2.4.26 For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list. Assuming that your formula or rule is correct, determine the next three terms of the sequence. We will refer to the terms as $a_1, a_2, a_3, ...$

a) 3, 6, 11, 18, 27, 38, 51, 66, 83, 102, ...
   The pattern is $a_{n+1} = a_n + 2n + 1$. The next three terms are 123, 146, 171.

b) 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, ...
   The pattern is $a_{n+1} = a_n + 4$ (which gives the formula $a_n = 4n + 3$). The next three terms are 47, 51, 55.

c) 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, ...
   The pattern is running through all binary strings in order. (Equivalently: it is the sequence of natural numbers, written in binary.) The next three terms are 1100, 1101, 1110.

d) 1, 2, 2, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5, ...
   This sequence is made up of Fibonacci numbers, starting with one 1, then three 2’s then five 3’s, and so on, increasing the number of copies by two each time. The next three terms are 8, 8, 8.

e) 0, 2, 8, 26, 80, 242, 728, 2186, 6560, 19682, ...
   The pattern is $a_{n+1} = 3a_n + 2$. The next three terms are 59048, 177146, and 531440.

f) 1, 3, 15, 105, 945, 10395, 135135, 2027025, ...
   The pattern is $a_{n+1} = (2n + 1) \cdot a_n$ (which gives the formula $a_n = (2n - 1) \cdot (2n - 3) \cdot \ldots \cdot 5 \cdot 3 \cdot 1$). The next three terms are 34459425, 654729075, 13749310575.
g) \( 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, \ldots \)
The pattern is one 1, two 0’s, three 1’s, four 0’s, and so on. The next three terms are 0, 0, 0.

h) \( 2, 4, 16, 256, 65536, 4294967296, \ldots \) The pattern is \( a_{n+1} = a_n^2 \). The next three terms are

\[
\begin{align*}
18446744073709551616 \\
340282366920938463463374607431768211456 \\
11579208923731619543570985008687907853269984665640564039457584007913129639936
\end{align*}
\]

2.4.32 Compute the value of each of these sums.

a) \[
\sum_{j=0}^{8} (1 + (-1)^j) = 1 + 0 + 1 + 0 + 1 + 0 + 1 + 0 + 1 = 5
\]

b) \[
\begin{align*}
\sum_{j=0}^{8} (3^j - 2^j) &= \left( \sum_{j=0}^{8} 3^j \right) - \left( \sum_{j=0}^{8} 2^j \right) \\
&= \frac{3^9 - 1}{3 - 1} - \frac{2^9 - 1}{2 - 1} \\
&= 9330.
\end{align*}
\]

c) \[
\begin{align*}
\sum_{j=0}^{8} (2 \cdot 3^j + 3 \cdot 2^j) &= 2 \left( \sum_{j=0}^{8} 3^j \right) + 3 \left( \sum_{j=0}^{8} 2^j \right) \\
&= 2 \cdot \frac{3^9 - 1}{3 - 1} + 3 \cdot \frac{2^9 - 1}{2 - 1} \\
&= 21215.
\end{align*}
\]
d)
\[
\sum_{i=0}^{2} \sum_{j=1}^{3} ij = \sum_{i=0}^{2} i \cdot (1 + 2 + 3) \\
= \sum_{i=0}^{2} 6i \\
= 6 \cdot (0 + 1 + 2) \\
= 18.
\]

2.4.34 Compute each of these double sums.

a)
\[
\sum_{i=1}^{3} \sum_{j=1}^{2} (i - j) = \sum_{i=1}^{3} ((i - 1) + (i - 2)) \\
= ((1 - 1) + (1 - 2)) + ((2 - 1) + (2 - 2)) + ((3 - 1) + (3 - 2)) \\
= 0 - 1 + 1 + 0 + 2 + 1 = 3.
\]

b)
\[
\sum_{i=0}^{3} \sum_{j=0}^{2} (3i + 2j) = \sum_{i=0}^{3} ((3i + 2 \cdot 0) + (3i + 2 \cdot 1) + (3i + 2 \cdot 2)) \\
= \sum_{i=0}^{3} (9i + 6) \\
= 9 \cdot 0 + 6 + 9 \cdot 1 + 6 + 9 \cdot 2 + 6 + 9 \cdot 3 + 6 \\
= 81.
\]

c)
\[
\sum_{i=1}^{3} \sum_{j=0}^{2} j = \sum_{i=1}^{3} (0 + 1 + 2) \\
= \sum_{i=1}^{3} 3 \\
= 3 + 3 + 3 = 9.
\]
d) 
\[
\sum_{i=0}^{2} \sum_{j=0}^{3} i^2 j^3 = \sum_{i=0}^{2} i^2 (0^3 + 1^3 + 2^3 + 3^3) = \sum_{i=0}^{2} 36i^2 = 36(0^2 + 1^2 + 2^2) = 180.
\]

2.4.46 Find \( \prod_{j=0}^{4} j! \). 

\[
\prod_{j=0}^{4} j! = 0! \cdot 1! \cdot 2! \cdot 3! \cdot 4! = 1 \cdot 1 \cdot 2 \cdot 6 \cdot 24 = 288.
\]

2.5.10 Give an example of two uncountable sets \( A \) and \( B \) such that \( A - B \) satisfies each of the following properties.

a.) Finite: Take \( A = [0, 1] \) and \( B = [1, 2] \), so that \( A \cap B = \{1\} \).

b.) Countably infinite: Take \( A = [0, 1] \cup \mathbb{Z} \) and \( B = [1, 2] \cup \mathbb{Z} \), so that \( A \cap B = \mathbb{Z} \).

c.) Uncountable: Take \( A = [0, 2] \) and \( B = [1, 3] \), so that \( A \cap B = [0, 1] \).

2.5.28 Show that the set \( \mathbb{Z}^+ \times \mathbb{Z}^+ \) is countable.

Solution. Make an infinite grid with \( \mathbb{Z}^+ \) on each axis, and fill in the \( mn^{th} \) square with \((m, n)\). Follow the snaking path as in Figure 3 on page 173 (where it was proved that \( \mathbb{Q} \) was countable). Following the path gives us the first element of \( \mathbb{Z}^+ \times \mathbb{Z}^+ \), then the second, and so on, giving us a bijection between \( \mathbb{Z}^+ \times \mathbb{Z}^+ \) and \( \mathbb{N} \). Thus \( \mathbb{Z}^+ \times \mathbb{Z}^+ \) is countable.

4.1.10 Find the quotient and remainder in each case.

a) When 44 is divided by 8: since 44 = 5 \cdot 8 + 4, the quotient is 5 and the remainder is 4.

b) When 777 is divided by 21: since 777 = 37 \cdot 21 + 0, the quotient is 37 and the remainder is 0.
c) When $-123$ is divided by 19: since $-123 = -7 \cdot 19 + 10$, the quotient is $-7$ and the remainder is 10.

d) When $-1$ is divided by 23: since $-1 = -1 \cdot 23 + 22$, the quotient is $-1$ and the remainder is 22.

e) When $-2002$ is divided by 87: since $-2002 = -24 \cdot 87 + 86$, the quotient is $-24$ and the remainder is 86.

f) When 0 is divided by 17: since $0 = 0 \cdot 17 + 0$, the quotient is 0 and the remainder is 0.

g) When 1,234,567 is divided by 1001: since $1,234,567 = 1233 \cdot 1001 + 334$, the quotient is 1233 and the remainder is 334.

h) When $-100$ is divided by 101: since $-100 = -1 \cdot 101 + 1$, the quotient is $-1$ and the remainder is 1.

4.1.16 Let $m$ be a positive integer. Show that $a \mod m = b \mod m$ if $a \equiv b \pmod{m}$.

Proof. Since $a \equiv b \pmod{m}$, we have that $a - b = km$ for some integer $k$. Write

$$a = q_1m + r_1$$
$$b = q_2m + r_2,$$

where $0 \leq r_1 \leq m - 1$ and $0 \leq r_2 \leq m - 1$ (so that $r_1 = a \mod m$ and $r_2 = b \mod m$). We wish to show that $r_1 = r_2$.

Subtracting the two equations, we have

$$a - b = (q_1 - q_2)m + r_1 - r_2.$$ 

Plugging in $a - b = km$, we have

$$km = (q_1 - q_2)m + r_1 - r_2,$$

which rearranges to

$$r_1 - r_2 = (q_1 - q_1 - k)m \equiv 0 \pmod{m}.$$ 

So $r_1 \equiv r_2 \pmod{m}$. Since $0 \leq r_1 \leq m - 1$ and $0 \leq r_2 \leq m - 1$, we have $r_1 = r_2$. We conclude that $a \mod m = b \mod m$. 

\[\square\]

4.1.20 Evaluate these quantities.
a) \(-17 \mod 2 = 1\) since \(-17 = -9 \cdot 2 + 1\).
b) \(144 \mod 7 = 4\) since \(144 = 20 \cdot 7 + 4\).
c) \(-101 \mod 13 = 3\) since \(-101 = -8 \cdot 13 + 3\).
d) \(199 \mod 19 = 9\) since \(199 = 10 \cdot 19 + 9\).

(*) 4.1.40 Prove that if \(n\) is an odd positive integer, then \(n^2 \equiv 1 \pmod{8}\).

Proof. Write \(n = 2k + 1\), where \(k \in \mathbb{Z}\). We have
\[
n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4k(k + 1) + 1.
\]
Either \(k\) is even, or \(k + 1\) is even. In either case, \(4k(k + 1)\) is divisible by 8. Thus we have
\[
n^2 = 4k(k + 1) + 1 \equiv 1 \pmod{8}.
\]

4.2.4 Convert the binary expansion of each of these integers to a decimal expansion.

a) \((1 1111)_2\) in base ten is \(2^0 + 2^4 + 2^3 + 2^4 = 31\).
b) \((10 0000 0001)_2\) in base ten is \(2^0 + 2^9 = 513\).
c) \((1 0101 0101)_2\) in base ten is \(2^0 + 2^2 + 2^4 + 2^6 + 2^8 = 341\).
d) \((11110000001 1111)_2\) in base ten is \(31 + 31 \cdot 2^{10} = 31775\) (based on part (a)).

4.2.7 Convert the hexadecimal expansion of each of these integers to a binary expansion.

a) \((80E)_{16}\) in binary is \((1000 0000 1110)_2\).
b) \((135AB)_{16}\) in binary is \((10011 0101 1010 1011)_2\).
c) \((ABBA)_{16}\) in binary is \((1010 1011 1011 1010)_2\).
d) \((DEFACE)D)_{16}\) in binary is \((1101 1110 1111 1010 1100 1110 1101)_2\).

4.2.10 Convert each integer from a binary expansion to its hexadecimal expansion.

a) \((1111 0111)_2\) in hexadecimal is \((F7)_{16}\).
b) \((1010\ 1010\ 1010)_{2}\) in hexadecimal is \((\text{AAA})_{16}\).

c) \((1110111\ 0111\ 0111)_{2}\) in hexadecimal is \((\text{7777})_{16}\).

d) \((1010\ 1010\ 1010\ 1010)_{2}\) in hexadecimal is \((5555)_{16}\).

4.28 Use Algorithm 5 to find \(123^{1001} \mod 101\).

Solution. Algorithm five initially sets \(x = 1\) and \(\text{power} = 123 \mod 101 = 22\). We need the binary expansion of 1001, which is \((1111101001)_{2}\) (so \(a_0 = 1, a_1 = 0, a_2 = 0, \text{ etc.}\)).

- \(i=0: \) Because \(a_0 = 1, \) we have \(x = 1 \cdot 22 \mod 101 = 22, \) and \(\text{power} = 22^2 \mod 101 = 80.\)
- \(i=1: \) Because \(a_1 = 0, \) we have \(x = 22, \) and \(\text{power} = 80^2 \mod 101 = 37.\)
- \(i=2: \) Because \(a_2 = 0, \) we have \(x = 22, \) and \(\text{power} = 37^2 \mod 101 = 56.\)
- \(i=3: \) Because \(a_3 = 1, \) we have \(x = 22 \cdot 56 \mod 101 = 20, \) and \(\text{power} = 56^2 \mod 101 = 5.\)
- \(i=4: \) Because \(a_4 = 0, \) we have \(x = 20, \) and \(\text{power} = 5^2 \mod 101 = 25.\)
- \(i=5: \) Because \(a_5 = 1, \) we have \(x = 20 \cdot 25 \mod 101 = 96, \) and \(\text{power} = 25^2 \mod 101 = 19.\)
- \(i=6: \) Because \(a_6 = 1, \) we have \(x = 96 \cdot 19 \mod 101 = 6, \) and \(\text{power} = 19^2 \mod 101 = 58.\)
- \(i=7: \) Because \(a_7 = 1, \) we have \(x = 6 \cdot 58 \mod 101 = 45, \) and \(\text{power} = 58^2 \mod 101 = 31.\)
- \(i=8: \) Because \(a_8 = 1, \) we have \(x = 45 \cdot 31 \mod 101 = 82, \) and \(\text{power} = 31^2 \mod 101 = 52.\)
- \(i=9: \) Because \(a_9 = 1, \) we have \(x = 82 \cdot 52 \mod 101 = 22.\)

Thus we have \(123^{1001} \mod 101 = 22\)

4.3.12 Prove that for every positive integer \(n, \) there are \(n\) consecutive composite integers.
Proof. Consider the $n$ consecutive integers

$$(n + 1)! + 2, (n + 1)! + 3, (n + 1)! + 4, \ldots, (n + 1)! + n, (n + 1)! + n + 1.$$ 

None of them are prime, since $(n+1)!$ is divisible by all integers from 2 up to $n+1$, meaning that $(n+1)!+k$ is divisible by $k$ for $2 \leq k \leq n+1$. Thus we have $n$ consecutive composite integers.

4.3.32 Use the Euclidean algorithm to find the following gcd’s.

a) $\gcd(1, 5)$

\[
5 = 4 \cdot 1 + 1 \\
4 = 4 \cdot 1 + 0
\]

By the Euclidean algorithm, $\gcd(1, 5) = 1$.

b) $\gcd(100, 101)$

\[
101 = 1 \cdot 100 + 1 \\
100 = 100 \cdot 1 + 0
\]

By the Euclidean algorithm, $\gcd(100, 101) = 1$.

c) $\gcd(123, 277)$

\[
277 = 2 \cdot 123 + 31 \\
123 = 3 \cdot 31 + 30 \\
31 = 1 \cdot 30 + 1 \\
30 = 30 \cdot 1 + 0
\]

By the Euclidean algorithm, $\gcd(123, 277) = 1$.

d) $\gcd(1529, 14039)$

\[
14039 = 9 \cdot 1529 + 278 \\
1529 = 5 \cdot 278 + 139 \\
278 = 2 \cdot 139 + 0
\]

By the Euclidean algorithm, $\gcd(1529, 14039) = 139$.

e) $\gcd(1000, 5040)$

\[
5040 = 5 \cdot 1000 + 40 \\
1000 = 25 \cdot 40 + 0
\]

By the Euclidean algorithm, $\gcd(1000, 5040) = 40$. 

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f) \( \gcd(11111, 111111) \)

\[
111111 = 10 \cdot 11111 + 1 \\
11111 = 11111 \cdot 1 + 0
\]

By the Euclidean algorithm, \( \gcd(11111, 111111) = 1 \).

4.3.44 \textit{Use the extended Euclidean algorithm to express} \( \gcd(1001, 100001) \) \textit{as a linear combination of} 1001, 100001.

\textit{Solution}. First we’ll perform the standard Euclidean algorithm.

\[
100001 = 99 \cdot 1001 + 902 \\
1001 = 1 \cdot 902 + 99 \\
902 = 9 \cdot 99 + 11 \\
99 = 9 \cdot 11 + 0
\]

So \( \gcd(1001, 100001) = 11 \). We wish to write 11 as a linear combination of 100001.

\[
11 = 902 - 9 \cdot 99 \\
= 902 - 9 \cdot (1001 - 902) \\
= 10 \cdot 902 - 9 \cdot 1001 \\
= 10 \cdot (100001 - 99 \cdot 1001) - 9 \cdot 1001 \\
= 10 \cdot 100001 - 999 \cdot 1001.
\]

\[\square\]

4.3.52 \textit{Prove or disprove that} \( p_1p_2...p_n + 1 \) \textit{is prime for every positive integer} \( n \), \textit{where} \( p_1, p_2, ..., p_n \) \textit{are the} \( n \) \textit{smallest prime numbers}.

\textit{Disproof}. This claim is true for \( n \leq 5 \). However, for \( n = 6 \), we have

\[
2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 30,031 = 59 \cdot 509.
\]

\[\square\]