Math 55: Discrete Mathematics

UC Berkeley, Fall 2011
Homework # 11, due Wednesday, April 25

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9.4.9 Find the symmetric closures of the relations with the directed graphs shown in Exercises 5 through 7.

Solution. (The added edges are shown in red.)

(5) We just need to add two:

(6) Here we need to add three:
(7) Here we’re only missing one:

A --*-- B

(9.4.10) Find the smallest relation containing the relation in Example 2 that is both reflexive and symmetric.

Solution. Recall that $R$ is a relation on $\mathbb{N}$ defined by $(a, b) \in R$ if and only if $a > b$. To make $R$ reflexive, we must add in all elements of the form $(a, a)$, giving us $R'$, where $(a, b) \in R'$ if and only if $a \geq b$. Now to make it symmetric, we must add in all elements of the form $(a, b)$, where $a \geq b$. This gives $R''$, where $(a, b) \in R''$ if and only if $a \leq b$ or $a \geq b$. This includes all pairs $(a, b) \in \mathbb{N}^2$, so $R'' = \mathbb{N}^2$ is the smallest relation containing $R$ that is both reflexive and symmetric.

9.4.17 Find all circuits of length three in the directed graph in Exercise 16.

Solution. • $(a, a, a, a)$
• $(a, b, e, a)$
• $(a, d, e, a)$
• $(b, e, a, b)$
• $(b, c, c, b)$
• $(c, c, c, c)$
• $(c, b, c, c)$
• $(c, c, b, c)$
• \((d, e, a, d)\) (isn’t math so cheerful?)
• \((d, e, e, d)\)
• \((e, e, e, e)\)
• \((e, d, e, e)\)
• \((e, e, d, e)\)
• \((e, a, b, e)\)
• \((e, a, d, e)\)

9.4.24 Suppose that the relation \(R\) is irreflexive. Is the relation \(R^2\) necessary irreflexive?

Solution. No. For instance, let \(R\) be the relation \(\{(0, 1), (1, 0)\}\) on the set \(\{0, 1\}\). It is irreflexive since \((0, 0)\) and \((1, 1)\) are not in \(R\). However, \(R^2 = \{(0, 0), (1, 1)\}\), which is not irreflexive. (Indeed, it is reflexive. There are also examples where \(R\) is irreflexive and \(R^2\) is neither irreflexive nor reflexive.)

9.5.6 Define three equivalence relations on the set of classes offered at your school. Determine the equivalence classes for each of these equivalence relations.

Solution.  
- Let \(R = \{(a, b) \mid \text{either } a \text{ and } b \text{ start before 12PM, or } a \text{ and } b \text{ start at or after 12PM}\}\). This is reflexive since \(a\) starts at the same time as \(a\). It is symmetric since switching \(a\) and \(b\) doesn’t change satisfying the property. Finally, it is transitive since if \((a, b), (b, c) \in R\), all three classes either all start before 12 or all start by or after 12, so \((a, c) \in R\). There are two equivalence classes: classes that start before noon, and classes that start after noon.
- Let \(R = \{(a, b) \mid a \text{ and } b \text{ meet the same number of times per week}\}\). This is reflexive since \(a\) meets the same number of times as itself per week. It is symmetric since flipping \(a\) and \(b\) still has them meeting the same number of times per week. Finally, it is transitive since if \((a, b), (b, c) \in R\), all three classes meet the same number of times per week, so \((a, c) \in R\). There are five equivalence classes: classes that meet once a week, twice a week,
all the way up five times a week. (Maybe if you’re crafty and are enrolled in a course that never meets, zero can be an equivalence class, too.)

Let \( R = \{ (a, b) \mid a \text{ and } b \text{ are taught by professors with the same last initial} \}. \) This is reflexive since the professor of \( a \) has the same initial as himself or herself. It is symmetric since flipping \( a \) and \( b \) still has the professors having the same last initial. Finally, it is transitive since if \((a, b), (b, c) \in R\), all three professors have the same last initial, so \((a, c) \in R\). There are 26 equivalence classes, given by the 26 letters of the alphabet. (All these equivalence classes are nonempty: the math department gives you at least one for everything besides Q, U, X, and Y, and you can get the Q and the U from Haas, some X’s from physics, and a Y from EECS.)

9.5.16 Let \( R \) be the relation on the set of ordered pairs of positive integers such that \((a, b), (c, d) \in R \) if and only if \( ad = bc \). Show that \( R \) is an equivalence relation.

Solution.  

\begin{itemize}
  \item To see that \( R \) is reflexive, note that \( ab = ba \), so \((a, b), (a, b) \in R\).
  \item To see that \( R \) is symmetric, assume that \((a, b), (c, d) \in R\), so that \( ad = bc \). This implies that \( cb = da \), so \((c, d), (a, b) \in R\).
  \item To see that \( R \) is transitive, assume that \((a, b), (c, d) \in R \) and \((c, d), (e, f) \in R \), so that \( ad = bc \) and \( cf = de \). This means that \( af = \frac{adcf}{dc} = \frac{bcde}{dc} = be \), so \((a, b), (e, f) \in R\).
\end{itemize}

Since \( R \) is reflexive, symmetric, and transitive, it is an equivalence relation.

9.5.44 Which of these collections of subsets are partitions of the set of integers?

(a) the set of even integers and the set of odd integers

Solution. Yes: every integer falls into one of these two sets, and they don’t overlap.
(b) the set of positive integers and the set of negative integers

Solution. No: the integer 0 doesn’t fall into either set.

(c) the set of integers divisible by 3, the set of integers leaving a remainder of 1 when divided by 3, and the set of integers that leave a remainder of 2 when divided by 3

Solution. Yes: as shown in Example 9, congruence mod m is an equivalence relation, and congruence mod 3 gives this partition.

(d) the set of integers less than $-100$, the set of integers with absolute value not exceeding 100, and the set of integers great than 100

Solution. Yes: every integer falls into one of these three sets, and they don’t overlap. (It’s worth being careful with ±100, but both fall only into the second set.)

(e) the set of integers not divisible by 3, the set of even integers, and the set of integers that leave a remainder of 6 when divided by 6

Solution. No: 2 falls into the first two sets, and all pairs of sets need to be disjoint.

9.5.55 Find the smallest equivalence relation on the set \{a, b, c, d, e\} containing the relation \{(a, b), (a, c), (d, e)\}.

Solution. First let’s make things symmetric, giving us

\{(a, b), (b, a), (a, c), (c, a), (d, e), (e, d)\}.

We need to make things transitive, and since you can go from b to c or from c to b, we’ll add these relations, giving

\{(a, b), (b, a), (a, c), (c, a), (b, c), (c, b), (d, e), (e, d)\}.

Things aren’t quite yet transitive, since you can go a to b to a but not yet a to a (the same is true for all other letters), so we’ll add in everything that looks like \((x, x)\). (This also takes care of reflexivity.) This gives us

\{(a, b), (b, a), (a, c), (c, a), (b, c), (c, b), (d, e), (e, d), (a, a), (b, b), (c, c), (d, d), (e, e)\}.

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This satisfies all three requirements, and so is an equivalence relation. Since we didn’t add anything we didn’t need, it is the smallest one containing the relation \{ (a, b), (a, c), (d, e) \}.

(Incidentally: the two equivalence classes we get are \{ a, b, c \} and \{ d, e \}.)

9.5.62 Determine the number of different equivalence relations on the set \( S = \{ a, b, c, d \} \) by listing them.

Solution. Let \( \Delta = \{ (x, x) \mid x \in S \} \)

- With four equivalence classes: \( \Delta \).
- With three equivalence classes (there will be \( \binom{4}{2} = 6 \) ways):
  - \( \Delta \cup \{ (a, b), (b, a) \} \)
  - \( \Delta \cup \{ (a, c), (c, a) \} \)
  - \( \Delta \cup \{ (a, d), (d, a) \} \)
  - \( \Delta \cup \{ (b, c), (c, b) \} \)
  - \( \Delta \cup \{ (b, d), (d, b) \} \)
  - \( \Delta \cup \{ (c, d), (d, c) \} \)
- With two equivalence classes (there will be 3 ways that put them in pairs, plus 4 ways that put three together and one by itself):
  - \( \Delta \cup \{ (a, b), (b, a), (c, d), (d, c) \} \)
  - \( \Delta \cup \{ (a, c), (c, a), (b, d), (d, b) \} \)
  - \( \Delta \cup \{ (a, d), (d, a), (b, c), (c, b) \} \)
  - \( \Delta \cup \{ (a, b), (b, a), (c, a), (a, c), (b, c), (c, b) \} \)
  - \( \Delta \cup \{ (a, b), (b, a), (a, d), (d, a), (b, d), (d, b) \} \)
  - \( \Delta \cup \{ (a, c), (c, a), (a, d), (d, a), (c, d), (d, c) \} \)
  - \( \Delta \cup \{ (b, c), (c, b), (b, d), (d, b), (c, d), (d, c) \} \)
- With one equivalence class: \( \Delta \cup \{ (a, b), (b, a), (b, c), (c, b), (b, d), (d, b), (c, d), (d, c) \} \)

10.1.13 The intersection graph of a collection of sets \( A_1, ..., A_n \) is the graph that has a vertex for each of these sets and has an edge connected the vertices representing two sets if these two sets have nonempty intersection. Construct the intersection graph of these collections of sets.
(a) $A_1 = \{0, 2, 4, 6, 8\}$
   $A_2 = \{0, 1, 2, 3, 4\}$
   $A_3 = \{1, 3, 5, 7, 9\}$
   $A_4 = \{5, 6, 7, 8, 9\}$
   $A_5 = \{0, 1, 8, 9\}$

Solution. The graph is

![Graph](image)

(b) $A_1 = \{..., -4, -3, -2, -1, 0\}$
   $A_2 = \{..., -2, -1, 0, 1, 2, ...\}$
   $A_3 = \{..., -6, -4, -2, 0, 2, 4, 6, ...\}$
   $A_4 = \{..., -5, -3, -1, 1, 3, 5, ...\}$
   $A_5 = \{..., -6, -3, 0, 3, 6, ...\}$

Solution. The graph is

![Graph](image)
(c) (I'm assuming these sets take $x \in \mathbb{R}$ satisfying the given properties.)

$A_1 = \{x \mid x < 0\}$
$A_2 = \{x \mid -1 < x < 0\}$
$A_3 = \{x \mid 0 < x < 1\}$
$A_4 = \{x \mid -1 < x < 1\}$
$A_5 = \{x \mid x > -1\}$
$A_6 = \mathbb{R}$

Solution. The graph is
10.1.14 Use the niche overlap graph in Figure 11 to determine the species that compete with hawks.

Solution. As explained in Example 11, the competitors of the hawk are precisely the animals whose vertices are connected to the hawk’s with an edge. Thus the competitors are raccoons, owls, and crows.

10.1.28 Describe a graph model that represents a subway system in a large city.

Solution. The vertices should represent subway stops, and the edges should represent subway lines that go from one station to the next. The edges should be directed, since each rail usual has trains going only one direction. Multiple edges should be allowed for when there are rails going both directions between two stations; also some routes are express, so this may facilitate the need for more than one edge in the same direction. Loops should not be allowed, since we don’t need to go from a station to itself. (Thus we want a directed multigraph, but not a pseudograph.)

10.2.10 For each of the graphs in Exercises 7-9 determine the sum of the in-degrees of the vertices and the sum of the out-degrees of the vertices directly. Show that they are both equal to the number of edges in the graph.

Solution. (7) The in-degrees are 3, 1, 2, and 1, which add to 7, which is the number of edges. The out-degrees are 1, 2, 1, and 3, which add to 7 as well.

(8) The in-degrees are 2, 3, 2, and 1, which add to 8, which is the number of edges. The out-degrees are 2, 4, 1, and 1, which add to 8 as well.

(9) The in-degrees are 6, 1, 2, 4, and 0, which add to 13, which is the number of edges. The out-degrees are 1, 5, 5, 2, and 0, which add to 13 as well.

10.2.20 Draw these graphs.

(a) $K_7$
Solution. This graph is

\[
\begin{array}{c}
\text{(b) } K_{1,8} \\
\text{Solution. This graph is}
\end{array}
\]

\[
\begin{array}{c}
\text{(c) } K_{4,4} \\
\text{Solution. This graph is}
\end{array}
\]
(d) $C_7$

Solution. This graph is

(e) $W_7$

Solution. This graph is
(f) $Q_4$

Solution. This graph is

10.2.26 For which values of $n$ are these graphs bipartite?

(a) $K_n$
Solution. For $n = 1$ and $n = 2$, $K_n$ is bipartite ($K_1$ is trivially so, and $K_2$ is two-colorable). For $n \geq 3$, though, there exist cycles of odd length; for instance, $(a, b, c, a)$ for any three distinct points $a, b, c$ in the graph, so when $n \geq 3$ $K_n$ is not bipartite.

(b) $C_n$

Solution. $C_n$ is bipartite for $n$ even, and not bipartite for $n$ odd. This is because all cycles are of even length when $n$ is even, but there exist cycles of odd length for $n$ odd.

(c) $W_n$

Solution. $W_n$ is not bipartite for any $n$, since it is not two-colorable: if the center point were, say blue, then all other points would be forced to be red, and then there would be adjacent reds.

(d) $Q_n$

Solution. $Q_n$ is bipartite for all $n$, since $Q_n$ is two-colorable for all $n$. This can be proven by induction. $Q_0$ is two-colorable since it just a point. If $Q_n$ is two-colorable, then make another copy of $Q_n$ and color it in the reversed way (points that were blue are now red and vice versa). To make $Q_{n+1}$ you attach each point to the corresponding point on the other copy, which only involves attaching reds to blues and blues to reds, so $Q_{n+1}$ will be two-colorable.

10.2.35 How many vertices and how many edges do these graphs have?

(a) $K_n$

Solution. There are $n$ vertices. There is an edge for every pair of vertices, and there are $\binom{n}{2}$ pairs of vertices, so there are $\binom{n}{2}$ edges.

(b) $C_n$

Solution. This is $n$ vertices in a cycle, so there are $n$ vertices and $n$ edges.
(c) $W_n$

Solution. This is the same as $C_n$ but with one more point and $n$ edges added in, so there are $n + 1$ vertices and $2n$ edges.

(d) $K_{m,n}$

Solution.

(e) $Q_n$

Solution. There are $2^n$ vertices, since $Q_0$ has one vertex and going from $Q_n$ to $Q_{n+1}$ doubles the number of vertices. There are $n2^{n-1}$ edges: this is true for $Q_0$ (zero edges), and if $Q_n$ has $n2^{n-1}$ edges, then since $Q_{n+1}$ is two copies of $Q_n$ with corresponding points having edges added between them, $Q_{n+1}$ has $n2^{n-1} + n2^{n-1} + 2^n = 2n2^{n-1} + 2^n = n2^n + 2^n = (n+1)2^n$, giving us the desired formula for $Q_{n+1}$. By induction, the claim holds.

10.2.42 Determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence.

(a) 5, 4, 3, 2, 1, 0

Solution. This sequence is not graphic. The sum of the degrees of the vertices is an odd number, but the handshake theorem says it must be even for a graph.

(b) 6, 5, 4, 3, 2, 1

Solution. This sequence is not graphic: there are six vertices, so none can have degree 6.

(c) 2, 2, 2, 2, 2

Solution. This sequence is graphic:
Solution. This sequence is not graphic. The sum of the degrees of the vertices is an odd number, but the handshake theorem says it must be even for a graph.

(e) 3, 3, 2, 2, 2

Solution. This sequence is graphic:

(f) 1, 1, 1, 1, 1
Solution. This sequence is graphic:

\begin{figure}
\centering
\includegraphics{figures/example_graph.png}
\end{figure}

(g) 5, 3, 3, 3, 3

Solution. This sequence is graphic:

\begin{figure}
\centering
\includegraphics{figures/example_graph.png}
\end{figure}

(h) 5, 5, 4, 3, 2, 1

Solution. This sequence is not graphic. The two 5’s mean two points connect to all other points, meaning that each point has at least two edges attached to it, contradicting the presence of a 1.