TROPICAL INTERSECTIONS: WHERE THEY GO WRONG, AND WHERE THEY GO RIGHT

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1. Introduction

Tropical geometry is a powerful tool for understanding curves and other varieties. Often described as a skeletonized version of algebraic geometry, it reduces potentially ineffable objects to piecewise-linear ones. These combinatorial objects are often much easier to analyze than the originals, and can be used to piece together information about the schemes from whence they came. Understanding precisely when information can be lifted is therefore of the utmost importance.

A question in this vein is when tropicalization commutes with intersection. For two curves (or varieties, or schemes, depending on your preference) $X$ and $X'$, we know that $\text{Trop}(X \cap X') \subset \text{Trop}(X) \cap \text{Trop}(X')$; that is, actual intersection points are still intersection points in the tropicalization. However, the opposite direction of containment does not always hold: tropical curves may have intersection points not corresponding to points of the original curves. After outlining when intersections are nice (essentially when codimensions are nice), we will delve into some not-so-nice intersections, which can include components of positive dimension and even noncompact components. Fortunately, the work of [OR] allows us to make some sense out of these “bad” cases, by use of judicious compactification and multiplicity counting. So perhaps a more accurate but less catchy title for this project would have been Tropical Curves: Where They Go Right, and Where They Go Wrong, and Where They Go Kind Of OK After All. After outlining these results and methods, we offer examples illustrating various cases of intersections, and explore the freedom and restrictions we have for where “actual intersections” map to in the tropical intersections.

We will first establish our notation and conventions in accordance with [OR], and then mention the specific conditions we’ll usually be keeping in mind throughout the paper. Let $K$ be a non-Archimedean field with valuation $\text{val} : K \to \mathbb{R} \cup \{\infty\}$, and take $K$ to be complete or algebraically closed. Let $\mathbf{T} \cong \mathbf{G}_m^n$ be a finite-rank split torus over $K$ with coordinate functions $x_1, \ldots, x_n$. We define the tropicalization map from the closed points of $\mathbf{T}$ to $\mathbb{R}^n$ by $\text{trop} : |\mathbf{T}| \to \mathbb{R}^n$, $\text{trop}(\xi) = (\text{val}(x_1(\xi)), \ldots, \text{val}(x_n(\xi)))$. For a closed subscheme $X \subset \mathbf{T}$, the tropicalization of $X$, written trop($X$), is the (Euclidean) closure of the set trop($|X|$) in $\mathbb{R}^n$.

For our intents and purposes, we will usually take $K$ to be the field of Puiseux series over $\mathbb{C}$ with uniformizer $t$. Recall that this is the set of all sums $\sum_{s \in S \subseteq \mathbb{Q}} c_k t^s$, where $S$ has a least element and there is a bound on the denominators of the elements of $S$, and where $c_k \in \mathbb{C}$. This field is algebraically closed, and has $\text{val}(0) = \infty$ and $\text{val} \left( \sum_{s \in S \subseteq \mathbb{Q}} c_k t^s \right) = \min \{s \mid s \in S\}$.
for all nonzero elements. We will also generally take \( n = 2 \). However, we will state the results of [OR] in their full generality.

As a general idea for the types of intersections we’ll be considering, Figure 1 shows three situations: a transverse intersection, an intersection in a compact, dimension 1 component, an an intersection in a noncompact dimension 1 component.

![Figure 1. The good (transverse intersection), the bad (positive dimension intersection), and the ugly (positive dimension noncompact intersection).](image)

For transverse intersections, everything is nicely behaved, with tropical intersections lifting to points of intersection of the original curves (this will be made more precise and more general in Section 2); in the first picture, the blue curve is a line and the red curve is a conic, and as expected there are two intersections. In the latter two cases, the intersection has a higher dimensional connected component, but we can still make sense out of the tropical intersection \( \text{Trop}(X) \cdot \text{Trop}(X') \) by translating \( \text{Trop}(X) \) by a generic vector \( v \) and taking the limit as \( v \) goes to 0 (see Section 3); for instance, in the second picture, these are the two points of convergence of rays. Unfortunately, \( \text{Trop}(X \cdot X') \) no longer need be equal to \( \text{Trop}(X) \cdot \text{Trop}(X') \). For instance, the third picture could have arisen from \( X = \{x + y = 1\} \) and \( X' = \{tx + y = 1\} \), giving a stable tropical intersection point at \((0,0)\) even though \( X \) and \( X' \) do not intersect in \( T \) (see Example 3.1). This situation is dealt with in [OR] using compactifying fans, though we lose the ability to say precisely where the point is located. In Section 3 we outline this process, and in Section 4 we explore the question of where the intersections of \( X \) and \( X' \) can map to in the connected components of \( \text{Trop}(X) \cap \text{Trop}(X') \) via examples.

2. Nice Intersections

Much work has been done on cases where \( \text{Trop}(X) \cap \text{Trop}(X') = \text{Trop}(X \cap X') \). A key result due to [BJSST] (their Lemma 3.2) is that in the case of transverse intersections, we have commutativity of tropicalization and intersection.
Theorem 2.1 (Bogart, Jensen, Speyer, Sturmfels, Thomas (2007)). Let \( I \) and \( J \) be ideals in \( \mathbb{C}[x_1, ..., x_n] \). If their tropical varieties meet at a point of \( \mathbb{R}^n \), then that point is contained in the tropicalization of \( I + J \).

Although this context and language are slightly different from those we’re considering, the result will still hold. Indeed, this result was generalized in [OP] (their Theorem 1.1).

We say that \( \text{Trop}(X) \) and \( \text{Trop}(X') \) meet properly at a point \( w \) if \( \text{Trop}(X) \cap \text{Trop}(X') \) has codimension \( \text{codim} X + \text{codim} X' \) in a neighborhood of \( w \).

Theorem 2.2 (Osserman, Payne (2011)). If \( \text{Trop}(X) \) meets \( \text{Trop}(X') \) properly at \( w \), then \( w \) is contained in the tropicalization of \( X \cap X' \).

Indeed, [OP] goes beyond this. Theorem 2.2 considers the codimensions of \( X \) and \( X' \) as they sit inside \( T \). If \( X \) and \( X' \) are both subvarieties of another variety \( Y \) inside \( T \), there is no hope for \( X \) and \( X' \) to have the desired codimensions. However, we can say that \( \text{Trop}(X) \) and \( \text{Trop}(X') \) meet properly at a point \( w \) in \( \text{Trop}(Y) \) if the intersection \( \text{Trop}(X) \cap \text{Trop}(X') \subset \text{Trop}(Y) \) has pure codimension \( \text{codim}_Y X + \text{codim}_Y X' \). Call a point in \( \text{Trop}(Y) \) simple if it lies in the interior of a facet of multiplicity 1 (considering \( \text{Trop}(Y) \) as the underlying set of a polyhedral complex of pure dimension \( \dim Y \)).

Theorem 2.3 (Osserman, Payne (2011)). If \( \text{Trop}(X) \) meets \( \text{Trop}(X') \) properly at a simple point \( w \) of \( Y \), then \( w \) is contained in the tropicalization of \( X \cap X' \).

These are the ideal cases, where we may lift tropical intersections without hindrance. The object of our focus will be the cases where intersections do not have this nice property. As we will see, even when points may not be lifted, we can still make sense out of the tropical intersections after suitable modifications.

3. Handling Not-so-nice Intersections

In [OR], the tropical intersections we are interested in are ones with \( \text{codim} X + \text{codim} X' = \dim(T) \), but where \( \text{Trop}(X) \cap \text{Trop}(X') \) may have higher dimensional connected components. The following motivating example is offered in this paper.

Example 3.1. Let \( X = \{x + y = 1\} \) and \( X' = \{tx + y = 1\} \), whose tropicalizations intersect in the unbounded say consisting of the nonnegative \( x \)-axis, as illustrated in Figure 2. These have stable tropical intersection at the point \((0, 0)\) with multiplicity 1 (intuitively, this is because a generic \( \varepsilon \)-shift gives an intersection near there), but \( X \cap X' \) is the empty set (as any intersection would require \( x = 0 \)).

To handle this situation, we wish to compactify in the direction of the the ray. To do this, we view \( X, X' \) as curves in \( \mathbb{A}^1 \times \mathbb{G}_m \) instead of \( \mathbb{G}_m^2 \) and extend the tropicalization map to \( \text{trop} : |\mathbb{A}^1 \times \mathbb{G}_m| \to (\mathbb{R} \cup \{\infty\}) \times \mathbb{R} \) (we’re adding infinity so that 0 may be the first component, as \( \text{val}(0) = \infty \)). Now we may solve \( X \) and \( X' \) simultaneously to find the intersection point \((0, 1)\) (now allowed), and its tropicalization, namely \((\infty, 0)\), is indeed contained in the closure of \( \text{Trop}(X) \cap \text{Trop}(X') \) in \( \mathbb{R} \cup \{\infty\} \times \mathbb{R} \).
Much of the work of [OR] is in making precise the notion of *compactification in the directions where the tropicalization is infinite*. In the end, we say that an integral pointed fan $\Delta$ is a *compactifying fan* for a polyhedral complex $\Pi$ if the recession cone of each cell of $\Pi$ is a union of cones in $\Delta$.

We now describe the set-up of their main theorem.

First some notation: let $X_1, \ldots, X_m \subset T$ be pure-dimensional subschemes whose codimensions add to $\dim(T)$, whose tropicalizations’ intersection contains a connected component $C$. We let $\Pi$ be the underlying polyhedral complex of $C$, and let $\Delta$ be a compactifying fan for $\Pi$. As usual, $M$ will denote the lattice of characters of $T$ and $N$ is its dual lattice, meaning that the $\text{Trop}(X_i)$’s live in $N_\mathbb{R} = N \otimes \mathbb{Z} \mathbb{R}$.

Now to start compactifying. We partially compactify the torus with the toric variety $X(\Delta)$, which contains $T$ as a dense open subscheme. The extended tropicalization $N_\mathbb{R}(\Delta)$ canonically contains $N_\mathbb{R}$ as a dense open subset, and we have a map $\text{trop} : |X(\Delta)| \to N_\mathbb{R}(\Delta)$ extending $\text{trop} : |T| \to N_\mathbb{R}$. We let $\overline{C}$ denote the closure of $C$ in $N_\mathbb{R}(\Delta)$. The fact that $\Delta$ is a compactifying fan for $\Pi$ means that $\overline{C}$ is compact (explained in detail in [OR]’s Remark 3.3). This $\overline{C}$ ends up being the appropriate compactification of the connected component $C$.

As one final piece of bookkeeping, we need to establish notation for counting intersection multiplicity. For an isolated point $\xi \in \bigcap_{i=1}^m X_i$, let $i_K(\xi, X_1, \ldots, X_m; X(\Delta))$ denote the multiplicity of $\xi$ in the intersection class $\overline{X_1} \cdots \overline{X_m}$. On the tropical side of things, for $v \in \bigcap_{i=1}^m \text{Trop}(X_i)$, let $i(v, \text{Trop}(X_1) \cdots \text{Trop}(X_m))$ denote the stable tropical intersection $\text{Trop}(X_1) \cdots \text{Trop}(X_m)$.

We are now ready to state the main result of [OR].
**Theorem 3.2** (Osserman, Rabinoff (2011)). If $X(\Delta)$ is smooth, and if there are only finitely many points of $|\overline{X_1} \cap \cdots \cap \overline{X_m}|$ mapping to $\overline{C}$ under $\text{trop}$, then

$$\sum_{\xi \in |\bigcap_{i=1}^{m} X_i| \cap \text{trop}(\xi) \cap \overline{C}} i_K(\xi, X_1, \ldots, X_m; X(\Delta)) = \sum_{v \in C} i(v, \text{Trop}(X_1) \cdots \text{Trop}(X_m)).$$

This theorem should be interpreted as a lifting theorem for unbounded components of tropical intersections, with the caveat that we might need to compactify to find a point (or points) to lift to, as in Example 3.1. A very nice corollary is the case where the connected component $C$ was already compact (for instance, the middle picture of Figure 1), in which case no compactification was required. In this case we have that there are only finitely many points of $|\overline{X_1} \cap \cdots \cap X_m|$ mapping to $C$ under $\text{trop}$, and the equation becomes

$$\sum_{\xi \in |\bigcap_{i=1}^{m} X_i| \cap \text{trop}(\xi) \cap \overline{C}} i_K(\xi, X_1, \ldots, X_m; \text{Trop}(\xi)) = \sum_{v \in C} i(v, \text{Trop}(X_1) \cdots \text{Trop}(X_m)).$$

## 4. Examples

In these examples we seek to explore where the points of $X \cap X'$ can map to in $\text{Trop}(X) \cap \text{Trop}(X')$. Generally we will look at a family of examples and observe the restrictions on configurations of points, and then give explicit examples to show that the possible situations do indeed occur. We will be working with $K$ the field of Puiseux series over $\mathbb{C}$, and with $n = 2$.

**Example 4.1.** We wish to consider curves that tropicalize to the same line $x \oplus y \oplus 0$. We will therefore consider $X : \{x + ay + b = 0\}$, $X' : \{x + cy + d = 0\}$, where $\text{val}(a) = \ldots = \text{val}(d) = 0$. (This is a very simple example, but we’ll go through it to set the stage for later examples.)

Under tropicalization these lines both map to $x \oplus y \oplus 0$, giving us a tropical line sitting on top of itself. Generically $X$ and $X'$ will have a single intersection point $p$ with nonzero coordinates so that $\text{trop}(p)$ lies on the tropical line. Solving our equations gives $p = (\frac{b-d}{a-c} - b, -\frac{b-d}{a-c})$. As expected based on what $\text{Trop}(X) \cap \text{Trop}(X') = \text{Trop}(X)$ looks like, there are several possibilities.

- If $\text{val}(\frac{b-d}{a-c}) = 0$, then we have $\text{val}(y(p)) = 0$ and $\text{val}(x(p)) = \min\{\text{val}(\frac{b-d}{a-c}), \text{val}(b)\} = 0$.
- If $\text{val}(\frac{b-d}{a-c}) > 0$, then we have $\text{val}(y(p)) > 0$ and $\text{val}(x(p)) = \text{val}(\frac{b-d}{a-c} - b) = \text{val}(b) = 0$.
- If $\text{val}(\frac{b-d}{a-c}) < 0$, then we have $\text{val}(y(p)) < 0$ and $\text{val}(x(p)) = \text{val}(\frac{b-d}{a-c} - b) = \text{val}(\frac{b-d}{a-c}) = \text{val}(y(p))$.

These three cases indeed all describe points on $\text{Trop}(X) \cap \text{Trop}(X')$. It is not too hard to construct explicit curves that give any desired point on this tropical line as $\text{trop}(p)$ (so long as that point is rational, since $\text{val}$ maps $K$ to $\mathbb{Q}$). Let $r \geq 0$ be a rational number.

- For $\text{trop}(p)$ to be at $(r, 0)$, let $X : \{x + y + 1 = 0\}$, $X' : \{x + 2y + 2 = 0\}$.
- For $\text{trop}(p)$ to be at $(0, r)$, let $X : \{x + y + 1 + t^r = 0\}$, $X' : \{x - y - 1 = 0\}$.
• For trop($p$) to be at $(-r, -r)$, let $X : \{x + (1 + t^r)y + 1 = 0\}$, $X' : \{x + y + 2 = 0\}$.

These examples are rigged to give the corresponding cases, and when the dust settles the valuation of the coordinates of $p$ are as desired.

Note that it is certainly possible to rig $X$ and $X'$ so that they have no intersection in $\mathbb{T}$; just make the equations be only solvable with a $p$-coordinate being 0 (say, $x + y + 1 = 0$ and $x + 2y + 1 = 0$).

**Example 4.2.** We wish to consider curves that tropicalize to $x \oplus (y \odot 1) \oplus -1$ and $x \oplus y \oplus (x \odot y)$, which intersect in the line segment $0 \times [-1, 0]$, as shown in Figure 3. We will therefore consider $X = \{x + aty + b = 0\}$ and $X' = \{x + cy + dxy = 0\}$, where val($a$) = ... = val($d$) = 0. (The set-up of this example was shown to me by Melody Chan, as relayed from a discussion with Joe Rabinoff, Matt Baker, and Bernd Sturmfels, amongst others.)

![Figure 3](image_url)

**Figure 3.** For clarity, the line and the conic are shown intersecting transversely, and then shifted to the case of Example 4.2. Their intersection, which is bounded as shown, is the rightmost image.

The curves $X$ and $X'$ intersect in two points in $\mathbb{T}$ (if one of their intersection points would have 0 as a coordinate, then this forces the point to be $(0, 0)$ from $x + cy + dxy = 0$, which is not a point of $X$). We wish to understand where these two points $p$ and $q$ can map in $\text{Trop}(X) \cap \text{Trop}(X')$. Solving $x + aty + b = 0$ for $x$ and plugging into $x + cy + dxy = 0$, we have

$$-adty^2 + (c - bd - at)y - b = 0.$$ 

The Newton polygon of this polynomial depends on the valuation of $c - bd - at$. In particular, if val($c - bd - at$) $\geq 1/2$, the Newton polygon is a straight line segment from $(0, 0)$ to $(1, 2)$, meaning the two roots have valuation $-1/2$ (negative the slope). If, however, val($c - bd - at$) = $r$ where $0 \leq r \leq 1/2$, the Newton polygon has a line segment from $(0, 0)$ to $(1, r)$ and a line segment from $(1, r)$ to $(2, r)$, giving that the two roots have valuation $-r$ and $-1 + r$.

Here we have nontrivial structure between the images of $p$ and $q$: their $y$ coordinates must...
satisfy a sort of balance, being symmetric about \((0, -1/2)\), the midpoint of the line segment making up \(\text{Trop}(X) \cap \text{Trop}(X')\). This is illustrated in Figure 4.

\[ -dx^2 + (at - c - bd)x - bc = 0, \]
whose Newton polygon is the horizontal line from \((0, 0)\) to \((2, 0)\), meaning both intersection points have \(x\)-coordinate with valuation 0.

Having shown these constraints exist, let us construct examples realizing every possible configuration. Let \(r\) between 0 and 1/2 be a rational number. Set

\[ X = \{x + ty + 1 = 0\} \]
\[ X' = \{x + (1 + t')y + xy = 0\}. \]

With the notation above, we have that \(c - bd - at = 1 + t'^2 - 1 - t^2 = t'^2 - t^2\) has valuation \(r\), which as argued above gives \(y\) coordinates with valuations \(-r\) and \(-1 + r\). This gives the points \((0, -r)\) and \((0, -1 + r)\) under tropicalization.

We now move on to some examples of higher degree (conic with conic and line with cubic), and the families of curves considered will not be as complete as in the previous two examples. In particular, there are many weightings we could choose on the coefficients of our polynomials to give the same Newton polytopes and hence the same tropical curves, at least set-theoretically. We focus on several particular choices of weights that are chosen for tractability and seeing what degrees of freedom we have.
Example 4.3. For this example we will consider the curves of the form $X = \{ x + ay + bxy = 0 \}$ and $X' = \{ x + cy + dxy + t(ex^2 + fy^2 + g) = 0 \}$, where $a$ through $g$ have valuation 1. As shown in Figure 6, Trop($X$) and Trop($X'$) intersect in a three pronged component that looks like a truncated, upside-down tropical line. We know that $X \cap X'$ consists of four points $p_1, ..., p_4$ (no possible intersections could have coordinate 0, since $(0,0)$ is the only such point satisfying the equation for $X$, and this point does not satisfy the equation for $X'$). We wish to find the possible configurations of the tropicalizations of these points in Trop($X$) $\cap$ Trop($X'$).

Solving $x + ay + bxy = 0$ for $x$ yields $x = \frac{-ay}{1+by}$. Plugging this into $x + cy + dxy + t(ex^2 + fy^2 + g) = 0$ and clearing denominators (by multiplying by $1 + by$) gives a quartic in $y$ with constant term $tg$ and lead term $tfb^2y^4$. Letting $y(p_i)$ denote the $y$-coordinate of $p_i$, we may alternately write this quartic as

$$tfb^2(y - y(p_1)) \cdot ... \cdot (y - y(p_4)) = 0.$$  

Matching constant terms, it follows that $tfb^2y(p_1) \cdot ... \cdot y(p_4) = tg$, implying that

$$\text{val}(y(p_1)) + ... + \text{val}(y(p_4)) = \text{val}(y(p_1) \cdot ... \cdot y(p_4)) = \text{val}(tg/tfb^2) = 0.$$  

A symmetric argument holds for the $x(p_i)$'s. Hence the points trop($p_1$),..., trop($p_4$) in Trop($X$) $\cap$ Trop($X'$) satisfy a balancing condition akin to that found in Example 4.2: the $x$-coordinates must add to 0, and the $y$-coordinates must add to 0.

Let’s figure out what configurations are possible with these restrictions. Certainly we could have all four points at the origin. If we’re not in this (somewhat boring) case, then a point must actually sit on one of the segments at a positive distance from the origin.
Whichever segment this point is on, it necessitates at least one point on each other segment to have all $x$-coordinates adding to 0 and all $y$-coordinates adding to 0. Hence each segment has a point; the only remaining question is whether the fourth point is on a segment or at the origin. All told, we have three possible situations, illustrated in Figure 6:

- All four points are at the origin.
- Exactly one point is at the origin, and the other points are at $(r, r)$, $(-r, 0)$, and $(0, -r)$ for some $r$ with $0 < r \leq 1$.
- One segment has two points and the others each have one. If the two are on the diagonal segment, we have points $(-r, 0)$, $(0, -r)$, $(r', r')$, and $(r'', r'')$, where $r', r''$ are positive numbers with $r' + r'' = r$. The other cases (two on horizontal segment; two on vertical segment) are similar.

![Figure 6. Possible configurations for the trop($p_i$)'s of Example 4.3. The $x$-coordinates add to 0, as do the $y$-coordinates.](image)

We can picture this nicely as having two degrees of freedom. We may intuitively start out with our points at $(-1, 0)$, $(0, 0)$, $(0, -1)$, and $(1, 1)$; we may slide any of the outside points inwards, and the other two are dragged along their segments as well. We may then choose one of the three segments (a 0-dimensional choice), and move our point at $(0, 0)$ along that segment, dragging the other point on that segment in the opposite direction.

We’d like to come up with explicit choices of $a, ..., g$ that give these various configurations. Let us focus on the case where the horizontal axis has two points on it, so that the points are at $(p, p)$, $(0, -p)$, $(-q, 0)$, and $(-r, 0)$, where $p, q, r$ are nonnegative rational numbers with $p \leq 1$, $q + r = p$, and $r \leq q$. Let

$$X = \{ x + y + xy = 0 \}$$

$$X' = \{ (1 + t^{1-p+r})x + (1 + t^{1-r})y + t(x^2 + y^2 + 1) = 0 \}.$$ 

Solving the first equation for $y$, plugging into the second, and clearing denominators yields the quartic $\sum_i a_i x^i$, where

$$a_0 = t$$
\[ a_1 = -t^{1-p} + t^{1-p+r} + 2t \]
\[ a_2 = -t^{1-p} + 2t^{1-p+r} + 3t - t^2 \]
\[ a_3 = t^{1-p+b} + 2t - t^2 \]
\[ a_4 = t. \]

Considering the Newton polygon of this polynomial, its roots have valuation \( p, 0, -r, \) and \( r - p = -q. \) Now solving the first equation for \( x, \) plugging into the second, and clearing denominators, we have the quartic \( \sum_i b_i y^i, \) where

\[ b_0 = t \]
\[ b_1 = t^{1-p} - t^{1-p+b} + 2t \]
\[ b_2 = 2t^{1-p} - t^{1-p+r} + 3t - t^2 \]
\[ b_3 = t^{1-p} + 2t - t^2 \]
\[ b_4 = t. \]

Considering the Newton polygon of this polynomial, its roots have valuation \( p, 0, 0, \) and \( -p. \) Considering \( \text{Trop}(X) \cap \text{Trop}(X'), -q \) and \( -r \) from the first polynomial must match with the two 0’s, the 0 on the \( x \) side of things must match with \( -p, \) and finally \( p \) must match with \( p. \) This gives us the desired images of the four points of \( X \cap X'. \)

(Similar examples work in the cases where the vertical or the diagonal axis has two points on it.)

**Example 4.4.** Let’s now intersect a cubic with a line in such a way that we get three unbounded components. There are many equations we could choose to give the desired picture; for the moment we’ll take \( X = \{ x + ky + \ell = 0 \} \) and \( X' = \{ t^3(ax^3 + by^3 + c) + t(dx + ex^2 + fy + gy^2 + hx^2y + ixy^2) + jxy = 0 \}. \) Tropically, these give the curves displayed in Figure 7, intersecting in three unbounded rays.

Let’s suppose for the moment that \( X \cap X' \subset T \) consists of three points (it is possible that intersections are lost since variables cannot take on the value 0). Since there are three distinct connected components of \( \text{Trop}(X) \cap \text{Trop}(X'), \) it’s not unreasonable to think that each should receive a point, and that these should behave independently due to the disconnectedness. Indeed, we can cook up examples that give any arrangement of points \( (p,0), (0,q), \) and \( (-r,-r), \) where \( p,q,r \) are all rational numbers greater than or equal to 1. Let

\[ X = \{ x + y + 1 = 0 \} \]
\[ X' = \{ t^3(x^3 + y^3 + 1) + t(dx + (1+t^p-1)x^2 + fy + (1+t^p-1)y^2 + (1+t^{p-1})x^2y + xy^2) + xy = 0 \}. \]

Plugging the first equation into the second and solving for \( x \) gives the cubic polynomial \( t'x^3 + t^p = 0. \) Doing the same but for \( y \) gives \( t'y^3 + t^q = 0. \) Some simple analysis of the Newton polygons of these polynomials shows that \( t'x^3 + t^p = 0 \) has a root of valuation \( p, \) a root of valuation 0, and a root of valuation \(-r; \) and the same for \( t'y^3 + t^q = 0, \) except a root of valuation \( q \) instead of a root of valuation \( p. \) Since the three intersection points must map somewhere on the tropical intersection, this gives us that the images of the intersection points are \( (p,0), (0,q) \) and \( (-r,-r), \) as desired.
Example 4.5 (A new example to try). So far, the previous example is the only one with disconnected intersection of tropicalizations. A new thing to try is to intersect a line with a cubic again, but to make the tropical line move $-2$ in the $y$-direction so that the intersection with a cubic looks like a tropical line with a finite segment missing from the middle of one of the three spokes. For instance, take $X = \{x + k t^2 y + \ell = 0\}$ and $X' = \{t^3 (a x^3 + b y^3 + c) + t (d x + e x^2 + f y + g y^2 + h x^2 y + i xy^2) + j xy = 0\}$, where $a, ..., k$ have valuation 0.

5. Where To Go From Here

First and foremost, it’s worth working out other tractable examples. For instance, we could replace the line in Example 4.4 with the conic $X = \{x + ky + \ell xy = 0\}$, giving an intersection consisting of three bounded, disjoint line segments (providing an example of a bounded intersection with multiple connected one dimensional components).

Also of interest are generalizations of Example 4.1, wherein we consider curves that have the same tropicalizations and observe the possible configurations of the tropicalizations of their intersections. For instance, intersecting $X = \{x + ay + bxy + t(cx^2 + dy^2 + c) = 0\}$ with $X' = \{x + f y + g xy + t(h x^2 + iy^2 + j) = 0\}$ should give four points, which must map somewhere onto $\text{Trop}(X) \cap \text{Trop}(X') = \text{Trop}(X)$. What configurations are possible? Will there be balancing conditions similar to those of Example 4.3?

The long term goal is to understand the possible connection between these configurations and chip firing. As explained to me by Matt Baker and Melody Chan, the balancing behavior exhibited in these examples is somewhat akin to chip firing scenarios. For instance, in Example 4.2, we may think of our intersections starting at $(0, 0)$ and $(0, -1)$, and then “firing” from there in some way that must balance out. A similar phenomenon is shown in

![Figure 7. The line and the cubic are shown intersecting transversely, and then shifted to the case of Example 4.4. Their intersection, consisting of three unbounded rays, is the rightmost image.](image_url)
Example 4.3. Understanding this connection could establish a very interesting mathematical link between these two contexts, hopefully letting us convert results about one to results about the other.

REFERENCES


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