Exercise 1. Let \( P(n) \) be the statement that \( 1^2 + 2^2 + \ldots + n^2 = n(n + 1)(2n + 1)/6 \) for a positive integer \( n \).

(a) What is the statement \( P(1) \)?

Solution. The statement \( P(1) \) is “\( 1^2 = 1(1 + 1)(2 \cdot 1 + 1)/6 \).”

(b) Show that \( P(1) \) is true, completing the basis step of the proof.

Solution. Since \( 1^2 = 1 \) and \( 1(1 + 1)(2 \cdot 1 + 1)/6 = 2 \cdot 3/6 = 1 \), \( P(1) \) is true.

(c) What is the inductive hypothesis?

Solution. The inductive hypothesis is “\( 1^2 + 2^2 + \ldots + k^2 = k(k + 1)(2k + 1)/6 \), where \( k \geq 1 \).”

(d) What do you need to prove in the inductive step?

Solution. We need to prove that, assuming the inductive hypothesis, we have

\[
1^2 + 2^2 + \ldots + (k + 1)^2 = (k + 1)((k + 1) + 1)(2(k + 1) + 1)/6;
\]

in other words, assuming \( P(k) \) is true, we need to prove that \( P(k + 1) \) is also true.

(e) Complete the inductive step, identifying where you use the inductive hypothesis.

Solution. Consider the sum

\[
1^2 + 2^2 + \ldots + k^2 + (k + 1)^2.
\]

BY THE INDUCTIVE HYPOTHESIS, we know that \( 1^2 + 2^2 + \ldots + k^2 = k(k + 1)(2k + 1)/6 \), so we can write

\[
1^2 + 2^2 + \ldots + k^2 + (k + 1)^2 = (k(k + 1)(2k + 1)/6) + (k + 1)^2.
\]

Manipulating this formula, we have

\[
1^2 + 2^2 + \ldots + k^2 + (k + 1)^2 = (k(k + 1)(2k + 1)/6) + (k + 1)^2
\]

\[
= k(k + 1)(2k + 1) + (k + 1)^2
\]

\[
= (k + 1)(k(2k + 1) + 6(k + 1))/6
\]

\[
= (k + 1)(2k^2 + k + 6k + 6)/6
\]

\[
= (k + 1)(2k^2 + 7k + 6)/6
\]

\[
= (k + 1)(k + 2)(2k + 3)/6
\]

\[
= (k + 1)((k + 1) + 1)(2(k + 1) + 1)/6,
\]

which is exactly what we wanted to prove.
(f) Explain why these steps show that this formula is true whenever \( n \) is a positive integer.

**Solution.** Since the base case holds (by parts (a) and (b)), and since the inductive step holds (part (e)), we conclude that \( P(n) \) is true for all positive integers \( n \); that is, for all positive integers \( n \), we have

\[
1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.
\]

\[\square\]

**Exercise 2.** Prove that 6 divides \( n^3 - n \) whenever \( n \) is a nonnegative integer.

**Proof By Induction.** We wish to prove that 6 divides \( n^3 - n \) for integers \( n \geq 0 \).

- **Base case:** \( n = 0 \). For \( n = 0 \), the claim is that “6 divides \( 0^3 - 0 = 0 \)”, which is true since all nonzero numbers divide 0. Thus the base case holds.

- **Inductive step:** assume that the claim is true for \( k \geq 0 \); that is, assume that 6 divides \( k^3 - k \). We wish to show that 6 divides \( (k+1)^3 - (k+1) \).

Note that

\[
(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1 = (k^3 - k) + 3k(k+1).
\]

We know that 6 divides \( k^3 - k \) BY THE INDUCTIVE HYPOTHESIS. We also know that 6 divides \( 3k(k+1) \) since 3 divides \( 3k(k+1) \) and since one of \( k \) and \( k+1 \) is divisible by 2. Since both \( k^3 - k \) and \( 3k(k+1) \) are divisible by 6, their sum is divisible by 6, so \( (k+1)^3 - (k+1) \) is divisible by 6, as desired.

Since the base case holds, and since the inductive step also holds, we know that 6 divides \( n^3 - n \) whenever \( n \) is a nonnegative integer. \[\square\]

**Exercise 3.** Let \( P(n) \) be the statement that a postage of \( n \) cents can be formed using just 3-cent stamps and 5-cent stamps. Follow this outline to prove that \( P(n) \) is true for \( n \geq 8 \).

(a) Show that the statements \( P(8) \), \( P(9) \), and \( P(10) \) are all true, completing the basis step of the proof.

**Solution.** We can write

\[
8 = 3 + 5
\]

\[
9 = 3 + 3 + 3
\]

\[
10 = 5 + 5,
\]

so the statements \( P(8) \), \( P(9) \), and \( P(10) \) are all true. \[\square\]

(b) What is the inductive hypothesis of the proof?

**Solution.** The inductive hypothesis of this proof, since we’ll be using strong induction, is that “\( P(n) \) is true for all \( n \) with \( 8 \leq n \leq k \), where \( k \geq 10 \) is some integer.” (We take \( k \) to be \( \geq \) the biggest base case, which in this case is 10.) \[\square\]
(c) What do you need to prove in the inductive step?

Solution. We need to prove that, assuming the inductive hypothesis for \( k \), we have that \( P(k+1) \) is true; that is, that \( k+1 \) can be written as a sum of 3’s and 5’s.

(d) Complete the inductive step for \( k \geq 10 \).

Solution. We wish to write \( k+1 \) as a sum of 3’s and 5’s. BY THE INDUCTIVE HYPOTHESIS, \( P(k-2) \) is true (this is why we had to cover 8, 9, and 10 as base cases), letting us write

\[
k - 2 = 3s + 5r.
\]

Adding 3 to both sides gives us

\[
k + 1 = 3s + 5r + 3 = 3(s + 1) + 5r,
\]

so the \( P(k+1) \) is true.

(e) Explain why these steps show that this statement is true whenever \( n \geq 8 \).

Solution. Since the base cases \( n = 8, n = 9, \) and \( n = 10 \) all held, and since the inductive step held for \( k \geq 10 \), \( P(n) \) holds for all \( n \geq 8 \). That is, if \( n \) is an integer \( \geq 8 \), we can make a postage of \( n \) cents just using 3-cent stamps and 5-cent stamps.

Exercise 4. Use strong induction to show that every positive integer \( n \) can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers \( 2^0 = 1, 2^1 = 2, 2^2 = 4, \) and so on. (Hint: For the inductive step, you might want to handle the cases of \( k+1 \) being even and \( k+1 \) being odd separately.)

Solution. We will do a proof by strong induction.

- Base case: \( n = 1 \). We can write \( 1 = 2^0 \), which is a sum of distinct powers of two. Thus the claim holds for \( n = 1 \).

- Inductive step: Let \( k \geq 1 \), and assume the claim holds for all \( n \) with \( 1 \leq n \leq k \). We wish to show that the claim holds for \( k+1 \). We will prove this by cases.

  - Case 1: \( k+1 \) is even. If \( k+1 \) is even, then \( (k+1)/2 \) is an integer. Moreover, it is an integer between 1 and \( k \). BY THE STRONG INDUCTIVE HYPOTHESIS, the claim holds for \( (k+1)/2 \). This lets us write

    \[
    \frac{k+1}{2} = 2^{a_1} + 2^{a_2} + ... + 2^{a_m},
    \]

    where \( a_1, ..., a_m \) are all distinct. Multiplying both sides by 2 yields

    \[
k + 1 = 2(2^{a_1} + 2^{a_2} + ... + 2^{a_m}) = 2^{a_1+1} + 2^{a_2+1} + ... + 2^{a_m+1}.
    \]

    Since \( a_1 + 1, ..., a_m + 1 \) are all distinct (because \( a_1, ..., a_m \) were all distinct), we have written \( k+1 \) as a sum of distinct powers of two. Thus the claim is true (in the case where \( k+1 \) is even).
– Case 2: \( k + 1 \) is odd. BY THE STRONG INDUCTIVE HYPOTHESIS, we may write

\[ k = 2^{b_1} + 2^{b_2} + \ldots + 2^{b_\ell}, \]

where \( b_1, \ldots, b_\ell \) are all distinct. Since \( k + 1 \) is odd, we know \( k \) is even. This implies that none of the \( b_i \)'s are equal to 0: if one were, then we would have

\[ k = 2^0 + (\text{higher powers of } 2) = 1 + (\text{an even number}), \]

meaning \( k \) would be odd, but it’s not. Thus we can write

\[ k + 1 = 1 + 2^{b_1} + 2^{b_2} + \ldots + 2^{b_\ell} = 2^0 + 2^{b_1} + 2^{b_2} + \ldots + 2^{b_\ell}, \]

where 0, \( b_1, \ldots, b_\ell \) are all distinct. This means we have written \( k + 1 \) as a sum of distinct powers of two, so the claim is true (in the case where \( k + 1 \) is odd).

Having proven the inductive step for the cases of \( k + 1 \) even and \( k + 1 \) odd, we have that the inductive step holds.

Since the base case holds, and since the inductive step holds, the claim is true for all positive integers \( n \); that is, any positive integer \( n \) can be written as a sum of distinct powers of 2. \( \Box \)