Ricci flows in higher dimensions

Richard H Bamler

UC Berkeley

September 2020
Bibliography

Advertisement

Online class on Ricci flow this fall semester

14:10–15:30 (Pacific time)
August 27 – December 3

email me (rbamler@berkeley.edu) or check my webpage (https://math.berkeley.edu/~rbamler/rfclass.html) for Zoom ID
Ricci flow \((M, (g_t)_{t\in[0,T]})\) on a compact manifold \(M^n\):

\[
\partial_t g_t = -2 \text{Ric}_{g_t}
\]

**Important Question:** Understand the singularity formation if \(T < \infty\) (and the long-time asymptotics if \(T = \infty\))

**Blow-up analysis:**
Choose \((x_i, t_i) \in M \times [0,T)\) s.t.:

\[
t_i \nearrow T \quad |\text{Rm}(x_i, t_i)| \to \infty
\]

Hope that for some \(\lambda_i \to \infty\):

\[
(M, (\lambda_i^2 g_{\lambda_i^{-2} t + t_i}), x_i) \xrightarrow[i \to \infty]{} (M_\infty, (g_\infty, t)_{t \leq 0}, x_\infty)
\]

**parabolic rescaling**


**“singularity model”**

So far: curvature bounds are necessary!
Dimension 2: singularity model = \((S^2, (2|t|g_{S^2})_{t<0})\) (Chow, Hamilton)

Dimension 3: singularity models are \(\kappa\)-solutions \ldots (Perelman)

Gradient shrinking soliton \((M, g, f)\): \[\text{Ric} + \nabla^2 f - \frac{1}{2} g = 0\]
\[\leadsto g_t := |t|\phi_t^*g \text{ is RF, where } \phi_t = \text{flow of } |t|\nabla f, \ t < 0\]
- \(|Rm| \sim C/|t|\) (Type I)
- The singularity model of \((M, (g_t)_{t<0})\) is the flow itself.

Type-I curvature bound \(|Rm| \leq C/(T - t))\): All singularity models are gradient shrinking solitons. (Sesum, Naber, Enders, Buzano, Topping)

Type-I scalar curvature bound \(R \leq C/(T - t))\): All singularity models are gradient shrinking solitons with codimension \(\geq 4\) singular set. (B., Chen, Hallgren, Wang, Zhang)

Folklore Conjecture

For a general Ricci flow “most” singularity models are gradient shrinking solitons.

Goal of this talk: Verify this conjecture in a certain (possibly optimal) sense.
Examples in higher dimensions

**Appleton:** ∃ RFs in dimension 4 whose blow-up limits are:

Eguchi-Hanson, \( \mathbb{R}^4 / \mathbb{Z}_2 \), (Bryant soliton/\( \mathbb{Z}_2 \), \( \mathbb{R}P^3 \times \mathbb{R} \))

\textit{Ricci flat} \quad \textit{singular}

\textit{gradient shrinking soliton}

**Stolarski:** ∃ RFs in dimensions \( n \geq 13 \) whose only gradient shrinking soliton blow-up limit is a Ricci flat cone

**Li, Tian, Zhu:** ∃ Kähler-RF that has to develop a singularity, but cannot converge to a smooth gradient shrinking soliton.

**Conclusion:** Need to allow singular set in Folklore Conjecture + Ricci flat cones
Recall: Einstein metrics

Consider a sequence of pointed, complete Einstein manifolds \((M^n_i, g_i, x_i)\), 
Ric = \(\lambda_i g_i\), \(|\lambda_i| \leq 1\). Then a subsequence Gromov-Hausdorff converges to a 
pointed metric length space:

\[
(M^n_i, g_i, x_i) \xrightarrow{GH \quad i \to \infty} (X, d, x_\infty).
\]

Suppose that the following non-collapsing condition holds:

\[|B(x_i, r)| \geq v > 0.\]

Then there is a regular-singular decomposition

\[X = \mathcal{R} \cup S\]

such that:

- \(\mathcal{R}\) is an open manifold and there is a smooth Einstein metric \(g_\infty\) on \(\mathcal{R}\) such 
  that \(d|_\mathcal{R} = d_{g_\infty}\). So \((X, d)\) is isometric to the metric completion of \((\mathcal{R}, g_\infty)\).
- \(\dim_M S \leq n - 4\)  \hspace{1cm} (Cheeger, Colding, Tian, Naber)
- Any tangent cone at any point of \(X\) is a metric cone.  \hspace{1cm} (Cheeger, Colding)
- There is a stratification \(S^0 \subset \ldots \subset S^{n-4} = S\) such that \(\dim_\mathcal{H} S^k \leq k\) and 
  every \(x \in S^k\) has a tangent cone that splits of an \(\mathbb{R}^k\)-factor. \hspace{1cm} (Cheeger, 
  Naber)
Main results of this talk

Similar theory for minimal surfaces, harmonic maps, mean curvature flow, harmonic map heat flow, . . .

Key points:

- There is a compactness and partial regularity theory for Ricci flow that is comparable to (and implies) that of Einstein metrics.
- This theory allows us to establish the Folklore Conjecture and several other related results.
- We need new, parabolic versions of notions such as: “metric space”, “Gromov-Hausdorff limit”, . . .
Theorem (B. 2020) Compactness theory of Ricci flows

Consider a sequence of $n$-dimensional, pointed Ricci flows:

$$(M_i, (g_i, t)_{t \in (-T_i, 0]}, (x_i, 0)), \quad T_{\infty} := \lim_{i \to \infty} T_i > 0.$$ 

Then a subsequence $F$-converges to a metric flow over $(-T_{\infty}, 0]$:

$$(M_i, (g_i, t)_{t \in (-T_i, 0]}, (\nu_{x_i}, 0)) \xrightarrow{F, i \to \infty} (\mathcal{X}, d, (\nu_{x_{\infty}})).$$

Suppose that the following non-collapsing condition holds:

$${\mathcal{N}}_{x_i, 0}(\tau_0) \geq -Y_0 > -\infty.$$ 

Then we have a regular-singular decomposition

$$\mathcal{X} = \mathcal{R} \cup \mathcal{S}$$

such that:

- $\mathcal{X}$ restricted to $\mathcal{R}$ is given by a smooth Ricci flow spacetime structure and $\mathcal{X}$ is uniquely determined by this structure.
- $\dim_{\mathcal{M}} \mathcal{S} \leq (n + 2) - 4$
- All tangent flows of $\mathcal{X}$ are gradient shrinking solitons with singularities.
- There is a filtration $\mathcal{S}^0 \subset \ldots \subset \mathcal{S}^{n-2} = \mathcal{S}$ such that $\dim_{\mathcal{H}} S^k \leq k$ and every $x \in S^k$ has a tangent flow that splits off an $\mathbb{R}^k$-factor or is static and splits off an $\mathbb{R}^{k-2}$-factor.
Consequences & Further results

Regarding Folklore Conjecture:

Theorem (B. 2020)

Consider a Ricci flow \((M, (g_t)_{t \in [0, T)\}), \ T < \infty\). Then there is a metric space \((M_T, d_T) = \lim_{t \searrow T}(M, g_t)\)

such that:

- If \(g_t \to g_T\) smoothly on \(U \subset M\), then \(U \subset M_T\) and \(d_T|_U\) is locally isometric to \(d_{g_t}|_U\).

- For any \("(x_i, t_i) \to (z, T)\)\), \(z \in M_T\), there is a sequence of blow-ups that converges to singular gradient shrinking soliton. This soliton can be viewed as the tangent flow at \((z, T)\).

In dimension 4:

Theorem (B. 2020)

In dimension 4 all tangent flows are given by singular gradient shrinking solitons on smooth orbifolds with conical singularities, i.e. \((M, g, f), \text{Ric} + \nabla^2 f - \frac{1}{2} g = 0\). Moreover, either \(R > 0\) or \((M, g) \cong \mathbb{R}^4/\Gamma\).
Regarding long-time asymptotics:

**Theorem (B. 2020)**

If \((M, (g_t)_{t \geq 0})\) is immortal, then for \(Y, t \gg 1\)

\[ M = M_{thick}(t) \cup M_{thin}(t) \]

such that:

- If \(x_i \in M_{thick}(t_i)\) and \(t_i \to \infty\), then \((M, (t_i^{-1} g_{t_i} t_i), x_i)\) converges to a singular, Einstein Ricci flow with \(\text{Ric} = -\frac{1}{2t} g_{\infty, t}\).
  - If \(n = 4\), then this flow is given by an Einstein orbifold.
- If \(x \in M_{thick}(t)\), then \(\mathcal{N}_{x, t}(t/2) \leq -Y\).
Application: Backwards Pseudolocality

Theorem (B. 2020)
If \([t_0 - r^2, t_0] \subset I\) and
\[
|B(x_0, t_0, r)| \geq \alpha r^n
\]
and
\[
|Rm| \leq (\alpha r)^{-2} \quad \text{on} \quad B(x_0, t_0, r),
\]
then
\[
|Rm| \leq (\varepsilon (n, \alpha) r)^{-2} \quad \text{on} \quad P(x_0, t_0; \varepsilon r, - (\varepsilon r)^2).
\]

Further Remarks:
- In dimension 3, this theory essentially recovers Perelman’s theory.
- Compactness theory (not assuming non-collapsing) also holds for super Ricci flows \(\partial_t g_t + 2 \text{Ric} \geq 0\).
Let \((M, (g_t)_{t \in I})\) be a Ricci flow and \(u, v \in C^2(M \times I)\).

**Heat equation:** \[\square u = (\partial_t - \triangle g_t)u = 0\]

**Conjugate heat equation:** \[\square^* v = (-\partial_t - \triangle g_t + R_{g_t})v = 0\]

**Heat kernel:** \[K(x, t; y, s), \quad x, y \in M, \quad s < t\]

for fixed \((y, s)\): \[
\square K(\cdot, \cdot; y, s) = 0, \quad \lim_{t \searrow s} K(\cdot, t; y, s) = \delta_y
\]

for fixed \((x, t)\): \[
\square^* K(x, t; \cdot, \cdot) = 0, \quad \lim_{s \nearrow t} K(x, t; \cdot, s) = \delta_x
\]

**Representation formulas:** If \(\square u = \square^* v = 0\), then
\[
u(x, t) = \int_M K(x, t; \cdot, s)u(\cdot, s)dg_s \quad v(y, s) = \int_M K(\cdot, t; y, s)v(\cdot, t)dg_t
\]

**Reproduction formula for heat kernel:** \(s < t' < t\)
\[
K(x, t; y, s) = \int_M K(x, t; \cdot, t')K(\cdot, t'; y, s)dg_{t'}
\]
Properties of heat equation:
- $u \leq C$ and $u \geq -C$ are preserved.
- $|\nabla u| \leq C$ is preserved
- Let $\Phi : \mathbb{R} \times \mathbb{R}_{\geq 0}$ be the solution to the 1-dimensional heat equation $\frac{\partial_t \Phi}{t} = \Phi''$ with initial condition $\Phi_0 = \chi_{[0, \infty)}$.

![Graph of \Phi_0 and \Phi_t, t > 0]

Improved gradient estimate (B. 2020)
If $0 < u(\cdot, t_0) < 1$, then for $t > t_0$

$$u_t(x) = \Phi_t(x') \implies |\nabla u_t|(x) \leq \Phi'_{t-t_0}(x') \quad (*)$$

Moreover, $(*)$ is preserved for any fixed $t_0$.

Properties of conjugate heat equation:
- $\nu \geq 0$ is preserved
- $\int_M \nu(\cdot, s) dg_s$ is constant in $s$ and $\int_M K(x, t; \cdot, s) dg_s = 1$
- Think of $\nu$ as $d\mu_s = \nu(\cdot, s) dg_s$. 
Conjugate heat kernel probability measure:

\[ d\nu_{x,t;s} := K(x, t; \cdot, s)dg_s, \quad \nu_{x,t;t} := \delta_x \]

Integral characterization of (conjugate) heat flows:

Heat flow:

\[ \Box u = 0 \iff u(x, t) = \int_M u(\cdot, s)d\nu_{x,t;s} \]

Conjugate heat flow:

\[ d\mu_s = \nu(\cdot, s)dg_s, \quad \Box^* \nu = 0 \iff \mu_s = \int_M \nu_{,t;s}d\mu_t \]

Reproduction formula:

\[ \nu_{x,t;s} = \int_M \nu_{,t';s}d\nu_{x,t;t'} \]
Metric flows

Metric flow over an interval $I$

$$\mathcal{X} = (\mathcal{X}, t, (d_t)_{t \in I}, (\nu_{x;s})_{x \in \mathcal{X}, s \in I, s \leq t(x)})$$

1. $\mathcal{X}$ is a set consisting of points
2. $t : \mathcal{X} \to I$ is the time-function and its level sets $\mathcal{X}_t := t^{-1}(t)$ are time-slices
3. $(\mathcal{X}_t, d_t)$ is a complete and separable metric space for all $t \in I$
4. $\nu_{x;s}$ are probability measures called conjugate heat kernel and satisfy $\nu_{x;t(x)} = \delta_x$ and the reproduction formula

$$\nu_{x;s} = \int_{\mathcal{X}_t} \nu_{.,t;s} d\nu_{x;t}$$

5. (Conjugate) heat flows are defined using the integral property as before.
6. We require that the improved gradient estimate holds for heat flows: If $u_{t_0} = \Phi_{t_0} \circ f_{t_0}$ for some 1-Lipschitz $f_{t_0} : \mathcal{X}_{t_0} \to \mathbb{R}$, then for all $t \geq t_0$ we have $u_t = \Phi_t \circ f_t$ for some 1-Lipschitz $f_t : \mathcal{X}_t \to \mathbb{R}$. 
Ricci flow \((M, (g_t)_{t \in I})\) \quad \longrightarrow \quad Metric flow \(\mathcal{X}\)

- \(\mathcal{X} := M \times I\)
- \(t :=\) projection onto second factor.
- \(d_t := d_{g_t}\) on \(\mathcal{X}_t = M \times \{t\}\)
- \(d\nu_{(x,t);s} := K(x, t; \cdot, s)dg_s\)

**Note:**

- The distance between points in different time-slices is not defined!
- This construction forgets worldlines \(t \mapsto (x, t)\).
  Instead: For \(x \in \mathcal{X}_t\) there is a probability distribution \(\nu_{x;s}\) of points \(y \in \mathcal{X}_s\) that lie in the “past” of \(x\).
Concentration property

**Variance of probability measure $\mu$ on a metric space $(X, d)$:**

\[
\text{Var}(\mu) := \int_X \int_X d^2(x, y) d\mu(x) d\mu(y)
\]

**Theorem (B. 2020)**

On any Ricci flow

\[
\text{Var}(\nu_{x, t; s}) \leq H_n(t - s),
\]

where $H_n := \frac{(n-1)\pi^2}{2} + 4$.

A metric flow $\mathcal{X}$ is called $H$-**concentrated** if $(\ast) + \ldots$ holds for $H_n = H$.

"The past in $\mathcal{X}_s$ of any point $x \in \mathcal{X}_t$ is determined up to an error of $\sim \sqrt{t - s}$."
1-Wasserstein distance

\( \mu_1, \mu_2 \) probability measures on complete, separable metric space \((X, d)\)

\[
d_{W_1}(\mu_1, \mu_2) := \inf_{q \text{ coupling between } \mu_1, \mu_2} \int_{X \times X} d \, dq = \sup_{f : X \to \mathbb{R}, 1-\text{Lipschitz}} \int_X f \, d(\mu_1 - \mu_2)
\]

**Lemma**

If \( x, y \in X_t \), then for \( s \leq t \) we have

\[
d_{W_1}(\nu_{x,s}, \nu_{y,s}) \leq d_t(x, y).
\]

Moreover, \( s \mapsto d_{W_1}(\nu_{x,s}, \nu_{y,s}) \) is non-decreasing and the same is true for any other pair of conjugate heat flows.

"Distances don't shrink on metric flows (in a probabilistic sense)"
Parabolic balls

Conventional parabolic ball in a Ricci flow:

\[ P(x_0, t_0; r) := B_{g_{t_0}}(x_0, r) \times [t_0 - r^2, t_0 + r^2] \]

\(P^*\)-parabolic ball in a metric flow:

\[
P^*(x_0; r) := \left\{ x \in X_{t_0} : \begin{array}{l}
t(x) \in [t_0 - r^2, t_0 + r^2] \\
d_{W_1}^{X_{t_0} - r^2} (\nu_{x_0; t_0 - r^2}, \nu_{x; t_0 - r^2}) < r
d_{W_1}^{X_{t_0} - r^2} (\nu_{x_0; t_0 - r^2}, \nu_{x; t_0 - r^2}) < r
d_{W_1}^{X_{t_0} - r^2} (\nu_{x_0; t_0 - r^2}, \nu_{x; t_0 - r^2}) < r
d_{W_1}^{X_{t_0} - r^2} (\nu_{x_0; t_0 - r^2}, \nu_{x; t_0 - r^2}) < r
d_{W_1}^{X_{t_0} - r^2} (\nu_{x_0; t_0 - r^2}, \nu_{x; t_0 - r^2}) < r
\end{array} \right\}
\]

- standard containment properties still hold for \(P^*\)-parabolic palls (e.g. \(P^*(x; r_1) \subset P^*(x; r_2)\) if \(r_1 \leq r_2\)
- Conventional and \(P^*\)-parabolic balls are comparable if curvature bounded.
- The natural topology on \(X\) is generated by the set of all \(P^*\)-parabolic balls.
- \(P^*\)-parabolic balls allow the definition of the parabolic Hausdorff and Minkowski dimension \(\dim\mathcal{H}^*\) and \(\dim\mathcal{M}^*\).

We count the time-direction twice!
Gromov-$W_1$-distance and convergence

**Gromov-$W_1$-distance**

If $(X_i, d_i, \mu_i), \ i = 1, 2,$ are two normalized metric measure spaces, then

$$d_{GW_1}((X_1, d_1, \mu_1), (X_2, d_2, \mu_2)) := \inf_{\varphi_1, \varphi_2, Z} d_{W_1}^Z((\varphi_1)_* \mu_1, (\varphi_2)_* \mu_2),$$

where the infimum is taken over all isometric embeddings $\varphi_i : (X_i, d_i) \to (Z, d_Z)$ into a common metric space $(Z, d_Z)$.

**Gromov-$W_1$-convergence**

$$\lim_{i \to \infty} (X_i, d_i, \mu_i) \xrightarrow{GW_1} (X_\infty, d_\infty, \mu_\infty)$$

**Important observation**

Compare with pointed Gromov-Hausdorff convergence: The probability measures $\mu_i$ take the role of the basepoint.
**$d_F$-distance and $F$-convergence**

**$d_F$-distance:**
Consider metric flows $\mathcal{X}_i$, $i = 1, 2$ equipped with conjugate heat flows $(\mu_i, t)_{t \in I}$. We define

$$d_F((\mathcal{X}_1^1, (\mu_1^1)_{t \in I}), (\mathcal{X}_2^2, (\mu_2^2)_{t \in I}))$$

to be the infimum over all $r > 0$ such that there are isometric embeddings

$$(\varphi_t^i : (\mathcal{X}_t^i, d_t^i) \to (Z_t, d_t^Z))_{t \in I \setminus E, i = 1, 2}$$

with:

1. $|E| \leq r^2$
2. $d_{W_1}^Z((\varphi_t^1)_* \mu_t^1, (\varphi_t^2)_* \mu_t^2) \leq r$ for all $t \in I \setminus E$
3. “integral $W_1$-closeness of conjugate heat kernels between times $s, t \in I \setminus E$”

**$F$-convergence**
If $d_F((\mathcal{X}_i^i, (\mu_i^i)_{t \in I}), (\mathcal{X}_\infty^\infty, (\mu_\infty^\infty)_{t \in I})) \to 0$, then we write

$$(\mathcal{X}_i, (\mu_i, t)_{t \in I_i}) \xrightarrow{i \to \infty} F (\mathcal{X}_\infty, (\mu_\infty, t)_{t \in I_i})$$

This implies Gromov-$W_1$-convergence at almost every time.
Let $F_I$ be the space of pairs $(\mathcal{X}, (\mu_t)_{t \in I})$.

**Theorem (B. 2020)**

$(F_I, d_F)$ is a complete metric space.

Suppose $I = (-T, 0]$. Fix $n$.

**Theorem (B. 2020)**

\[
\left\{ (\mathcal{X}, (\mu_t)_{t \in I}) \text{ corresponding to Ricci flows } (M^n, (g_t)_{t \in I}, (\nu_{x,0}; t)_{t \in I}) \right\} \subset F_I \quad \text{is precompact.} \tag{\star}
\]

**Corollary**

For any sequence of $n$-dimensional, pointed Ricci flows $(M_i, (g_{i,t})_{t \in (-T, 0]}, (x_i, 0))$ there is a subsequence such that:

\[
(M_i, (g_{i,t})_{t \in (-T_i, 0]}, (\nu_{x_i,0})) \xrightarrow{\text{F}} (\mathcal{X}, (\nu_{x_\infty})).
\]

**Remark:** There is a compact subset $F_I^*(H) \subset F_I$, essentially corresponding to all $H$-concentrated metric flows, that contains the subset from $(\star)$.
If we assume curvature bounds, then: $\mathbb{F}$-convergence $\iff$ local smooth convergence in the sense of Cheeger, Gromov, Hamilton.

**Example:** Bryant soliton $(M_{\text{Bry}}, (g_{\text{Bry}}, t)_{t \in \mathbb{R}}, x_{\text{Bry}})$

- rotational symmetric
- $g_{\text{Bry}, t} = dr^2 + f^2(r)g_{S^2}$, where $f(r) \sim \sqrt{r}$
- steady gradient soliton $\implies$ all time-slices are isometric

Consider blow-downs $(M_{\text{Bry}}, (\lambda_i^2 g_{\text{Bry}}, \lambda_i^{-2} t)_{t \in \mathbb{R}}, x_{\text{Bry}})$ for $\lambda_i \to 0$.

- Gromov-Hausdorff limit at any fixed time: $[0, \infty)$
- $\mathbb{F}$-limit:
  round shrinking cylinder $(S^2 \times \mathbb{R}, (g_t = 2|t|g_{S^2} + g_{\mathbb{R}})_{t < 0})$
  this is the asymptotic soliton!
Ricci flow spacetimes

Ricci flow spacetime over an interval \( I \):

\[
\mathcal{M} = (\mathcal{M}, t, \partial_t, g)
\]

1. \( \mathcal{M} \) is a smooth \((n + 1)\)-manifold, called **spacetime manifold**
2. \( t : \mathcal{M} \to I \) is a smooth map whose level sets \( \mathcal{M}_t := t^{-1}(t) \) are called **time-slices**.
3. \( \partial_t \) is a smooth vector field on \( \mathcal{M} \) with \( \partial_t t = 1 \). Its trajectories are **worldlines**.
4. \( g \) is a metric on the horizontal distribution \( \ker dt \subset T\mathcal{M} \)
5. **Ricci flow equation**: \( \mathcal{L}_{\partial_t} g = -2 \text{Ric}_g \)

**Ricci flow** \((\mathcal{M}, (g_t)_{t \in I}) \) \(\longrightarrow\) **Ricci flow spacetime** \( \mathcal{M} \)

- \( \mathcal{M} := \mathcal{M} \times I \)
- \( t := \) projection onto second factor
- \( \partial_t := \) std. vector field on \( I \)
- \( g := g_t \) on \( \mathcal{M}_t = \mathcal{M} \times \{t\} \)
Structure of non-collapsed $\mathbb{F}$-limits

Let $\mathcal{X}$ be a $\mathbb{F}$-limit of smooth Ricci flows over $I$.
Assume the non-collapsing condition $N_{x_i,0}(\tau_0) \geq -Y_0 > -\infty$.

**Theorem (B. 2020)**

There is a decomposition

$$\mathcal{X} = \mathcal{R} \cup \mathcal{S}$$

and a smooth Ricci flow spacetime structure $(\mathcal{R}, t, \partial_t, g)$ on $\mathcal{R}$ such that:

- $\mathcal{R} \subset \mathcal{X}$ is open and dense.
- For any $t \in I$ the time-slice $(\mathcal{X}_t, d_t)$ is the metric completion of $(\mathcal{R}_t, g_t)$.
- (Conjugate) heat flows restricted to $\mathcal{R}$ are uniquely characterized by $\Box u = 0$ and $\Box^* v = 0$ on $\mathcal{R}$.
- $\dim_{\mathcal{M}^*} S \leq (n + 2) - 4$
- Tangent flows at any $x \in \mathcal{X}$ (= $\mathbb{F}$-limits of blow-ups of $(\mathcal{X}, (\nu_{x,t}))$) are singular gradient shrinking solitons.
- There is a filtration $S^0 \subset \ldots \subset S^{n-2} = S$ such that $\dim_{\mathcal{M}^*} S^k \leq k$ and every $x \in S^k$ has a tangent flow that splits off an $\mathbb{R}^k$-factor or is static and splits off an $\mathbb{R}^{k-2}$-factor.
Theorem (B. 2020)

If $\mathcal{X}$ is a gradient shrinking soliton, then there is an identification

$$\mathcal{X} = X \times I$$

for a metric space $(X, d)$ with regular part $R_X \subset X$ such that:

- $(\mathcal{X}_t, d_t) = (X, |t|^{1/2}d)$
- $(R_t, g_t) = (R_X, |t|g_{R_X})$
- The soliton equation holds on $R_X$.

If $n = 4$, then $(X, d)$ is the length space of a smooth orbifold.
Outstanding promise: Non-collapsing condition

**Pointed Nash entropy:**

Fix $(x_0, t_0) \in M \times I$ and write $\tau := t_0 - t$, $K(x_0, t_0; \cdot, \cdot) = (4\pi \tau)^{-n/2} e^{-f}$

$$\mathcal{N}_{x_0, t_0}(\tau) := \int_M f(\cdot, t_0 - \tau) d\nu_{x_0, t_0; t_0 - \tau} - \frac{n}{2}$$

**Basic properties:**

- $\mathcal{N}_{x_0, t_0}(\tau) \leq 0$
- $\frac{d}{d\tau} \mathcal{N}_{x_0, t_0}(\tau) \leq 0$

There is a relation between $\mathcal{N}$ and Perelman’s $\mu$-entropy that implies: If $I = [0, T)$, then

$$\mathcal{N}_{x_0, t_0}(\tau) \geq \mu[M, g_0, T] > -\infty.$$ 

So a non-collapsing condition always holds on a fixed flow with $T < \infty$. 
Guiding principle: On a manifold with $\text{Ric} \geq -g$: 
\[ \frac{|B(x, r)|}{r^n} \approx e^{N_x(r^2)} \]

Theorem (B. 2020)

Suppose that $R \geq R_{\text{min}}$. Set $N^*_s(x, t) := N_{x,t}(t - s)$.

1. $|\nabla N^*_s| \leq \sqrt{\frac{n}{2(t-s)} - R_{\text{min}}}

2. $-\frac{n}{2(t-s)} \leq \Box N^*_s \leq 0$

3. $(1)+(2)$ imply a bound on $\text{osc}\, N^*_s$ over $P^*$-parabolic neighborhoods.

4. For any $(x, t)$, $s < t$, there is a point $z$ near the “center” of $\nu_{x,t;s}$ such that

\[ K(x, t; y, s) \leq \frac{C(\varepsilon)}{(t-s)^{n/2}} \exp \left( -\frac{d^2_s(y, z)}{(8 + \varepsilon)(t - s)} \right) \]

5. $|B(x, t, r)| \leq C(R_{\text{min}}) \exp(N_{x,t}(r^2))$

6. Reverse lower volume bound holds near concentration centers of conjugate heat kernels and under scalar curvature bounds.

7. 

---

Richard Bamler (UC Berkeley)  
Ricci flows in higher dimensions  
September 2020  
30 / 31
The picture at the first singular time

Suppose that \((M, (g_t)_{t \in [0, T)})\) develops a singularity at time \(T < \infty\).

**Singular time-slice** \((M_T, d_T)\):

\[
M_T := \{ \text{conjugate heat flows}(\mu_t)_{t \in [0, T)} : \Var(\mu_t) \leq H_n(T - t) \}
\]

\[
d_T((\mu^1_t), (\mu^2_t)) := \lim_{t \to T} d_{W^1}^{g_t}(\mu^1_t, \mu^2_t)
\]

**Theorem**

- \((M_T, d_T)\) is a complete metric space.
- If \(g_t \to g_T\) on \(U\) as \(t \to T\), then \(U \leftrightarrow U' \subset M_T\) and \(d_{g_T} \cong d_T\) locally.
- For any \(p := (\mu_t)\) any blow-ups of \((M, (g_t)_{t \in [0, T)}, (\mu_t)_{t \in [0, T]}))\) subsequentially \(\mathbb{F}\)-converge to a singular gradient shrinking soliton.