TORIC VARIETIES IN ALGEBRAIC AND SYMPLECTIC GEOMETRY

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1. Introduction

In algebraic geometry, toric varieties form a special family of algebraic varieties. In symplectic geometry, toric manifolds form a special family of symplectic manifolds. In both aspects, toric varieties (manifolds) can be constructed combinatorially, thus are easy to work with, and provide a testing ground for abstract theories. The algebraic construction is more general. For our purpose, when we say toric varieties, we mean normal toric varieties over $\mathbb{C}$.

In more details, toric varieties can be constructed from fans via the fan construction, which we will introduce in Section 2. We will concentrate more on the special case when when the fan arises from a rational polytope, in which case the resulting toric variety is equivariantly projective. On the symplectic side, toric manifolds can be constructed from Delzant polytopes (a special kind of rational polytopes) via the Delzant construction, which we will present without proof in Section 3. On the symplectic side there is an inverse to the Delzant construction by taking the image of the moment map. On the algebraic side, if the rational polytope is latticial, then the polytope can also be recovered by taking the image of the (algebraic) moment map. Moreover, in Section 4 we prove that the algebraic and symplectic constructions agree on the overlap.

The relation between the constructions can be summarized as follows.

$$\begin{tikzcd}
\left\{\begin{matrix}\text{lattice}\\ \text{polytopes}\end{matrix}\right\} \arrow[dr, hook] & & & \\
& \left\{\begin{matrix}\text{rational}\\ \text{polytopes}\end{matrix}\right\} \arrow[r] & \{\text{fans}\} \arrow[r, two heads, "\text{fan}", "\text{construction}"'] & \left\{\begin{matrix}\text{toric}\\ \text{varieties}\end{matrix}\right\} \\
\left\{\begin{matrix}\text{Delzant}\\ \text{polytopes}\end{matrix}\right\} \arrow[ur, hook] & & &
\end{tikzcd} \tag{1}$$

$$\begin{tikzcd}
\left\{\begin{matrix}\text{lattice}\\ \text{polytopes}\end{matrix}\right\} \arrow[r, shift left, "\text{fan construction}"] & \left\{\begin{matrix}\text{equivariantly projective}\\ \text{toric varieties}\end{matrix}\right\} \arrow[l, shift left, dotted, "\text{moment map}"] \\
\left\{\begin{matrix}\text{Delzant lattice}\\ \text{polytopes}\end{matrix}\right\} \arrow[u, hook] \arrow[d, hook] \arrow[r, shift left, "\substack{\text{fan/Delzant}\\ \text{construction}}"] & \left\{\begin{matrix}\text{equivariantly proj-}\\ \text{ective toric varieties}\end{matrix}\right\} \cap \left\{\begin{matrix}\text{symplectic}\\ \text{toric manifolds}\end{matrix}\right\} \arrow[l, shift left, "\text{moment map}"] \arrow[u, hook] \arrow[d, hook] \\
\left\{\begin{matrix}\text{Delzant}\\ \text{polytopes}\end{matrix}\right\} \arrow[r, shift left, "\text{Delzant construction}"] & \left\{\begin{matrix}\text{symplectic}\\ \text{toric manifolds}\end{matrix}\right\} \arrow[l, shift left, "\text{moment map}"]
\end{tikzcd} \tag{2}$$

It should be pointed out that there is essentially nothing original in this note. The algebraic part credits to [6] and the symplectic part to [1]. See also [2].

2. Algebraic perspective

2.1. **The fan construction.** We begin with an abstract definition of toric varieties. However, it will turn out that all toric varieties can be constructed via the fan construction, which we will explain in a moment.

Definition 2.1. *A* ***toric variety*** *of dimension n is a triple (X, T, ρ) where X is a normal algebraic variety over $\mathbb{C}$, $T \cong (\mathbb{C}^*)^n$ is an algebraic n-torus contained in X as a dense open subscheme, and ρ is a T-action on X that extends the T-action on itself. A* ***toric morphism*** *between toric varieties (X, T, ρ) and (X', T', ρ') is a morphism of schemes $\phi\colon X \to X'$ such that $\phi(T) \subset T'$ and $\phi|_T\colon T \to T'$ is a morphism of algebraic groups.*

In particular, a morphism of toric varieties is equivariant with respect to the toric actions (equivariance when restricting to the torus $T \subset X$ is clear, and the general statement follows from the denseness of T).

Let N be a lattice of dimension n and M be its dual lattice. A (strongly convex, rational) **cone** σ in N is a subset of $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ that is the $\mathbb{R}_+$-span of some lattice points, such that it does not contain any nonzero subspace of $N_{\mathbb{R}}$. A **face** of a cone is either the cone itself, or the intersection of the it with one of its supporting hyperplane. In particular it is a cone itself. A (nonempty, finite) **fan** in N is a nonempty finite set of cones in N closed under taking faces and intersections.

For a cone σ in N, its **dual cone** in M is defined to be

$$\sigma^\vee := \{u \in M_{\mathbb{R}} \colon \langle u, v\rangle \geq 0 \text{ for all } v \in \sigma\}.$$

(Note this need not be a cone by our definition because it contains a subspace of $N_{\mathbb{R}}$ of dimension $n - \dim \sigma$, where $\dim \sigma = \dim \operatorname{span}_{\mathbb{R}}(\sigma)$.) Define the **affine toric variety of** σ to be $U_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$.

If σ' is a face of σ, then $\sigma^\vee \subset \sigma'^\vee$. This inclusion induces a ring homomorphism $\mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[\sigma'^\vee \cap M]$, which then induces a morphism of schemes $U_{\sigma'} \to U_\sigma$. This is in fact an inclusion of a distinguished open subscheme: write $\sigma' = \sigma \cap u^\perp$ for some $u \in \sigma^\vee$ such that $u^\perp$ is a supporting hyperplane of σ, then $\mathbb{C}[\sigma' \cap M] = \mathbb{C}[\sigma \cap M]_u$, so $U_{\sigma'} \to U_\sigma$ is the inclusion of $D(u) \subset U_\sigma$. Moreover, from the naturality of this construction we see if Δ is a fan in N, then all affine varieties U_σ, $\sigma \in \Delta$ glue to a scheme, which we denote as $X(\Delta)$, called the **toric variety of** Δ.

We shall check $X(\Delta)$ is a toric variety in the sense of Definition 2.1. Since $\{0\}$ is always a cone in Δ, we see $U_{\{0\}} = \operatorname{Spec} \mathbb{C}[M] = T_N \subset X(\Delta)$ is a dense n-torus. Its action on each U_σ, $\sigma \in \Delta$ is defined by $U_\sigma \times T_N \to U_\sigma$ induced by $\mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[\sigma^\vee \cap M] \otimes \mathbb{C}[M]$, $\chi \mapsto \chi \otimes \chi$. These glue to an action on $X(\Delta)$ that extends the usual T_N-action on itself. Finally, $X(\Delta)$ is reduced because on each affine it is; it is irreducible because it contains the dense open torus T_N; it is separated because for each $\sigma, \sigma' \in \Delta$, $U_{\sigma\cap\sigma'} \to U_\sigma \times U_{\sigma'}$ is a closed embedding as $\mathbb{C}[\sigma^\vee \cap M] \otimes \mathbb{C}[(\sigma')^\vee \cap M] \to \mathbb{C}[(\sigma \cap \sigma')^\vee \cap M]$ is surjective (c.f. [6, Section 1.2]).

The construction $\Delta \mapsto X(\Delta)$, known as the **fan construction**, is natural in the fan Δ in the following sense: if Δ' is a fan in another lattice N', $f\colon N \to N'$ is a lattice map such that for every cone $\sigma \in \Delta$ we have $f(\sigma) \subset \sigma'$ for some cone $\sigma' \in \Delta'$, then f induces a map $f_*\colon X(\Delta) \to X(\Delta')$ of schemes that restricts to a morphism $T_N \to T_{N'}$ of algebraic tori, thus is a toric morphism. This can be first seen on affines, and the gluing is straightforward. See Example 2.2 for an illustration.

Theorem 2.2. *Every toric variety of dimension n is $X(\Delta)$ for some fan Δ in N.*

Proof. See [3, Theorem 1.3.5]. □

Example 2.1. Let $N = M = \mathbb{Z}^{n+1}$ with the natural pairing and $\Delta = \{\sigma_I\colon I \subsetneq \{0,1,\cdots,n\}\}$ where $\sigma_I = \{(x_0,\cdots,x_n) \in \mathbb{R}^{n+1}\colon x_i \geq 0 \text{ for } i \in I,\ x_j = 0 \text{ for } j \notin I\}$. Then Δ is a fan in N. For each I, $\sigma_I^\vee = \{(t_0,\cdots,t_n) \in \mathbb{R}^{n+1}\colon t_i \geq 0 \text{ for } i \in I\}$, $\sigma_I^\vee \cap M$ is the submonoid of $\mathbb{Z}^{n+1}$ generated by $e_i^\vee$, $i \in I$, $\pm e_j^\vee$, $j \notin I$, where $e_0^\vee,\cdots,e_n^\vee$ is the dual basis of the standard basis in $\mathbb{Z}^{n+1}$. Therefore $U_I := U_{\sigma_I} = \operatorname{Spec}\mathbb{C}[Z_i\colon i \in I; Z_j^{\pm}\colon j \notin I]$, which can be realized as the distinguished open subset $D(\prod_{j\notin I} Z_j)$ of $\mathbb{A}^{n+1}_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[Z_0,\cdots,Z_n]$. For I, J, σ_I is a face of σ_J if and only if $I \subset J$, in which case the open embedding $U_I \subset U_J$ is compatible with the above inclusion into $\mathbb{A}^{n+1}_{\mathbb{C}}$. It follows that $X(\Delta) = \cup_I D(\prod_{j\notin I} Z_j) = \mathbb{A}^{n+1}_{\mathbb{C}}\backslash\{0\}$. In this case, the dense open torus is $U_\emptyset = (\mathbb{A}^1_{\mathbb{C}}\backslash\{0\})^{n+1}$ and the toric action is the usual multiplication.

Example 2.2. Let N be any n-dimensional lattice and let $v_0,\cdots,v_n \in N$ be a generating set with $\sum_i v_i = 0$. Let $\Delta = \{\sigma_I\colon I \subsetneq \{0,1,\cdots,n\}\}$ where $\sigma_I \subset N_{\mathbb{R}}$ is the $\mathbb{R}_+$-span of v_i, $i \in I$. We claim that $X(\Delta) = \mathbb{P}^n_{\mathbb{C}} = \operatorname{Proj}\mathbb{C}[T_0,\cdots,T_n]$. For each subset $\{i\}^c \subset \{0,1,\cdots,n\}$, v_j, $j \neq i$ form a basis for N. Let $u_j^{(i)} \in M$, $j \neq i$ be the dual basis. Then $\sigma_{\{i\}^c}^\vee$ is the $\mathbb{R}_+$-span of $u_j^{(i)}$, $j \neq i$. Write U_I for U_{σ_I}, then $U_{\{i\}^c} = \operatorname{Spec}\mathbb{C}[Z_{0/i},\cdots,Z_{n/i}] = \mathbb{A}^n_{\mathbb{C}}$. For $i \neq i'$, $U_{\{i,i'\}^c} \subset U_{\{i\}^c}$ can be realized as the open subscheme $D(Z_{i'/i})$. Therefore the gluing of $U_{\{i\}^c}$ and $U_{\{i'\}^c}$ on the overlap $U_{\{i,i'\}^c} = \operatorname{Spec}\mathbb{C}[Z_{0/i},\cdots,Z_{n/i}]_{Z_{i'/i}} \cong \operatorname{Spec}\mathbb{C}[Z_{0/i'},\cdots,Z_{n/i'}]_{Z_{i/i'}}$ is given by $Z_{j/i'} \mapsto Z_{j/i}Z_{i'/i}^{-1}$, $Z_{i/i'}^{-1} \mapsto Z_{i'/i}$. It follows that we can identify $U_{\{i\}^c}$ as $D_+(T_i) \subset \mathbb{P}^n_{\mathbb{C}}$ by $T_{j/i} \mapsto Z_{j/i}$, compatibly in i, meaning that we obtain an isomorphism $X(\Delta) \cong \mathbb{P}^n_{\mathbb{C}}$. In general, $U_I = D_+(\prod_{j\notin I} T_j)$, so in particular the dense open torus is $D_+(T_0\cdots T_n) = \operatorname{Spec}\mathbb{C}[T_0^{\pm},\cdots,T_n^{\pm}]_0$, whose closed points are exactly those $[z_0 : \cdots : z_n]$ with $z_i \neq 0$ for all i. The toric action is by multiplication.

Let Δ', σ_I''s be the Δ, σ_I's in the previous example for the lattice $\mathbb{Z}^{n+1}$. Then the lattice map $f\colon \mathbb{Z}^{n+1} \to N$, $e_i \mapsto v_i$ maps each σ_I bijectively onto σ_I'. In particular it induces a toric morphism $f_*\colon X(\Delta') \to X(\Delta)$. This is nothing but the quotient map $\mathbb{A}^{n+1}_{\mathbb{C}}\backslash\{0\} \to \mathbb{P}^n_{\mathbb{C}}$, because each $U_{\sigma_I'} \to U_{\sigma_I}$ is the map $\mathbb{A}^{n+1}_{\mathbb{C}} \supset D(\prod_{j\notin I} Z_j) \to D_+(\prod_{j\notin I} T_j) \subset \mathbb{P}^n_{\mathbb{C}}$ induced by $T_{i/j} \mapsto Z_i Z_j^{-1}$, $j \notin I$.

2.2. **Fans from polytopes.** Let N, M be as before. For our purpose, all polytopes are assumed to be convex, and not contained in a hyperplane. A **rational polytope** in $M_{\mathbb{R}}$ is a polytope $P \subset M_{\mathbb{R}}$, satisfying the rationality condition: at each vertex $p \in P$, all edges of P has the form $p + tv$, $t \in [0,\ell]$ for some $v \in M$. A **face** of such a polytope is either the cone itself, or the intersection of it with one of its supporting hyperplanes. A **lattice polytope** in $M_{\mathbb{R}}$ is a polytope whose vertices lie in M, which is in particular a rational polytope.

Let P be a rational polytope in M. It associates a fan in N

$$\Delta_P := \{\sigma_Q : Q \text{ is a face of } P\}$$

where $\sigma_Q = \{v \in N_{\mathbb{R}} : \langle u, v\rangle \le \langle u', v\rangle \text{ for all } u \in Q, u' \in P\}$ is a cone with codimension $\dim Q$. Then the fan construction gives a toric variety $X(\Delta_P)$, which we call the **toric variety of** P. So far we have explained diagram (1) (except the Delzant polytope part, which will be clear from its definition in Section 3).

Recall that in Example 2.2 we realized $\mathbb{P}^n$ as a toric variety. In general, a toric variety is **equivariantly projective** if it admits a toric embedding (i.e. a toric morphism that is an embedding) into some $\mathbb{P}^N$.

Theorem 2.3. *A toric variety is equivariantly projective if and only if it is $X(\Delta_P)$ for some rational polytope P.*

Proof. See [5, Theorem VII.3.11]. The backward direction is also proved in Remark 2.9. □

Example 2.3. The toric variety $X(\Delta) = \mathbb{A}^{n+1}\backslash\{0\}$ in Example 2.1 is not proper over $\mathbb{C}$, so it is not projective, or equivariantly projective. We can also see this using Theorem 2.3 and noting the fan Δ is not of the form Δ_P for any P because it does not contain a cone of full dimension.

Example 2.4. The toric variety $X(\Delta) = \mathbb{P}^n$ in Example 2.2 is trivially equivariantly projective. We can also see this by noting the fan Δ equals Δ_P for $P = K^0$ where $K \subset N_{\mathbb{R}}$ is the convex hull of $v_0, \cdots, v_n$ and $K^0 := \{u \in M_{\mathbb{R}} : \langle u, v\rangle \ge -1 \text{ for all } v \in K\}$, which is an n-simplex. In fact, the faces of P are

$$Q_I := \{u \in M_{\mathbb{R}} : \langle u, v_i\rangle = -1 \text{ for all } i \in I\},\ I \subsetneqq \{0, \cdots, n\},$$

and $\sigma_{Q_I} = \sigma_I$.

2.3. **Orbit decomposition; Divisors.** In this section we present a few more algebraic geometric aspects of toric varieties that enable us to define the moment map in the next section.

Let Δ be a fan in M and let $X = X(\Delta)$ be the corresponding toric variety. For each affine $\sigma \in \Delta$ the affine $U_\sigma \subset X$ has a distinguished closed point x_σ, defined by the $\mathbb{C}$-algebra map $\mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}$ induced by the monoid map

$$\sigma^\vee \cap M \to \mathbb{C},\ u \mapsto \begin{cases} 1, & u \in \sigma^\perp \\ 0, & u \notin \sigma^\perp \end{cases}.$$

The T_N-orbit containing it is denoted O_σ, whose closure is denoted $V(\sigma)$. For example, for Example 2.1, x_{σ_I} is the point whose i-th coordinate is 0 if $i \in I$ and 1 if $i \notin I$; O_{σ_I} is the set of points whose i-th coordinate is 0 if $i \in I$ and nonzero if $i \notin I$; $V(\sigma)$ is the set of points whose i-th coordinate is 0 if $i \in I$. For Example 2.2 we have a completely similar description, using projective coordinate instead of the Euclidean one. In general, $O_\sigma \cong \operatorname{Spec} \mathbb{C}[M(\sigma)] = T_{N(\sigma)}$ where $M(\sigma) = \sigma^\perp \cap M$ is a lattice of codimension $\dim \sigma$ in M and $N(\sigma) = N/(\operatorname{span}(\sigma) \cap N)$ is its dual, so $\dim O_\sigma = \dim V(\sigma) = n - \dim \sigma$. Moreover, X is the disjoint union of the O_σ's, and this gives a stratification of X (c.f. [6, Section 3.1]).

Next we consider Weil divisors and Cartier divisors. In our toric setup, we are naturally interested in those divisors that are $T = T_N$-invariant; we call them T-Weil divisors and T-Cartier divisors, respectively. By the orbit decomposition, all T-Weil divisors are linear combinations of the $D_i := V(\sigma_i)$'s, where σ_i run over all rays (i.e. 1-dimensional cones) in Δ. Since X is integral, noetherian, and normal, the Cartier divisor group embeds into the Weil divisor group.

Fix $\sigma \in \Delta$, let $u \in M$ and let $\chi^u \in \mathbb{C}[M] = K(X)$ be the element defined by u. Then $\mathrm{div}(\chi^u)$ is a T-Weil divisor that corresponds to a T-Cartier divisor. Also, $\mathrm{div}(\chi^u) = \mathrm{div}(\chi^{u'})$ if and only if $u - u' \in \sigma^\perp \cap M = M(\sigma)$ because $\mathcal{O}^\times(U_\sigma) = \mathbb{C}[M(\sigma)]$. Conversely, we have the following.

Lemma 2.4. *A T-Cartier divisor on U_σ is $\mathrm{div}(\chi^u)$ for some $u \in M/M(\sigma)$.*

Proof. Let $I \subset K(X) = \mathbb{C}[M]$ denote the fractional ideal of a Cartier divisor D. By T-invariance, the map $\mathbb{C}[M] \xrightarrow{1\otimes 1} \mathbb{C}[M] \otimes \mathbb{C}[M] \xrightarrow{\Delta\circ(1\otimes x_v)} \mathbb{C}[M]$ maps I to itself for all $v \in N$, where $x_v \colon \mathbb{C}[M] \to \mathbb{C}$ is induced by $v \in N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. It follows that I is a direct sum of some $\mathbb{C} \cdot \chi^u$, $u \in M$. Let $\mathfrak{m}$ denote the maximal ideal at the distinguished point $x_\sigma \in U_\sigma$. Since I is locally principal, localizing at x_σ shows that $I/\mathfrak{m}I$ is a 1-dimensional complex vector space, which implies I is generated by a single χ^u as a $\mathbb{C}[\sigma^\vee \cap M]$-algebra. This generator u is uniquely determined up to $M(\sigma)$. □

Therefore by a usual gluing argument we see the following.

Proposition 2.5. *A T-Cartier divisor on $X(\Delta)$ is the same as an element $(u(\sigma)) \in \varprojlim_{\sigma\in\Delta} M/M(\sigma)$. More explicitly, given such an element, the corresponding T-Cartier divisor over U_σ is $\mathrm{div}(\chi^{-u(\sigma)})$ (minus sign is to match the convention).* □

Explicitly, the T-Weil divisor corresponding to the T-Cartier divisor defined by $(u(\sigma))$ is given by

$$D = -\sum_i \langle u(\sigma_i), v_i \rangle D_i, \tag{3}$$

where v_i is the primitive lattice point on the ray σ_i. To see this, it suffices to check on each U_{σ_i}. By a choice of basis we may assume $M = N = \mathbb{Z}^n$ and $v_i = e_1$ is the first basis vector. Then $U_{\sigma_i} = \mathrm{Spec}\, \mathbb{C}[Z_1, Z_2^\pm, \cdots, Z_n^\pm]$, $D_i = V(Z_1)$, and the order of vanishing $\chi^{-u(\sigma_i)}$ along $V(Z_1)$ is $-\langle u(\sigma_i), e_1 \rangle = -\langle u(\sigma_i), v_i \rangle$.

The **support function** of a Cartier divisor $D = (u(\sigma))$ is

$$\psi = \psi_D \colon |\Delta| \to \mathbb{R},\ v \mapsto \langle u(\sigma), v \rangle,\ v \in \sigma \in \Delta.$$

Here $|\Delta| = \cup_{\sigma\in\Delta}\sigma$. The support function is linear on each cone in Δ, and satisfies the integrability condition that $\psi(|\Delta|) \subset \mathbb{Z}$. Conversely, given such a function, we can recover the T-Cartier divisor from it by

$$D = D_\psi = -\sum_i \psi(v_i) D_i.$$

These two constructions are inverses to each other.

From now on, for our purpose, we restrict ourselves to the special case $\Delta = \Delta_P$ for some lattice polytope $P \subset M_{\mathbb{R}}$, so that $|\Delta| = N_{\mathbb{R}}$ and X is proper ([6, Section 2.4]).

The **associated polytope** of a T-Cartier divisor D is

$$P_D := \{u \in M_{\mathbb{R}} \colon u \geq \psi_D \text{ as functions on } N_{\mathbb{R}}\}. \tag{4}$$

Under our assumption it is a rational polytope.

Lemma 2.6.

$$\Gamma(X, \mathcal{O}(D)) = \bigoplus_{u \in P_D \cap M} \mathbb{C} \cdot \chi^u.$$

Proof. By T-invariance as in the proof of Lemma 2.4, we can deduce $\Gamma(X, \mathcal{O}(D))$ is the direct sum of some $\mathbb{C} \cdot \chi^u$, $u \in M$. Now that $\chi^u \in \Gamma(\mathcal{O}(D)) \iff \operatorname{div}(\chi^u) \geq -D \iff \langle u, v_i \rangle \geq \psi_D(v_i)$ for all $i \iff u \geq \psi_D \iff u \in P_D \cap M$, the statement follows. □

There is a distinguished T-Cartier divisor D_P on $X = X(\Delta_P)$, whose support function is defined by

$$\psi_P := \min_{u \in P \cap M} \langle u, \cdot \rangle \colon N_{\mathbb{R}} \to \mathbb{R}.$$

Its associated polytope is exactly P, i.e. we have

Lemma 2.7. $P_{D_P} = P$.

Proof. We have $u \in P_{D_P} \iff u \geq \psi_P \iff \langle u, v \rangle \geq \min_{u' \in P \cap M} \langle u', v \rangle$ for all $v \in N_{\mathbb{R}} \iff u \in P$, because P is convex. □

Proposition 2.8. *The line bundle $\mathcal{O}(D_P)$ is globally generated and ample.*

Proof. We first prove globally generation. It suffices to show for each n-dimensional $\sigma \in \Delta_P$ there exists a global section χ^u, $u \in P \cap M$, that is nonvanishing at the closed point $V(\sigma)$, because by T-invariance and the topology of stratification, such a section is also nonvanishing on any orbit closure $V(\tau)$ containing $V(\sigma)$. Since $V(\sigma) = \cap_{\sigma_i \subset \sigma} V(\sigma_i)$, it suffices to check χ^u is nonvanishing on each $V(\sigma_i)$ for those rays $\sigma_i \subset \sigma$. By (3), this is equivalent to $\langle u, v_i \rangle = \langle u(\sigma_i), v_i \rangle$ for those i. This always has a unique solution, namely $u = u(\sigma)$, since P is convex and σ is n-dimensional. Note also $u(\sigma)$ is a vertex for P and

$$\sigma^\vee = \{u' \in M_{\mathbb{R}} \colon u(\sigma) + tu' \in P \text{ for some } t > 0\}. \tag{5}$$

Next we prove ampleness. Globally generation gives us a map $\phi \colon X \to \mathbb{P}^{r-1}$ defined by sections χ^u, $u \in P \cap M$, where $r = \#(P \cap M)$. For each $\sigma \in \Delta$ as above, we have shown $\chi^{u(\sigma)}$ is nonvanishing on U_σ, thus we get a map $\phi|_{U_\sigma} \colon U_\sigma \to D_+(\chi^{u(\sigma)}) \cong \mathbb{A}^{r-1}$ induced by the ring map $\mathbb{C}[x_1, \cdots, x_{r-1}] \to \mathbb{C}[\sigma^\vee \cap M]$, $x_i \mapsto \chi^{u_i - u(\sigma)}$ where u_i runs over $P \cap M \backslash \{u(\sigma)\}$. If this were surjective for all σ, then ϕ would be an embedding, proving very-ampleness. In general, this need not be true, but is true upon replacing P by mP for some $m \in \mathbb{Z}_+$, in view of (5). Finally, note by definition we can check $\psi_{mP} = m\psi_P$, thus $D_{mP} = mD_P$. This shows mD_P is very ample for some $m \in \mathbb{Z}_+$, proving the statement. □

Remark 2.9. We can now prove the backward direction of Theorem 2.3. Pick an embedding $i \colon X \hookrightarrow \mathbb{P}^N$ defined by some very ample mD_P. Its restriction to T_N is given by $\operatorname{Spec} \mathbb{C}[M] \to \operatorname{Spec} \mathbb{C}[T_0^{\pm}, \cdots, T_N^{\pm}]_0$ induced by the lattice map

$$\{(k_0, \cdots, k_N) \in \mathbb{Z}^{N+1} \colon \sum k_i = 0\} \to M, \ e_i - e_j \mapsto u_i - u_j,$$

thus is a morphism of algebraic tori. Here $u_0, \cdots, u_N$ denote the elements in $mP \cap M$. Therefore, i is a toric embedding.

2.4. **The moment map.** Let the notations be as in the previous section. Upon choosing a basis, the globally generated line bundle $\mathcal{O}(D_P)$ defines a map $X = X(\Delta_P) \to \mathbb{P}^{r-1}$ where $r = \#(P \cap M)$. The **(algebraic) moment map** of X is defined to be

$$\mu\colon X(\mathbb{C}) \to M_{\mathbb{R}},\ x \mapsto \frac{1}{\sum_{u\in P\cap M} |\chi^u(x)|^2} \sum_{u\in P\cap M} |\chi^u(x)|^2 u. \tag{6}$$

This is independent of the choice of $\mathcal{O}(D)_x \cong \mathcal{O}_x$ used to define $\chi^u(x) \in \mathbb{C}$.

Proposition 2.10. *The image of μ is P.*

Proof. For each face Q of P, we show that μ maps the orbit O_{σ_Q} onto Q. First, for $x \in O_{\sigma_Q}(\mathbb{C})$, we have $\chi^u \in \Gamma(X, \mathcal{O}(D_P))$ is nonvanishing at x if and only if $u \in Q \cap M$, as shown in the proof of Proposition 2.8. In such case, its value at x up to a scalar is $\chi^{u-u(\sigma)}(x)$. Thus $\mu|_{O_{\sigma_Q}(\mathbb{C})}$ maps into Q. Note by the identification $O_{\sigma_Q} = \operatorname{Spec}\mathbb{C}[M(\sigma_Q)]$ where $M(\sigma_Q) = \operatorname{span}(Q - u(\sigma)) \cap M$, a closed point $x \in O_{\sigma_Q}$ can be regarded as a character of $M(\sigma_Q)$, or alternatively an element $[v(x)] \in N(\sigma_Q)_{\mathbb{C}}/N(\sigma)$ via the exponential map $e^{2\pi i \cdot}$. Now the surjectivity of the map

$$\mu|_{O_{\sigma_Q}(\mathbb{C})}\colon N(\sigma_Q)_{\mathbb{C}}/N(\sigma) \to Q,\ [v] \mapsto \frac{1}{\sum_{u\in Q\cap M} |e^{2\pi i\langle u-u(\sigma), v\rangle}|^2} \sum_{u\in Q\cap M} |e^{2\pi i\langle u-u(\sigma), v\rangle}|^2 u$$

is elementary. See e.g. [6, Section 4.2]. □

We have now established the first row of diagram (2).

3. Symplectic perspective

Definition 3.1. *A* ***symplectic toric manifold*** *of dimension $2n$ is a tuple $(X, \omega, T, \rho, \mu)$ where (X, ω) is a compact connected $2n$-dimensional symplectic manifold, $T \cong (S^1)^n$ is an n-torus, ρ is an effective Hamiltonian T-action on (X, ω) with a moment map $\mu\colon X \to \mathfrak{t}^*$, where $\mathfrak{t}$ is the Lie algebra of T and $\mathfrak{t}^*$ is its dual.*

Explicitly, the Hamiltonian action and moment map conditions require that

- for any $X \in \mathfrak{t}$, let $\xi^\#$ be the associated vector field on X of the T-action and let $\mu^\xi = \langle \mu(\cdot), \xi\rangle\colon X \to \mathbb{R}$, then $\iota_{\xi^\#}\omega = -d\mu^\xi$;
- μ is T-invariant.

Definition 3.2. *For a lattice M, a* ***Delzant polytope*** *P in $M_{\mathbb{R}}$ is a polytope in $M_{\mathbb{R}}$ satisfying:*

- *Rationality: at each vertex $p \in P$, all edges has the form $p + tv$, $t \in [0, \ell]$ for some $v \in M$.*
- *Simplicity: at each vertex of P there are exactly n edges.*
- *Smoothness: at each vertex of P, write the n edges as $p + tv_i$, $t \in [0, \ell_i]$, $v_i \in M$ primitive, $i = 1, \cdots, n$. Then $v_1, \cdots, v_n$ form a $\mathbb{Z}$-basis for M.*

Let T be an n-torus, then the kernel of $exp\colon \mathfrak{t} \to T$ is an n-dimensional lattice $N \subset \mathfrak{t}$, so that $\mathfrak{t} = N_{\mathbb{R}}$ and $\mathfrak{t}^* = M_{\mathbb{R}}$ where M is the dual of N. With this natural choice of M, N, we can talk about Delzant polytopes in $\mathfrak{t}^*$.

Toric manifolds are classified by the following theorem, which explains the third row of diagram (2).

Theorem 3.3 (Delzant)**.** *Let $(X, \omega, T, \rho, \mu)$ be a toric manifold. Then $\mu(X) \subset \mathfrak{t}^*$ is a Delzant polytope. Moreover, for fixed T, the map*

$$\{\textit{toric manifolds, torus} = T\} \to \{\textit{Delzant polytopes in } \mathfrak{t}^*\},$$
$$(M, \omega, T, \rho, \mu) \mapsto \mu(M)$$

is bijective.

Proof. See [4]. □

Example 3.1. Let $\mathbb{CP}^n$ be equipped with the Fubini-Study symplectic form $\omega_{FS} = \frac{i}{2\pi}\partial\bar{\partial}\log|z|^2$ ($z = (z_0, \cdots, z_n)$ is any local lift into $\mathbb{C}^{n+1}$). The torus $T = (S^1)^{n+1}/S^1$ (S^1 acts diagonally on $(S^1)^{n+1}$) acts on $\mathbb{CP}^n$ effectively by $[\lambda_0 : \cdots : \lambda_n] \cdot [z_0 : \cdots : z_n] = [\lambda_0 z_0 : \cdots : \lambda_n z_n]$. Identify $\mathfrak{s}^1 = \mathbb{R}$ by writing $S^1 = \mathbb{R}/\mathbb{Z}$. Then $\mathfrak{t} = \mathbb{R}^{n+1}/\mathbb{R}$ and $\mathfrak{t}^* = \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} \colon x_0 + \cdots + x_n = 0\} =: \mathbb{R}_0^{n+1}$.

We claim the T-action is Hamiltonian, and a moment map is given by $\mu \colon \mathbb{CP}^n \to \mathbb{R}_0^{n+1}$, $[z_0 : \cdots : z_n] \mapsto \left(\frac{|z_0|^2}{|z|^2} - \frac{1}{n+1}, \cdots, \frac{|z_n|^2}{|z|^2} - \frac{1}{n+1}\right)$. The map μ is clearly T-invariant, so it remains to check

$$\iota_{\xi_j^\#}\omega_{FS} = -d\mu^{\xi_j} \tag{7}$$

for $j = 0, 1, \cdots, n$, where ξ_j is the image of the i-th coordinate vector in $(\mathfrak{s}^1)^{n+1}$ in $\mathfrak{t}$. Let $p \colon \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{CP}^n$ denote the projection map, then it suffices to check (7) after pullingback to $\mathbb{C}^{n+1}\backslash\{0\}$. A lift of $\xi_j^\#$ is given by $\tilde{\xi}_j^\# = 2\pi i(z_j\partial_{z_j} - \bar{z}_j\partial_{\bar{z}_j})$. The pullback of ω_{FS} and the pullback of μ have the same expressions as given above, respectively. Therefore we only need to check

$$2\pi i\iota_{z_j\partial_{z_j} - \bar{z}_j\partial_{\bar{z}_j}}\left(\frac{i}{2\pi}\partial\bar{\partial}\log|z|^2\right) = -d\left(\frac{|z_j|^2}{|z|^2}\right),$$

which is a straightforward verification. The corresponding Delzant polytope for the toric manifold $\mathbb{CP}^n$ is thus $\{(x_0, \cdots, x_n) \in \mathbb{R}_0^{n+1} \colon x_i \geq -\frac{1}{n+1} \text{ for all } i\} \subset \mathbb{R}_0^{n+1}$.

4. Two constructions agree

Lemma 4.1. *If P is Delzant, then $X(\Delta_P)$ is smooth.*

Proof. All maximal cones in Δ_P has the form σ_u for some vertex $u \in P$, thus equals the dual cone of $\{u' \in M_{\mathbb{R}} \colon u + tu' \in P \text{ for some } t > 0\}$. By the Delzant assumption, we see σ_u is spanned by a basis of N. By a change of coordinate we can assume $N = \mathbb{Z}^n$ and σ_u is spanned by the basis vectors $e_1, \cdots, e_n$. It follows that $U_{\sigma_u} \cong \mathbb{A}^n$. Since smoothness is local this proves the lemma. □

Remark 4.2. The converse is also true: if $X(\Delta_P)$ is smooth, then P is Delzant. Since we don't need this fact, we refer readers to Section 2 of [6]. In view of the agreement of algebraic geometric and symplectic geometric constructions, the Delzant condition on polytopes in symplectic geometry is really a smoothness condition. For general rational polytopes, one can still expect a Delzant construction, yielding some symplectic toric manifolds with singularities. In fact, by a translation and rescaling of the polytope (which, under the Delzant correspondence, corresponds to adding a constant to the moment map, and scaling the symplectic form and moment map by the same constant), we may restrict ourselves to lattice polytopes, and we may further assume the

T-Cartier divisor associated to this polytope is very ample (c.f. Proposition 2.8). Then Theorem 4.3 exactly gives such a generalized Delzant construction.

From now on, let P be a Delzant lattice polytope. We show that the Delzant construction and the fan construction for P agree, finishing the diagram (2). For simplicity assume $M = N = \mathbb{Z}^n$, and write X for the underlying complex manifold constructed from P via the fan construction (i.e. $X = X(\Delta_P)(\mathbb{C})$). Under our Delzant assumption, in the proof of Proposition 2.8 we may choose $m = 1$. In other words D_P is very ample, thus defines an equivariant embedding $i\colon X \hookrightarrow \mathbb{CP}^{r-1}$ (see Remark 2.9 for equivariance), which is an embedding of complex manifolds. Therefore $i^*\omega_{FS}$ is nondegenerate, thus defines a symplectic structure on X. The toric action on X restricts to a $T = (S^1)^n$-action. Now the agreement of the two constructions is a consequence of the following theorem together with Proposition 2.10.

Theorem 4.3. *The T-action on $(X, i^*\omega_{FS})$ is Hamiltonian. Moreover, the algebraic moment map μ_{alg} for X as in (6) is also a moment map in the symplectic sense for this Hamiltonian action.*

We need the following symplectic geometric lemma whose proof is straightforward.

Lemma 4.4. *If $f\colon (X, f^*\omega) \to (Y, \omega)$ is a map of symplectic manifolds, equivariant with respect to a G-action on X, a Hamiltonian H-action on Y with moment map $\mu\colon Y \to \mathfrak{h}^*$, and a Lie group homomorphism $\alpha\colon G \to H$, then the G-action on X is Hamiltonian with a moment map $\nu = \alpha^* \circ \mu \circ f$.* □

Proof of Theorem 4.3. In view of the argument in Proposition 2.10, the restriction of the toric embedding i to the dense open torus of X is a map

$$i|_{T_1}\colon T_1 \hookrightarrow T_2,\ x \mapsto [e^{2\pi i\langle u_1, v\rangle} : \cdots : e^{2\pi i\langle u_r, v\rangle}]$$

of algebraic tori, where T_1, T_2 are the dense open tori of $X, \mathbb{CP}^r$, respectively, and $[v] = [v](x) \in \mathbb{C}^n/\mathbb{Z}^n \cong (\mathbb{C}^*)^n = T_1$ via $e^{2\pi i\cdot}$. Identify $T_1 \cong \mathbb{C}^n/\mathbb{Z}^n$ and $T_2 \cong (\mathbb{C}^r/\mathbb{Z}^r)/(\mathbb{C}/\mathbb{Z})$, the map $i|_{T_1}$ restricts to maximal compact tori:

$$i|_{\mathbb{R}^n/\mathbb{Z}^n}\colon (\mathbb{R}/\mathbb{Z})^r \to (\mathbb{R}/\mathbb{Z})^r/(\mathbb{R}/\mathbb{Z}),\ [v] \mapsto [\langle u_1, v\rangle, \cdots, \langle u_r, v\rangle].$$

Ther corresponding Lie algebra map is $i_*\colon \mathbb{R}^n \to \mathbb{R}^r/\mathbb{R},\ v \mapsto [\langle u_1, v\rangle, \cdots, \langle u_n, v\rangle]$, and the dual Lie algebra map is $i^*\colon \mathbb{R}^r_0 \to \mathbb{R}^n,\ (x_1, \cdots, x_r) \mapsto \sum_{i=1}^r x_i u_i$.

Now by Lemma 4.4 and Example 3.1, the $T = (\mathbb{R}/\mathbb{Z})^r$-action on X is Hamiltonian with a moment map

$$\mu\colon X \to \mathbb{R}^n,\ x \mapsto \sum_{i=1}^r \left(\frac{|\chi^{u_i}(x)|^2}{\sum_{j=1}^r |\chi^{u_j}(x)|^2} - \frac{1}{r} \right) u_i = \mu_{alg}(x) - \frac{\sum_{i=1}^r u_i}{r}.$$

Since the moment map for a Hamiltonian toric action is unique up to an additive constant, $\mu_{alg} = \mu + \sum_{u \in P\cap M} u/r$ is also a moment map. □

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