(10.3.9) Represent each of these graphs with an adjacency matrix.

(a) $K_4$

Solution. 
\[
\begin{pmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0
\end{pmatrix}
\]

(b) $K_{1,4}$

Solution. 
\[
\begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

(c) $K_{2,3}$
Solution.
\[
\begin{pmatrix}
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0
\end{pmatrix}
\]

(d) $C_4$

Solution.
\[
\begin{pmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0
\end{pmatrix}
\]

(e) Solution. $W_4$

\[
\begin{pmatrix}
0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0
\end{pmatrix}
\]

(f) $Q_3$

Solution.
\[
\begin{pmatrix}
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0
\end{pmatrix}
\]
(10.3.22) **Represent the graph represented by the adjacency matrix**

\[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

**Solution.** The graph is

(10.3.44) **Determine if the two graphs are isomorphic. Prove your answer.**

**Solution.** They are not isomorphic.

From Exercise 10.3.46, we know that two graphs are isomorphic if and only if their complements are isomorphic. Taking the complement of the first graph gives a disconnected graph with two components, each a copy of $C_4$. Taking the complement of the second graph gives a connected graph which is isomorphic to $C_8$. These complements are not isomorphic (a disconnected graph cannot be equal to a connected graph), so the original graphs are not isomorphic.

(10.3.58) **Determine whether the graphs without loops with these incidence matrices are isomorphic.**

(a)

\[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 0
\end{pmatrix}, \begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix}
\]
Solution. Yes they’re isomorphic. They have the same columns up to reordering, so they encode the same edge data. (Note that if we DIDN’T have the same columns up to reordering, they still might be isomorphic; this is not an “if and only if.”)

(b)  
\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 0
\end{pmatrix}
\quad \begin{pmatrix}
0 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1
\end{pmatrix}
\]

Solution. Yes, for the same reason as part (a).  

(10.3.68) NOTE: These solutions are not yet complete. We’ve only counted up undirected graphs at this point. How many nonisomorphic directed simple graphs are there with \( n \) vertices, when \( n \) is

(a) 2?  

Solution. There are 2. Any graph on two points has either one edge or zero edges. Any pair of graphs on two points with zero edges are isomorphic: send a point to a point, and the other point to the other point. A similar argument holds for one edge. These two cases are not isomorphic since they don’t have the same number of edges

(b) 3?  

Solution. There are 4. This is because up to isomorphism, there is exactly one graph on three vertices for any number of edges between 0 and 3, giving three isomorphism classes.

(c) 4?  

Solution. There are 10:  
- With 0 edges, we get one graph (four vertices, no edges).  
- With 1 edge, we get one graph up to isomorphism (an edge between two vertices; doesn’t matter which).  
- With 2 edges, we get two (a line of three with one vertex left out; vertices paired off by an edge for each pair).
With 3 edges, we get two (a triangle formed by three vertices, and a chain of length three formed by all four).

With four edges, we get two (a square, and a triangle with an edge poking out to the fourth vertex).

With five edges, we get one (a complete graph with one edge deleted).

With six edges, we get one (a complete graph on four vertices).

Adding all these up gives 10.

(10.4.8) What do the connected components of a collaboration graph represent?

Solution. Connected components represent groups of people who are connected by some string of collaborators. (For instance, Ralph worked with Colin Adams worked with Frank Morgan worked with Zoltan Fredi worked with Paul Erdos, so Ralph and Erdos are in the same component of a collaboration graph of math papers.)

(10.4.14) Find the strongly connected components of each graph.

(a) Solution. The strongly connected components are

- \{c\}
- \{d\}
- \{a, b, e\}

(b) Solution. The strongly connected components are

- \{f\}
- \{a\}
- \{b\}
- \{c, d, e\}

(c) Solution. The strongly connected components are

- \{a, b, c, d, f, g, h, i\}
- \{e\}
(10.4.22) Use paths to show that these graphs are not isomorphic or to find an isomorphism between them.

Solution. We notice that there are two four cycles in each graph: 

\((u_2, u_3, u_4, u_5, u_2)\) and \((u_4, u_5, u_6, u_7, u_4)\) for the first, and \((v_1, v_3, v_4, v_8, v_1)\) and \((v_4, v_5, v_7, v_8, v_1)\). Notice that there each graph has two points that appear in both four cycles: \(u_4\) and \(u_5\) in the first, and \(v_4\) and \(v_8\) in the second. This suggests we should send, say \(u_4\) to \(v_4\) and \(u_5\) to \(v_8\). Moving along the four cycle leads us to send \(u_3\) to \(v_3\) and \(u_2\) to \(v_1\), \(u_1\) to \(v_2\), \(u_8\) to \(v_6\), \(u_7\) to \(v_5\), and \(u_6\) to \(v_7\).

The adjacency matrix for the first graph is

\[
\begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}
\]

Calculating the adjacency matrix for the second graph, with the ordering of the vertices rearranged based on our map sending \(u_i\)'s to \(v_j\)'s, gives the same matrix. Hence the map we came up with is an isomorphism.

(10.4.38) Show that an edge in a simple graph is a cut edge if and only if this edge is not a part of any simple circuit in this graph.

Solution. Let \(G\) be a connected graph, let \(e\) connected \(a_1\) and \(a_2\), and let \(G'\) be \(G\) with \(e\) deleted.

Assume \(e\) is a cut edge of \(G\). Suppose for the sake of contradiction that \(e\) is part of a simple circuit in \(G\), say \((a_1, a_2, \ldots, a_n, a_1)\). Since \(e\) is a cut edge, \(a_1\) and \(a_2\) are in different components of \(G'\). However, there is a path \((a_2, \ldots, a_n, a_1)\) connecting \(a_2\) and \(a_1\) in \(G'\) (since \((a_1, a_2, \ldots, a_n, a_1)\) as a simple circuit, the edge \(e\) is not used outside of connecting \(a_1\) to \(a_2\)). This is a contradiction to \(a_1\) and \(a_2\) being in different components. Hence \(e\) is NOT part of any simple circuit in \(G\).

Assume that \(e\) is not part of any simple circuit \(G\). We claim that \(a_1\) and \(a_2\) are in different components of \(G'\). Suppose not, for the sake
of contradiction. Then there is a path \((a_2, a_3, a_4, ..., a_n, a_1)\) from \(a_2\) to \(a_1\) in \(G'\), meaning it cannot use the edge \(e\). Moreover, we may assume this path is simple: if it ever hits the same point twice, just shorten the path so that it doesn’t. Then \((a_1, a_2, ..., a_n, a_1)\) is a simple circuit, a contradiction. Hence \(a_1\) and \(a_2\) are in different components of \(G'\), so \(e\) is a cut edge.

(10.5.8) \textbf{Determine whether this graph has an Euler circuit. If yes, construct one. If no, determine if there is an Euler path, and construct it if so.}

\textit{Solution.} There must exist an Euler circuit by Theorem 1 in 10.5, since each vertex has even degree.

To construct this, we’ll use the method suggested by Algorithm 1 on page 696. First we take a nice obvious cycle from our graph, say the outer rectangle:

\((a, b, c, d, e, j, o, n, m, l, k, f, a)\).

Delete the edges from this cycle to give a simpler graph. On this graph, there’s another nice cycle which we have starting from a vertex in the above cycle, namely \((b, d, i, n, l, g, b)\). We insert this into our original cycle where \(b\) was:

\((a, b, d, i, n, l, g, b, c, d, e, j, o, n, m, l, k, f, a)\).

Finally, we are left with a graph that is yet another nice cycle that starts and ends at a vertex in our cycle, namely \((g, f, m, h, c, j, i, h, g)\). This is inserted in place of \(g\):

\((a, b, d, i, n, l, g, f, m, h, c, j, i, h, g, b, c, d, e, j, o, n, m, l, k, f, a)\).

We’ve obtained our Euler circuit.

(10.5.10) \textbf{Can someone cross all the bridges shown in this map exactly once and return to the starting point?}

\textit{Solution.} No. Let \(G\) be the graph whose vertices are the banks and islands, and whose edges are the bridges. Crossing all bridges exactly once and returning to the starting position would be an Euler circuit on \(G\). But \(G\) has two vertices with odd degree, and by Theorem 1 in 10.5, such a graph cannot have an Euler circuit.
For which values of $n$ do these graphs have an Euler circuit?

(Recall Theorem 1 from 10.5: a connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.)

(a) $K_n$

Solution. $K_n$ has an Euler circuit for $n$ odd, and no Euler circuit for $n$ even.

$K_1$ must be handled separately, and trivially has an Euler circuit since it has no edges. For $n \geq 2$, we may apply Theorem 1: $K_n$ has an Euler circuit if and only if each of its vertices has even degree, which holds if and only if $n - 1$ (the degree of each vertex) is even, if and only if $n$ is odd.

(b) $C_n$

Solution. $C_n$ has an Euler circuit for all $n$ (here $n \geq 3$ by definition).

This may be seen by applying Theorem 1 and noting that each vertex has degree 2. Also, we may give an explicit description of the Euler circuit: just go around the “circle” that makes up the graph.

(c) $W_n$

Solution. $W_n$ does not have an Euler circuit for any $n$.

This may be seen by applying Theorem 1 and noting that all external points on $W_n$ have odd degree.

(d) $Q_n$

Solution. $Q_n$ has an Euler circuit for $n$ even, and no Euler circuit for $n$ odd.

This is because the valence of each vertex of $Q_n$ is $n$, and so by Theorem 1 $Q_n$ has an Euler circuit if and only if $n$ is even.

Show that the Petersen graph does not have a Hamilton circuit, but that the subgraph obtained by deleting a vertex $v$, and all edges incident with $v$, does have a Hamilton circuit.
Solution. Suppose there were a Hamilton circuit. This means that we can color the edges of the Petersen graph with three colors so that no two adjacent edges are the same color: for instance, alternate red and blue for the Hamilton circuit, and let the rest be colored green. We’ll prove that this is a contradiction.

By the pigeonhole principle, two of the exterior edges must be the same color, so without loss of generality (a, b) and (c, d) are red. This means (a, e) must have a different color, say blue, forcing the other two outer edges (b, c) and (d, e) to be green. These colors force each edge connecting the outside to the inside to be a particular color (three blues, a red, and a green), and once we reach the inside star we reach a contradiction in terms of what can be what color. Hence there is no Hamilton circuit.

Now for the deletion-of-a-vertex part. By symmetry, it does not matter which vertex we delete. (This takes some arguing to see that, say, a and f “look the same”; this has to do with the fact that the inside portion and the outside portion both look like $C_5$.) Deleting a for simplicity gives us a graph that, up to isomorphism, looks like

![Graph Image]

This has a Hamilton circuit, for instance $(e, j, g, b, c, h, i, d)$.  

(10.6.4) Find the shortest path between a and z in the weighted graph.

Solution. (Solution for this one yet to be written)
(10.6.18) *Is the shortest path between two vertices in a weighted graph unique if the weights of edges are distinct?*

*Solution.* No. Consider a graph with vertices \{a, b, c\} and edges \{(a, b), (b, c), (a, c)\}. Say we weight \((a, c)\) as 3 and weight \((a, b)\) as 1 and \((b, c)\) as 2. Then there are two shortest paths from \(a\) to \(c\), namely \((a, c)\) (which has total weight 3), and \((a, b)\) followed by \((b, c)\) (which again has total weight 3). \qed