in \mathbb{F}_2^n : x_1 + \ldots + x_n = 0\}$

Rate: $n-1/n$ Dist: 2   00..011

Tanner code: C_0 "base code"   blocklength n_0



reg. graph view:
if right deg = 2
⟺



n_0-reg. graph

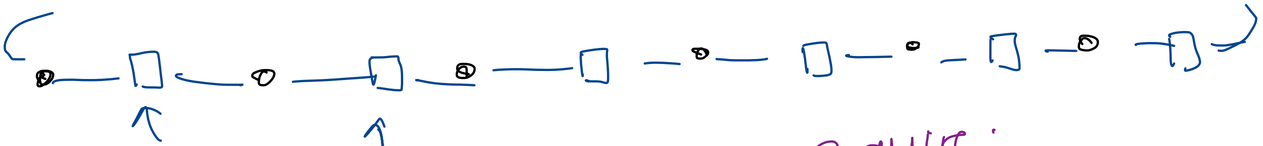
Blocklength = $|E|$ in the regular graph view

If G expanding, C_0 is "good"

$\Rightarrow$ C is also "good"

How about Quantum?

classical $C = \bigcap \{\text{subspaces } S_i : v \in \mathbb{F}_2^n \text{ that satisfy check } i\}$



$x_1 + x_2 = 0$ induce a subspace.

$x_2 + x_3 = 0$

Require: All P_i, P_j commute

Quantum: $C = \bigcap \{\text{subspaces } S_i : \forall v \in S_i \quad P_i v = v\}$

$P_1^{(i)} \otimes \underbrace{P_2^{(i)} \otimes \ldots}_{} \otimes P_n^{(i)}$

"constraints"

"Stabilizer code"

n qubits

2^n dim space.

$\boxed{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}$

Qubit: $|0\rangle \quad |1\rangle$

$\alpha|0\rangle + \beta|1\rangle \quad |\alpha|^2 + |\beta|^2 = 1$

$|0\rangle \otimes |0\rangle \Leftarrow: |00\rangle \qquad |1\rangle \otimes |0\rangle \Leftarrow: |10\rangle$

What are Ps? Pauli operators. 2x2 complex matrices

- Eigenvalues are all $1, -1$
- Generators: $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ "bit flips"

$X|0\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$ $\quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$X|1\rangle = |0\rangle$

"phase shift" $\rightarrow Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$Z|0\rangle = |0\rangle$

$Z|1\rangle = -|1\rangle$

Obs: $Z^2 = X^2 = \mathbb{I}$

$XZ = -ZX$

Error types:

{CSS codes} $\subseteq$ {stabilizer code}

very structured version

Q: Quantum Repetition Code?

Rate: $1/3$

$|0\rangle \quad |1\rangle \qquad |0\rangle \mapsto |000\rangle \qquad |1\rangle \mapsto |111\rangle$

Errors: $P_1 \otimes P_2 \otimes P_3$

$(\mathbb{I} \otimes X \otimes \mathbb{I})|000\rangle = |010\rangle$

$(\mathbb{I} \otimes \mathbb{I} \otimes Z)|111\rangle = -|111\rangle$

Reality: Distribution of errors.

$$\boxed{\sum \alpha_i P_i} \rightarrow \begin{array}{l} \text{error} \\ P_1 \text{ wp } \alpha_1^2 \\ P_2 \text{ wp } \alpha_2^2 \\ \vdots \end{array}$$

$Z \otimes Z \otimes I\,|000\rangle = |000\rangle$

$Z \otimes Z \otimes I\,|111\rangle = |111\rangle$

Stabilizers?

$$\left.\left\{\begin{array}{l} Z \otimes Z \otimes I. \\ Z \otimes \mathbb{I} \otimes Z \\ \mathbb{I} \otimes Z \otimes Z \end{array}\right.\right\} \text{checks.}$$

$\underbrace{\mathbb{I} X \mathbb{I}\,|000\rangle}_{|010\rangle} \Rightarrow$ apply Z check

$(Z \otimes Z \otimes \mathbb{I})(\mathbb{I} \otimes X \otimes \mathbb{I})|000\rangle$

$= Z Z \mathbb{I}\,|010\rangle$

$= -|010\rangle$

$(Z \otimes Z \otimes \mathbb{I})(\mathbb{I} \otimes X \otimes \mathbb{I})|\psi\rangle$

stab.

$= -(\mathbb{I} \otimes X \otimes \mathbb{I})\underbrace{(Z \otimes Z \otimes \mathbb{I})|\psi\rangle}_{|\psi\rangle}$

$= -(\mathbb{I} \otimes X \otimes \mathbb{I})|\psi\rangle$

Ex error:

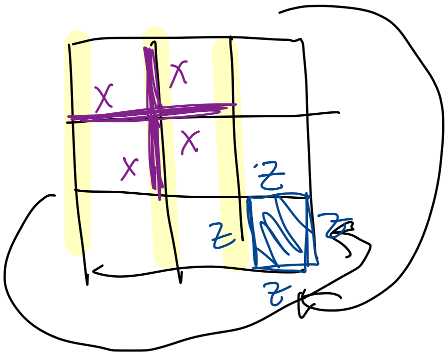
$\mathbb{I}\,\mathbb{I}\,Z$ ← none of the stabilizers detect this error.

$\Rightarrow$ dist $= 0$

$E|\psi\rangle \qquad P_i E|\psi\rangle = \pm 1 \cdot E P_i |\psi\rangle$

$= \pm 1 \cdot E|\psi\rangle$

"True" Quantum Rep. Code : Toric code (Kitaev)

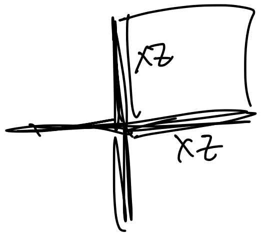


qubits live on edges

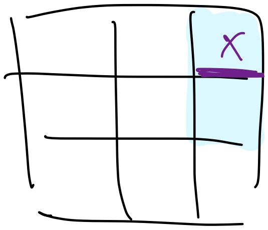
Z checks : "plaquettes"
X checks : "vertices"

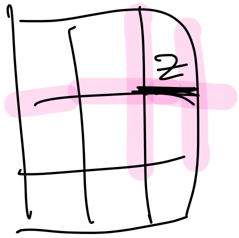
generators of stabilizer group.

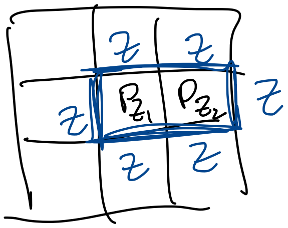
Check: X and Z checks commute.



Error Detection:



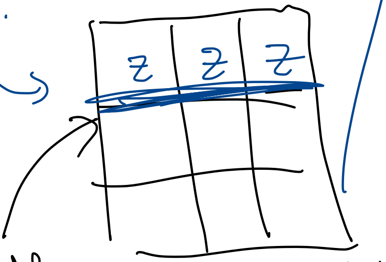




$P_{Z_1} P_{Z_2} |\psi\rangle = |\psi\rangle$

⟸ valid, since it is stabilized.

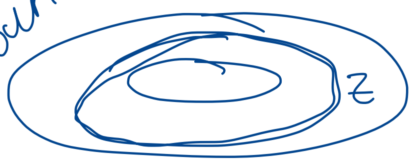
cycle.

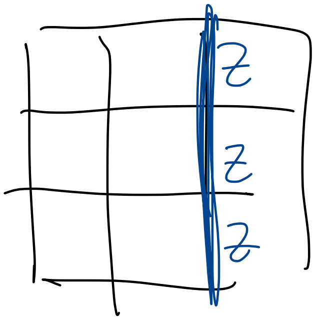


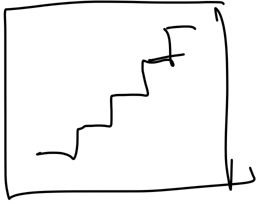
Pcycle

Pcycle $|\psi\rangle \neq |\psi\rangle$

/ boundaries







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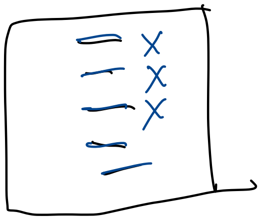
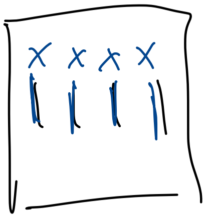
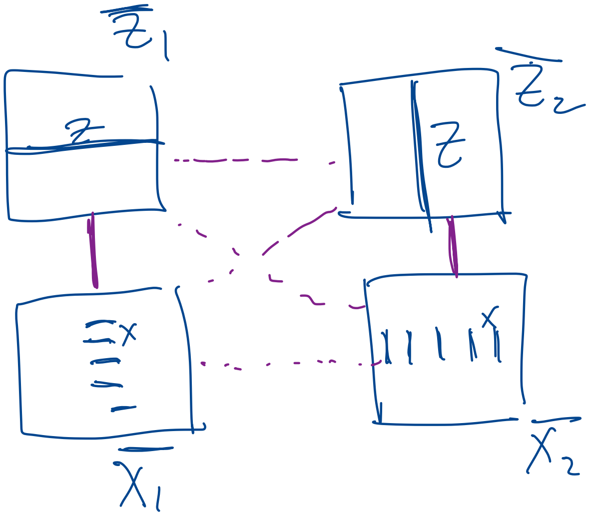






$$\cong \left\{ \begin{array}{l} X\otimes \mathbb{I} \\ \mathbb{I}\otimes X \\ Z\otimes \mathbb{I} \\ \mathbb{I}\otimes Z \end{array}\right\} \begin{array}{l} X_1 \\ X_2 \\ Z_1 \\ Z_2 \end{array} \otimes \mathbb{I}^{N-2}$$

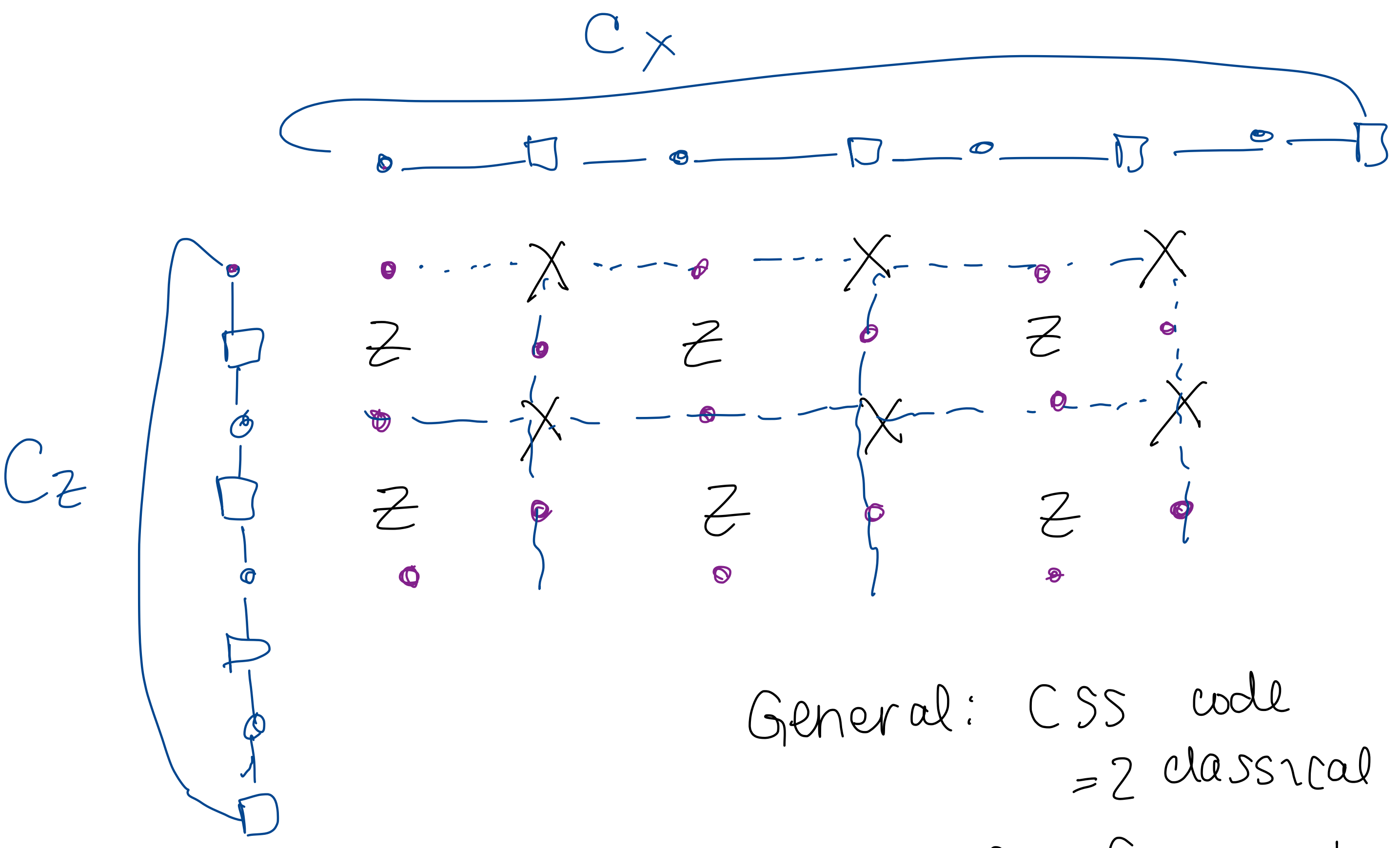
proj caused by stabilizers

$\Downarrow$ $\Downarrow$

$$\text{codespace} \xrightarrow[\cong]{} \begin{matrix} |00\rangle & |01\rangle \\ |10\rangle & |11\rangle \end{matrix} \otimes \mathbb{I}^{N-2}$$

$\exists$ U unitary st $\bar{X}_1 = U X_1 U^\dagger$

$\bar{Z}_1 = U Z_1 U^\dagger$



General: CSS code

= 2 classical codes

C_X C_Z $C_X^{\perp} \subseteq C_Z$