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Online class on Ricci flow this fall semester

14:10–15:30 (Pacific time)
August 27 – December 3

email me (rbamler@berkeley.edu) or check my webpage
(https://math.berkeley.edu/~rbamler) for further details
Consider a Ricci flow \((M, (g_t)_{t \in [0, T)})\) on a compact manifold \(M^n\):

\[
\partial_t g_t = -2 \text{Ric}_g_t
\]

**Important Question:** Understand the singularity formation if \(T < \infty\) (and the long-time asymptotics if \(T = \infty\))

**Blow-up analysis:**

Choose \((x_i, t_i) \in M \times [0, T)\) s.t.:

\[
t_i \nearrow T \quad |\text{Rm}(x_i, t_i)| \to \infty
\]

Hope the for some \(\lambda_i \to \infty\):

\[
(M, (\lambda_i^2 g_{\lambda_i^{-2} t + t_i}), x_i) \xrightarrow{i \to \infty} (M_\infty, (g_\infty, t)_{t \leq 0}, x_\infty)
\]

parabolic rescaling

"singularity model"

So far: curvature bounds are necessary!
singularity model = $(S^2, (2|t|g_{S^2})_{t<0})$  \( (Chow, Hamilton) \)

singularity models are $\kappa$-solutions \( \ldots \) \( (Perelman) \)

**Gradient shrinking soliton** \((M, g, f)\):
\[
\text{Ric} + \nabla^2 f - \frac{1}{2} g = 0
\]
\[
\leadsto g_t := |t|\phi^*_t g \text{ is RF, where } \phi_t = \text{flow of } |t| \nabla f, \ t < 0
\]

- $|R_m| \sim C/|t|$ (Type I)
- The singularity model of $(M, (g_t)_{t<0})$ is the flow itself.

**Type-I curvature bound** \((|R_m| \leq C/(T-t))\):
All singularity models are gradient shrinking solitons. \( (Sesum, Naber, Enders, Buzano, Topping) \)

**Type-I scalar curvature bound** \((R \leq C/(T-t))\):
All singularity models are gradient shrinking solitons with codimension 4 singular set. \( (B., Chen, Hallgren, Wang, Zhang) \)

**Folklore Conjecture**
For a general Ricci flow “most” singularity models are gradient shrinking solitons.

Goal of this talk: Verify this conjecture in a certain (possibly optimal) sense.
Examples in higher dimensions

**Appleton:** \( \exists \) RFs in dimension 4 whose blow-up limits are:

Eguchi-Hanson, \( \mathbb{R}^4/\mathbb{Z}_2 \), (Bryant soliton/\( \mathbb{Z}_2 \), \( \mathbb{R}P^3 \times \mathbb{R} \))

- **Ricci flat**
- **singular**

- **gradient shrinking soliton**

**Stolarski:** \( \exists \) RFs in dimensions \( n \geq 13 \) whose only gradient shrinking soliton blow-up limit is a Ricci flat cone

**Li, Tian, Zhu:** \( \exists \) Kähler-RF that has to develop a singularity, but cannot converge to a smooth gradient shrinking soliton.

**Conclusion:** Need to allow singular set in Folklore Conjecture + Ricci flat cones
Recall: Einstein metrics

Consider a sequence of pointed, complete Einstein manifolds \((M^n_i, g_i, x_i)\), \(\text{Ric} = \lambda_i g_i, \ |\lambda_i| \leq 1\). After passing to a subsequence we have Gromov-Hausdorff convergence to a pointed metric length space:

\[
(M^n_i, g_i, x_i) \xrightarrow{GH} (X, d, x_\infty).
\]

Suppose that the following non-collapsing condition holds:

\[|B(x_i, r)| \geq v > 0.\]

Then there is a regular-singular decomposition

\[X = \mathcal{R} \cup \mathcal{S}\]

such that:

- \(\mathcal{R}\) is an open manifold and there is a smooth Einstein metric \(g_\infty\) on \(\mathcal{R}\) such that \(d|_{\mathcal{R}} = d_{g_\infty}\). So \((X, d)\) is isometric to the metric completion of \((\mathcal{R}, d_{g_\infty})\).
- \(\dim_{\mathcal{M}} \mathcal{S} \leq n - 4\) \hspace{1cm} (Cheeger, Colding, Tian, Naber)
- Any tangent cone at any point of \(X\) is a metric cone. \hspace{1cm} (Cheeger, Colding)
- There is a filtration \(\mathcal{S}^0 \subset \ldots \subset \mathcal{S}^{n-4} = \mathcal{S}\) such that \(\dim_{\mathcal{M}} \mathcal{S}^k \leq k\) and every \(x \in \mathcal{S}^k\) has a tangent cone that splits of an \(\mathbb{R}^k\)-factor. \hspace{1cm} (Cheeger, Naber)
Main results of this talk

Similar theory for minimal surfaces, harmonic maps, mean curvature flow, harmonic map heat flow, . . .

**Key points:**

- There is a compactness and partial regularity theory for Ricci flow that is comparable to that of Einstein metrics.
- This theory allows us to establish the Folklore Conjecture and several other related results.
- We need new, parabolic versions of notions such as: “metric space”, “Gromov-Hausdorff limit”, . . .
Consider a sequence of $n$-dimensional, pointed Ricci flows:

$$(M_i, (g_i, t)_{t \in (-T_i, 0]}, (x_i, 0)), \quad T_\infty := \lim_{i \to \infty} T_i > 0.$$  

Then a subsequence $F$-converges to a metric flow over $(-T_\infty, 0]$:

$$\left(M_i, (g_i, t)_{t \in (-T_i, 0]}, (\nu_{x_i}, 0)\right) \xrightarrow{F} \left(\mathcal{X}, d, (\nu_{x_\infty})\right).$$

Suppose that the following non-collapsing condition holds:

$$\mathcal{N}_{x_i, 0}(\tau_0) \geq -Y_0 > -\infty.$$

Then we have a regular-singular decomposition

$$\mathcal{X} = \mathcal{R} \cup \mathcal{S}$$

such that:

- $\mathcal{X}$ restricted to $\mathcal{R}$ is given by a smooth Ricci flow spacetime structure and $\mathcal{X}$ is uniquely determined by this structure.
- $\dim_{\mathcal{M}^*} \mathcal{S} \leq (n + 2) - 4$.
- All tangent flows of $\mathcal{X}$ are gradient shrinking solitons with singularities.
- There is a filtration $\mathcal{S}^0 \subset \ldots \subset \mathcal{S}^{n-2} = \mathcal{S}$ such that $\dim_{\mathcal{M}^*} \mathcal{S}^k \leq k$ and every $x \in \mathcal{S}^k$ has a tangent flow that splits off an $\mathbb{R}^k$-factor or is static and splits off an $\mathbb{R}^{k-2}$-factor.
Regarding Folklore Conjecture:

**Theorem (B. 2020)**

Consider a Ricci flow \((M, (g_t)_{t \in [0, T)})\), \(T < \infty\). Then there is a metric space \((M_T, d_T) = \lim_{t \to T} (M, g_t)\) such that:

- If \(g_t \to g_T\) smoothly on \(U \subset M\), then \(U \subset M_T\) and \(d_T|_U\) is locally isometric to \(d_{g_t}|_U\).
- For any \(z \in M_T\) there is a sequence \("(x_i, t_i) \to (z, T)"\) whose "blow-up sequence produces a singular gradient shrinking soliton". Vice versa, any such sequence has a subsequence that "corresponds to a point \(z \in M_T\)."

**In dimension 4:**

**Theorem (B. 2020)**

In dimension 4 all singular gradient shrinking solitons are given by \((M, g, f)\), \(\text{Ric} + \nabla^2 f - \frac{1}{2}g = 0\), where \(M\) is an orbifold with conical singularities. Moreover, either \(R > 0\) or \((M, g) \cong \mathbb{R}^4/\Gamma\).
Regarding long-time asymptotics:

**Theorem (B. 2020)**

Suppose that \((M, (g_t)_{t \geq 0})\) is immortal and consider \((x_i, t_i) \in M \times [0, \infty), t_i \to \infty\). Then after passing to a subsequence, one of the following holds:

- \((M, (t_i^{-1} g_{t_i} t), x_i)\) converges to a singular, Einstein Ricci flow with \(\text{Ric} = -\frac{1}{2t} g_{\infty, t}\).
  - If \(n = 4\), then this flow is given by an Einstein orbifold.
- We have collapsing \(\mathcal{N}_{x_i, t_i}(t_i/2) \to -\infty\)

**Picture in dimension 4:**

If \(t \gg 1\), then

\[
M = M_{\text{thick}}(t) \cup M_{\text{almost.sing.}}(t) \cup M_{\text{thin}}(t)
\]

where:

- \(\text{Ric} \approx -\frac{1}{2t} g_t\) on \(M_{\text{thick}}(t)\)
- \(M_{\text{almost.sing.}}(t)\) consists of components \(\Omega\) with \(\partial \Omega \approx S^3/\Gamma\) and \(\partial \Omega \subset \partial M_{\text{thick}}(t)\) and diam \(\partial \Omega \ll \sqrt{t}\)
- \(M_{\text{thin}}(t)\) is locally collapsed \(\implies |B(x, t, At^{1/2})| \ll t^{n/2}\) for any \(A < \infty\).
Application: Backwards Pseudolocality

**Theorem (B. 2020)**

If \([t_0 - r^2, t_0] \subset I\) and

\[
|B(x_0, t_0, r)| \geq \alpha r^n, \quad |Rm| \leq (\alpha r)^{-2} \text{ on } B(x_0, t_0, r),
\]

then \(|Rm| \leq (\varepsilon(n, \alpha) r)^{-2} \text{ on } P(x_0, t_0; \varepsilon r, -\varepsilon r^2)\).

**Further Remarks:**

- In dimension 3, this theory essentially recovers Perelman’s theory.
- Compactness theory (not assuming non-collapsing) also holds for super Ricci flows \(\partial_t g_t + 2 \text{Ric} \geq 0\).
Let \((M, (g_t)_{t \in I})\) be a Ricci flow and \(u, v \in C^2(M \times I)\).

**Heat equation:** \(\Box u = (\partial_t - \triangle_{g_t})u = 0\)

**Conjugate heat equation:** \(\Box^* v = (-\partial_t - \triangle_{g_t} + R_{g_t})v = 0\)

**Heat kernel:** \(K(x, t; y, s), \quad x, y \in M, \quad s < t\)

for fixed \((y, s)\): \(\Box K(\cdot, \cdot; y, s) = 0, \quad \lim_{t \searrow s} K(\cdot, t; y, s) = \delta_y\)

for fixed \((x, t)\): \(\Box^* K(x, t; \cdot, \cdot) = 0, \quad \lim_{s \nearrow t} K(x, t; \cdot, s) = \delta_x\)

**Representation formulas:** If \(\Box u = \Box^* v = 0\), then
\[
\begin{align*}
  u(x, t) &= \int_M K(x, t; \cdot, s)u(\cdot, s)dg_s \\
  v(y, s) &= \int_M K(\cdot, t; y, s)v(\cdot, t)dg_t
\end{align*}
\]

**Reproduction formula for heat kernel:** \(s < t' < t\)
\[
K(x, t; y, s) = \int_M K(x, t; \cdot, t')K(\cdot, t'; y, s)dg_{t'}
\]
Properties of heat equation:

- $u \leq C$ and $u \geq -C$ are preserved.
- $|\nabla u| \leq C$ are preserved.
- Let $\Phi : \mathbb{R} \times \mathbb{R}_{\geq 0}$ be the solution of the 1-dimensional heat equation $\partial_t \Phi_t = \Phi''_t$ with initial condition $\Phi_0 = \chi[0, \infty)$.

Improved gradient estimate (B. 2020)

If $0 < u(\cdot, t_0) < 1$, then for $t > t_0$

$$u_t(x) = \Phi_t(x') \implies |\nabla u_t|(x) \leq \Phi_{t-t_0}'(x') \quad (\ast)$$

Moreover, $(\ast)$ is preserved for any fixed $t_0$.

Properties of conjugate heat equation:

- $v \geq 0$ is preserved.
- $\int_M v(\cdot, s)dg_s$ is constant in $s$ and $\int_M K(x, t; \cdot, s)dg_s = 1$.
- Think of $v$ as $\mu_s = v(\cdot, s)dg_s$.
Conjugate heat kernel probability measure:

\[ d\nu_{x,t; s} := K(x, t; \cdot, s) dg_s, \quad \nu_{x,t; t} := \delta_x \]

Integral characterization of (conjugate) heat flows:

Heat flow: \( \Box u = 0 \iff u(x,t) = \int_M u(\cdot, s) d\nu_{x,t; s} \)

Conjugate heat flow:

\[ d\mu_s = \nu(\cdot, s) dg_s, \quad \Box^* \nu = 0 \iff \mu_s = \int_M \nu(\cdot,t; s) d\mu_t \]

Reproduction formula:

\[ \nu_{x,t; s} = \int_M \nu_{\cdot,t'; s} d\nu_{x,t; t'} \]
Metric flows

Metric flow over an interval $I$

$$\mathcal{X} = (\mathcal{X}, t, (d_t)_{t \in I}, (\nu_{x;s})_{x \in \mathcal{X}, s \in I, s \leq t(x)})$$

1. $\mathcal{X}$ is a set consisting of points
2. $t : \mathcal{X} \to I$ is the time-function and its level sets $\mathcal{X}_t := t^{-1}(t)$ are time-slices
3. $(\mathcal{X}_t, d_t)$ is a complete and separable metric space for all $t \in I$
4. $\nu_{x;s}$ are probability measures called conjugate heat kernel and satisfy $\nu_{x;t(x)} = \delta_x$ and the reproduction formula

$$\nu_{x;s} = \int_{\mathcal{X}_t} \nu_{x,t;s} d\nu_{x;t}$$

5. (Conjugate) heat flows are defined using the integral property as before.
6. We require that the improved gradient estimate holds for heat flows:
   If $u_{t_0} = \Phi_{t_0} \circ f_{t_0}$ for some 1-Lipschitz $f_{t_0} : \mathcal{X}_{t_0} \to \mathbb{R}$, then for all $t \geq t_0$ we have $u_t = \Phi_t \circ f_t$ for some 1-Lipschitz $f_t : \mathcal{X}_t \to \mathbb{R}$. 

**Ricci flow** \((M,(g_t)_{t \in I}) \rightarrow \text{Metric flow } \mathcal{X}\)

- \(\mathcal{X} := M \times I\)
- \(t := \text{projection onto second factor.}\)
- \(d_t := d_{g_t} \text{ on } \mathcal{X}_t = M \times \{t\}\)
- \(d\nu_{(x,t);s} := K(x, t; \cdot, s)dg_s\)

**Note:**

- The distance between points in different time-slices is not defined!
- This construction forgets worldlines \(t \mapsto (x, t)\).
  Instead: For \(x \in \mathcal{X}_t\) there is a probability distribution \(\nu_{x; s}\) of points \(y \in \mathcal{X}_s\) that lie in the “past” of \(x\).
Concentration property

Variance of probability measure $\mu$ on a metric space $(X, d)$:

$$\text{Var}(\mu) := \int_X \int_X d^2(x, y) d\mu(x) d\mu(y)$$

Theorem (B. 2020)

On any Ricci flow

$$\text{Var}(\nu_{x,t;s}) \leq H_n(t - s),$$

where $H_n := \frac{(n-1)\pi^2}{2} + 4$.

A metric flow $\mathcal{X}$ is called $H$-concentrated if $(\ast) + \ldots$ holds for $H_n = H$.

"The past in $\mathcal{X}_s$ of any point $x \in \mathcal{X}_t$ is determined up to an error of $\sim \sqrt{t - s}$."

Richard Bamler (UC Berkeley)  
Ricci flows in higher dimensions  
August 2020  
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1-Wasserstein distance

$\mu_1, \mu_2$ probability measures on complete, separable metric space $(X, d)$

$$d_{W_1}(\mu_1, \mu_2) := \inf_{q \text{ coupling btw } \mu_1, \mu_2} \int_{X \times X} d \, dq = \sup_{f : X \to \mathbb{R}, \text{1-Lipschitz}} \int_X f \, d(\mu_1 - \mu_2)$$

Lemma

If $x, y \in \mathcal{X}_t$, then for $s \leq t$ we have

$$d_{W_1}^{\mathcal{X}_s}(\nu_{x;s}, \nu_{y;s}) \leq d_t(x, y).$$

Moreover, $s \mapsto d_{W_1}^{\mathcal{X}_s}(\nu_{x;s}, \nu_{y;s})$ is non-decreasing and the same is true for any other pair of conjugate heat flows.

“Distances don’t shrink on metric flows (in a probabilistic sense)”
Parabolic balls

Conventional parabolic ball in a Ricci flow:

\[ P(x_0, t_0; r) := B_{g_{t_0}}(x_0, r) \times [t_0 - r^2, t_0 + r^2] \]

\( P^\ast \)-parabolic ball in a metric flow:

\[ P^\ast(x_0; r) := \left\{ x \in X_{t_0} : \begin{array}{l} t(x) \in [t_0 - r^2, t_0 + r^2] \\ d_{W_1}^{X_{t_0-r^2}}(\nu_{x_0};t_0-r^2,\nu_x;t_0-r^2) < r \end{array} \right\} \]

- standard containment properties still hold for \( P^\ast \)-parabolic balls (e.g. \( P^\ast(x; r_1) \subset P^\ast(x; r_2) \) if \( r_1 \leq r_2 \))
- Conventional and \( P^\ast \)-parabolic balls are comparable if curvature bounded.
- The natural topology on \( X \) is generated by the set of all \( P^\ast \)-parabolic balls.
- \( P^\ast \)-parabolic balls allow the definition of the parabolic Hausdorff and Minkowski dimension \( \dim_{\mathcal{H}^*} \) and \( \dim_{\mathcal{M}^*} \).

We count the time-direction twice!
Gromov-\(W_1\)-distance and convergence

**Gromov-\(W_1\)-distance**

If \((X_i, d_i, \mu_i), \ i = 1, 2\), are two normalized metric measure spaces, then

\[
d_{GW_1}((X_1, d_1, \mu_1), (X_2, d_2, \mu_2)) := \inf_{\varphi_1, \varphi_2, Z} d_{W_1}^Z((\varphi_1)_* \mu_1, (\varphi_2)_* \mu_2),
\]

where the infimum is taken over all isometric embeddings \(\varphi_i : (X_i, d_i) \to (Z, d_Z)\) into a common metric space \((Z, d_Z)\).

**Gromov-\(W_1\)-convergence**

\[
(X_i, d_i, \mu_i) \xrightarrow{GW_1} (X_\infty, d_\infty, \mu_\infty)
\]

Compare with pointed Gromov-Hausdorff convergence: The probability measures \(\mu_i\) take the role of the basepoint.
$d_F$-distance and $F$-convergence

$d_F$-distance:
Consider metric flows $\mathcal{X}_i$, $i = 1, 2$ equipped with conjugate heat flows $(\mu_i, t)_{t \in I}$. We define

$$d_F(\mathcal{X}^1_t, (\mu^1_t)_{t \in I}, \mathcal{X}^2_t, (\mu^2_t)_{t \in I})$$

to be the infimum over all $r > 0$ such that there are isometric embeddings

$$(\varphi^i_t : (\mathcal{X}^i_t, d^i_t) \to (Z_t, d^Z_t))_{t \in I \setminus E, i = 1, 2}$$

with:

1. $|E| \leq r^2$
2. $d^Z_t((\varphi^1_t)_* \mu^1_t, (\varphi^2_t)_* \mu^2_t) \leq r$ for all $t \in I \setminus E$
3. “integral $W_1$-closeness of conjugate heat kernels between times $s, t \in I \setminus E$”

$F$-convergence
If $d_F(\mathcal{X}^i_t, (\mu^i_t)_{t \in I}, \mathcal{X}^\infty_t, (\mu^\infty_t)_{t \in I}) \to 0$, then we write

$$(\mathcal{X}_i, (\mu_i, t)_{t \in I}) \xrightarrow{\mathbb{F}} (\mathcal{X}^\infty_t, (\mu^\infty_t)_{t \in I})$$

This implies Gromov-$W_1$-convergence at almost every time.
Let $F_I$ be the space of pairs $(X, (\mu_t)_{t \in I})$.

**Theorem (B. 2020)**

$(F_I, d_F)$ is a complete metric space.

Suppose $I = (-T, 0]$. Fix $n$.

**Theorem (B. 2020)**

\[
\left\{ (X, (\mu_t)_{t \in I}) \text{ corresponding to Ricci flows } (M^n, (g_t)_{t \in I}, (\nu_{x,0}; t)_{t \in I}) \right\} \subset F_I \text{ is precompact.} \quad (*)
\]

**Corollary**

For any sequence of $n$-dimensional, pointed Ricci flows $(M_i, (g_{i,t})_{t \in (-T,0]}, (x_i,0))$ there is a subsequence such that:

\[
(M_i, (g_{i,t})_{t \in (-T,0]}, (\nu_{x_i,0})) \xrightarrow{F \quad i \rightarrow \infty} (X, (\nu_{x_{\infty}})).
\]

**Remark:** There is a compact subset $F^*_I(H) \subset F_I$, essentially corresponding to all $H$-concentrated metric flows, that contains the subset from $(*)$. 
Digesting $F$-convergence

If we assume curvature bounds, then: $F$-convergence $\iff$ local smooth convergence in the sense of Cheeger, Gromov, Hamilton.

**Example:** Bryant soliton $(M_{Bry}, (g_{Bry},t)_{t \in \mathbb{R}}, x_{Bry})$

- rotational symmetric
- $g_{Bry,t} = dr^2 + f^2(r)g_{S^2}$, where $f(r) \sim \sqrt{r}$
- steady gradient soliton $\implies$ all time-slices are isometric

Consider blow-downs $(M_{Bry}, (\lambda_i^2 g_{Bry}, \lambda_i^{-2}t)_{t \in \mathbb{R}}, x_{Bry})$ for $\lambda_i \to 0$.

- Gromov-Hausdorff limit at any fixed time: $[0, \infty)$
- $F$-limit: round shrinking cylinder $(S^2 \times \mathbb{R}, (g_t = 2|t|g_{S^2} + g_{\mathbb{R}})_{t<0})$ this is the asymptotic soliton!
Ricci flow spacetimes

Ricci flow spacetime over an interval $I$:

$$\mathcal{M} = (\mathcal{M}, t, \partial_t, g)$$

1. $\mathcal{M}$ is a smooth $(n+1)$-manifold, called **spacetime manifold**
2. $t: \mathcal{M} \rightarrow I$ is a smooth map whose level sets $\mathcal{M}_t := t^{-1}(t)$ are called **time-slices**.
3. $\partial_t$ is a smooth vector field on $\mathcal{M}$ with $\partial_t t = 1$. Its trajectories are **worldlines**.
4. $g$ is a metric on the horizontal distribution $\ker dt \subset T\mathcal{M}$
5. **Ricci flow equation:** $\mathcal{L}_{\partial_t} g = -2 \text{Ric}_g$

Ricci flow $\left(\mathcal{M}, (g_t)_{t \in I}\right) \rightarrow$ Ricci flow spacetime $\mathcal{M}$

- $\mathcal{M} := \mathcal{M} \times I$
- $t :=$ projection onto second factor
- $\partial_t :=$ std. vector field on $I$
- $g := g_t$ on $\mathcal{M}_t = \mathcal{M} \times \{t\}$
Structure of non-collapsed $\mathbb{F}$-limits

Let $\mathcal{X}$ be a $\mathbb{F}$-limit of smooth Ricci flows over $I$.
Assume the non-collapsing condition $\mathcal{N}_{x_i,0}(\tau_0) \geq -Y_0 > -\infty$.

**Theorem (B. 2020)**

There is a decomposition

$$\mathcal{X} = \mathcal{R} \cup \mathcal{S}$$

and a smooth Ricci flow spacetime structure $(\mathcal{R}, t, \partial_t, g)$ on $\mathcal{R}$ such that:

- $\mathcal{R} \subset \mathcal{X}$ is open and dense.
- For any $t \in I$ the time-slice $(\mathcal{X}_t, d_t)$ is the metric completion of $(\mathcal{R}_t, d_{g_t})$.
- (Conjugate) heat flows restricted to $\mathcal{R}$ are uniquely characterized by $\Box u = 0$ and $\Box^* \nu = 0$ on $\mathcal{R}$.
- $\dim_{\mathcal{M}^*} \mathcal{S} \leq (n + 2) - 4$
- Tangent flows at any $x \in \mathcal{X}$ (\(= \mathbb{F}\)-limits of blow-ups of $(\mathcal{X}, (\nu_{x,t}))$) are singular gradient shrinking solitons.
- There is a filtration $\mathcal{S}^0 \subset \ldots \subset \mathcal{S}^{n-2} = \mathcal{S}$ such that $\dim_{\mathcal{M}^*} \mathcal{S}^k \leq k$ and every $x \in \mathcal{S}^k$ has a tangent flow that splits off an $\mathbb{R}^k$-factor or is static and splits off an $\mathbb{R}^{k-2}$-factor.
Theorem (B. 2020)

If $\mathcal{X}$ is a gradient shrinking soliton, then there is an identification

$$\mathcal{X} = \mathcal{X} \times I$$

for a metric space $(\mathcal{X}, d)$ with regular part $\mathcal{R}_X \subset \mathcal{X}$ such that:

- $(\mathcal{X}_t, d_t) = (\mathcal{X}, |t|^{1/2} d)$
- $(\mathcal{R}_t, g_t) = (\mathcal{R}_X, |t|g_{\mathcal{R}_X})$

The soliton equation holds on $\mathcal{R}_X$.

If $n = 4$, then $(\mathcal{X}, d)$ is the length space of a smooth orbifold.
Outstanding promise: Non-collapsing condition

**Pointed Nash entropy:** \((\text{Perelman, Topping, Hein, Naber})\)

Fix \((x_0, t_0) \in M \times I\) and write \(\tau := t_0 - t\),

\[ K(x_0, t_0; \cdot, \cdot) =: (4\pi\tau)^{-n/2}e^{-f} \]

\[
\mathcal{N}_{x_0, t_0}(\tau) := \int_M f(\cdot, t_0 - \tau)d\nu_{x_0, t_0; t_0 - \tau} - \frac{n}{2}
\]

**Basic properties:**

- \(\mathcal{N}_{x_0, t_0}(\tau) \leq 0\)
- \(\frac{d}{d\tau}\mathcal{N}_{x_0, t_0}(\tau) \leq 0\)
- There is a relation between \(\mathcal{N}\) and Perelman’s \(\mu\)-entropy that implies: If \(I = [0, T]\), then

\[
\mathcal{N}_{x_0, t_0}(\tau) \geq \mu[M, g_0, T] > -\infty.
\]

So a non-collapsing condition always holds on a fixed flow with \(T < \infty\).
Guiding principle: On a manifold with $\text{Ric} \geq -g$: 

$$\frac{|B(x, r)|}{r^n} \approx e^{N_x(r^2)}$$

Theorem (B. 2020)

Suppose that $R \geq R_{\text{min}}$. Set $N^*_s(x, t) := N_{x, t}(t - s)$.

1. $|\nabla N^*_s| \leq \sqrt{\frac{n}{2(t - s)}} - R_{\text{min}}$

2. $-\frac{n}{2(t - s)} \leq \Box N^*_s \leq 0$

3. (1)+(2) imply a bound on $\text{osc} N^*_s$ over $P^*$-parabolic neighborhoods.

4. For any $(x, t)$, $s < t$, there is a point $z$ near the “center” of $\nu_{x, t; s}$ such that

$$K(x, t; y, s) \leq \frac{C(\varepsilon)}{(t - s)^{n/2}} \exp \left( -\frac{d^2_s(y, z)}{(8 + \varepsilon)(t - s)} \right)$$

5. $|B(x, t, r)| \leq C(R_{\text{min}}) \exp(N_{x, t}(r^2))$

6. Reverse lower volume bound holds near concentration centers of conjugate heat kernels and under scalar curvature bounds.

7. ...
The picture at the first singular time

Suppose that \((M,(g_t)_{t\in[0,T)})\) develops a singularity at time \(T < \infty\).

**Singular time-slice** \((M_T,d_T)\):

\[
M_T := \{ \text{conjugate heat flows}(\mu_t)_{t\in[0,T)} : \text{Var}(\mu_t) \leq H_n(T-t) \}
\]

\[
d_T((\mu^1_t),(\mu^2_t)) := \lim_{t \uparrow T} d_{W_1}^g(\mu^1_t, \mu^2_t)
\]

**Theorem**

- \((M_T,d_T)\) is a complete metric space.
- If \(g_t \to g_T\) on \(U\) as \(t \uparrow T\), then \(U \leftrightarrow U' \subset M_T\) and \(d_{g_T} \cong d_T\) locally.
- For any \(p := (\mu_t)\) any blow-ups of \((M,(g_t)_{t\in[0,T)}),(\mu_t)_{t\in[0,T)}\) subsequentially \(\mathbb{F}\)-converge to a singular gradient shrinking soliton.