Algorithms for Mumford Curves (Part II)

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Set-up

Recall from Qingchun’s talk:

- The field $K$ is algebraically closed with a complete non-Archimedean valuation, like $\mathbb{C}_p$.
  - $B = B(a, r) = \{ z \in K : |z - a| < r \}$
  - $B^+ := \{ z \in K : |z - a| \leq r \}$

- **Schottky groups** $\Gamma = \langle \gamma_1, \ldots, \gamma_g \rangle \leq PGL(2, K)$ act on $\mathbb{P}^1$ by
  \[
  \begin{bmatrix} a & b \\ c & d \end{bmatrix} : z \mapsto \frac{az + b}{cz + d}.
  \]

- If $\Sigma \subset \mathbb{P}^1$ is the set of “bad points” for $\Gamma$, then
  \[
  (\mathbb{P}^1 \setminus \Sigma)/\Gamma \cong C,
  \]
  where $C$ is a smooth curve of genus $g$ called a **Mumford curve**.
Good starting data for algorithms

Recall our algorithms:

<table>
<thead>
<tr>
<th>Algorithm</th>
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<tbody>
<tr>
<td><strong>Input:</strong> A set of generators of a Schottky group.</td>
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<tr>
<td><strong>Output:</strong></td>
</tr>
<tr>
<td>(1) The minimal skeleton of the Mumford curve.</td>
</tr>
<tr>
<td>(2) The period matrix of its Jacobian.</td>
</tr>
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<td>(3) Points in the image of the canonical embedding.</td>
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</table>

These goals are more feasible if we have more information than an arbitrary set of generators.

Our algorithms need generators in good position, together with a good fundamental domain.
Good fundamental domains

**Definition**

The generators $\gamma_1, \ldots, \gamma_g$ of $\Gamma$ are in **good position** if there exist $2g$ open balls $B_1, \ldots, B_g, B'_1, \ldots, B'_g$ in $\mathbb{P}^1$ such that

- the corresponding closed balls are pairwise disjoint, and
- $\gamma_i(\mathbb{P}^1 \setminus B'_i) = B_i^+$ and $\gamma_i^{-1}(\mathbb{P}^1 \setminus B_i) = B'_i^+$.

The set $\mathbb{P}^1 \setminus (B_1 \cup \ldots \cup B_g \cup B'_1 \cup \ldots \cup B'_g)$ is called a **good fundamental domain** for $\Gamma$.

The curve $(\mathbb{P}^1 \setminus \Sigma)/\Gamma$ can be obtained from a good fundamental domain by gluing boundaries together.

**Key fact:** Every Schottky group has a set of generators in good position.
Good fundamental domains

The picture for a genus 2 Mumford curve:

We remove $2g$ open balls from $\mathbb{P}^1$. 

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JMM 1/16/2014 5 / 15
Good fundamental domains

The picture for a genus 2 Mumford curve:

The good fundamental domain is shown in blue.
Good fundamental domains

The picture for a genus 2 Mumford curve:

The matrix $\gamma_1$ maps $\mathbb{P}^1 \setminus B'_1$
Good fundamental domains

The picture for a genus 2 Mumford curve:

The matrix $\gamma_1$ maps $\mathbb{P}^1 \setminus B'_1$ into $B_1^+$. 
Good fundamental domains for trees

Schottky groups also act on the tree $(\mathbb{P}^1)^{an}$.

There is a corresponding notion of good fundamental domains for trees.
Example: a genus 2 Mumford curve

Let \( K = \mathbb{C}_3 \), \( \Gamma = \langle \gamma_1, \gamma_2 \rangle \).

<table>
<thead>
<tr>
<th>Matrices:</th>
<th>( \gamma_1 = \begin{pmatrix} -5 &amp; 32 \ -8 &amp; 35 \end{pmatrix} )</th>
<th>( \gamma_2 = \begin{pmatrix} -13 &amp; 80 \ -8 &amp; 43 \end{pmatrix} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eigenvectors:</td>
<td>( (1 : 1), (4 : 1) )</td>
<td>( (2 : 1), (5 : 1) )</td>
</tr>
<tr>
<td>Eigenvalues:</td>
<td>( 27, 3 )</td>
<td>( 27, 3 )</td>
</tr>
</tbody>
</table>

These matrices are in good position.

\[
B_1 = B(4, 1/9), B'_1 = B(1, 1/9)
\]

\[
B_2 = B(5, 1/9), B'_2 = B(2, 1/9)
\]

So \( \gamma_1(\mathbb{P}^1 \setminus B'_1) = B_1^+ \), as well as three similar statements.
Example: a genus 2 Mumford curve

\[ B_1 = B(4, 1/9), \quad B_1' = B(1, 1/9), \quad B_2 = B(5, 1/9), \quad B_2' = B(2, 1/9) \]

Each open ball \( B \) has a corresponding closed ball \( B^+ \), which has a corresponding point \( P \) in \((\mathbb{P}^1)^{an}\). Those points span a subtree:

Write \( c \) for the smallest distance between distinct endpoints. In this case, \( c = 2 \).
Role of good fundamental domains

Compute skeleta by gluing:

\[
\begin{array}{c}
\begin{array}{c}
1 \\
1 \\
1
\end{array}
\end{array}
\]

To approximate formulas like

\[
\prod_{\gamma \in \Gamma} \frac{(z - \gamma a)(\gamma_j z - \gamma \gamma_i a)}{(z - \gamma \gamma_i a)(\gamma_j z - \gamma a)}
\]

with error term \(O(p^n)\), let \(m \geq n/c\) and only multiply over \(\Gamma_m\), the set of words of length at most \(m\) in the generators.
Finding good fundamental domains

We start with matrices $\gamma_1, \ldots, \gamma_g \in PGL(2, \mathbb{Q}_p)$, generating a group $\Gamma$. Let $m = 1$, and as $m$ increases compute $\Gamma_m$ and run three processes:

1. Check if the identity matrix shows up more than once.
2. Check if any non-identity matrices have eigenvalues with the same valuation.
3. Let $a$ be a fixed point of some $\gamma_i$, and look at the subtree $T$ of $(\mathbb{P}^1)^{an}$ spanned by $\Gamma_m a$. See if $T$ contains a tree good fundamental domain, and find a corresponding set $A$ of generators.

Eventually one process will terminate.

- If (1), then $\gamma_1, \ldots, \gamma_g$ are not free generators of $\Gamma$.
- If (2), then $\Gamma$ is not Schottky.
- If (3), then $\Gamma$ is Schottky, $A$ is a set of generators in good position, and $B_i, B_i^+$ can be found from $T$. 
Given $\Gamma$ with $(\mathbb{P}^1 \setminus \Sigma)/\Gamma$ hyperelliptic, what are the ramification points?

Schottky groups for genus $g$ hyperelliptic Mumford curves arise in this way:

- Let $a_0, \ldots, a_g, b_0, \ldots b_g \in K$, with $a_i$ and $b_i$ in a closed ball containing no other $a_j$ or $b_j$.
- Let $s_i = \begin{bmatrix} a_i & b_i \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a_i & b_i \\ 1 & 1 \end{bmatrix}$.
- Let $\Gamma = \langle s_0s_1, s_0s_2, \ldots, s_0s_g \rangle$.

The group $\Gamma$ is Schottky, $(\mathbb{P}^1 \setminus \Sigma)/\Gamma$ is hyperelliptic, and the $2g + 2$ ramification points are $\Theta(a_i)$ and $\Theta(b_i)$, where $\Theta$ is a certain analytic function.
Thanks for your attention!

For more information, check out *Algorithms for Mumford curves*, available at http://arxiv.org/abs/1309.5243