Spectral Properties of the Laplacian on Compact Lie Groups

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Project 1 for 18.821

Abstract

By using Fourier series, we show that the Laplace-Beltrami operator on a compact Lie group equipped with a bi-invariant metric has a complete basis of eigenfunctions. We also explicitly compute the spectrum of the Laplace operator.

Contents

1 Introduction

The Laplacian

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

on $\mathbb{R}^n$ is an important operator for mathematicians and physicists. In physics, it arises naturally when one consider the heat flow. In mathematics, it is indispensable in differential geometry and motivates the study of elliptic PDE.

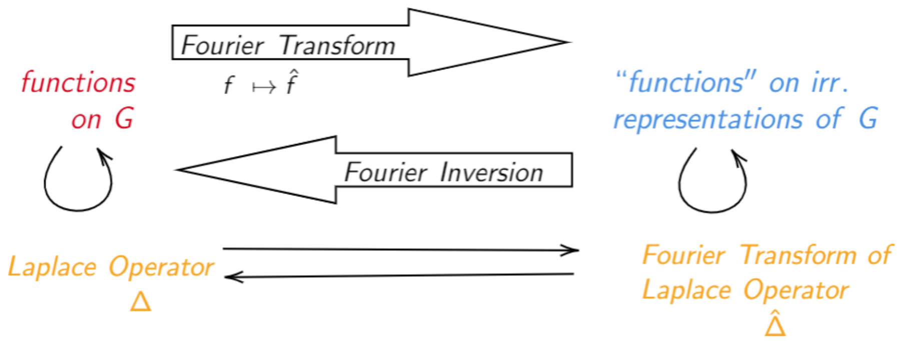

Figure 1: Big picture of the argument

To study a differential operator, it is often useful to understand its spectral properties (eigenfunctions, eigenvalues, and spectrum). We first define these terms in the following simplest case.

Consider the one-dimensional Laplacian $\Delta = d^2/dx^2$ on the real line $\mathbb{R}$. For our purpose, we consider its restriction to the linear operator Δ acting on the space of 1-periodic complex valued smooth functions on $\mathbb{R}$.

Definition 1.1. An 1-periodic (complex valued) smooth function f is called an *eigenfunction* of Δ with *eigenvalue* $\lambda \in \mathbb{C}$ if $\Delta f = \lambda f$. The *spectrum* of Δ, denoted $\sigma(\Delta)$, is the set of all eigenvalues of Δ.

A 1-periodic function on $\mathbb{R}$ is the same as a function on the circle $S^1 = \mathbb{R}/\mathbb{Z}$. By standard Fourier series theory on the circle, it is not hard to write out all eigenfunctions and eigenvalues of Δ, as we shall do in Section 2.1. We will explain that in some sense the set of eigenfunctions are complete.

We can generalize the problem by considering the Laplacian on doubly periodic functions on $\mathbb{R}^2$, or even higher dimension analogs. Formally, we call a subgroup $\Gamma \subset \mathbb{R}^n$ a *lattice* in $\mathbb{R}^n$ if it is an additive subgroup of $\mathbb{R}^n$ generated by n elements, such that the $\mathbb{R}$-linear span of the n generators is $\mathbb{R}^n$. (A standard example is $\Gamma = \mathbb{Z}^n$.) A function f on $\mathbb{R}^n$ is Γ-*periodic* if $f(x+k) = f(x)$ for all $x \in \mathbb{R}^n$, $k \in \Gamma$. We can consider the Laplace operator on the space of (complex valued) smooth Γ-periodic functions on $\mathbb{R}^n$, and ask for its spectral properties. The definitions of eigenfunction, eigenvalue, and (in our situation) spectrum extend in the obvious way to Δ on higher dimensional spaces. A Γ-periodic function on $\mathbb{R}^n$ is the same as a function on the torus $T = \mathbb{R}^n/\Gamma$, so we may rephrase our question as determining the spectral properties of Laplacian on the torus T. In this more general setting, it turns out that the Fourier series technique is still valid with mild modification, which immediately gives the answer to our question.

Any torus is a compact Lie group. A natural generalization of the previous question is to examine the spectral properties of Laplacian on an arbitrary compact Lie group G. In this case, the Laplacian Δ is given by the Beltrami-Laplace operator on Riemannian manifolds (see Appendix A) for a prescribed bi-invariant metric on G. By taking the identity component, we may without loss of generality assume G is connected. Our main theorem is the following.

Theorem 1.2. *Fix an ad-invariant inner product on B and equip G with the corresponding metric. For each irreducible representation $\pi\colon G \to GL(V)$ of G, every matrix coefficient of π is an eigenfunction of Δ with eigenvalue $c_\pi(B)$. Every eigenvalue of Δ is $c_\pi(B)$ for some irreducible representation π. Here $c_\pi(B)$ is the image of the Casimir element of $\mathfrak{g}$ with respect to B under π in $\mathfrak{gl}(V)$, identified with a scalar.*

In particular, the set of eigenfunctions of Δ will still be complete, due to the Peter-Weyl Theorem 4.5. This is expected, since a more general theorem says that the set of eigenfunctions of the Laplace-Beltrami operator on any compact Riemannian manifold is complete [4, Theorem 10.4.19].

The paper is structured as follows. In Section 2, we carefully carry out the Fourier series argument for the torus case and determine $\sigma(\Delta)$ in terms of the lattice Γ (Theorem 2.2). Before we go further, we give a brief review to general facts we will need about Lie theory in Section 3. In Section 4, we describe a generalization of Fourier series on torus to Fourier series on arbitrary compact Lie groups, called the Peter-Weyl Theorem (Theorem 4.5). We will also do some spectral analysis of left invariant differential operators. Finally, in Section 5 we show that the set of eigenfunctions of Δ are complete and give the expression of $\sigma(\Delta)$ for an arbitrary compact connected Lie group G equipped with a bi-invariant metric in terms of irreducible representations of G (Theorem 1.1).

2 Laplacian on Tori

2.1 Laplacian on the Circle

In this section we examine the spectral properties of the Laplacian on the circle $S^1 = \mathbb{R}/\mathbb{Z}$. It serves as a motivation to later generalizations.

By standard Fourier analysis, a (complex valued) smooth function f on S^1 can be uniquely written as a summation

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}$$

for some coefficients $a_n \in \mathbb{C}$. This summation is known as the *Fourier series* of f. By smoothness of f, the coefficients a_n decay rapidly at infinity in the sense that

$$|n|^k |a_n| \to 0 \text{ as } n \to \infty, \text{ for all } k > 0. \tag{1}$$

Conversely, any given sequence $\{a_n\}$ in $\mathbb{C}$ satisfying (1) determines a smooth function f with period 1. Explicitly, the coefficients a_n are determined by the *Fourier inversion formula*

$$a_n = \hat{f}(n) = \int_{S^1} f(x) e^{-2\pi i n x} dx.$$

By the rapid decay of a_n, differentiation commutes with summation. It follows that

$$\Delta f = -\sum_{n=-\infty}^{\infty} 4\pi^2 n^2 a_n e^{2\pi i n x}.$$

From this we deduce that $(ae^{2\pi i n x} + be^{-2\pi i n x}, -4\pi^2 n^2)$, a, b not all zero, $n \in \mathbb{Z}_{\geq 0}$ are all pairs of eigenfunctions, eigenvalues of Δ and that $\sigma(\Delta) = \{-4n^2\pi^2 \colon n \in \mathbb{Z}\}$.

Moreover, the set of eigenfunctions are complete in the sense that every smooth functions on S^1 is a linear combination of some eigenfunctions.

2.2 Laplacian on General Tori

Let $\Gamma \subset \mathbb{R}^n$ be a lattice. Then $T = \mathbb{R}^n/\Gamma$ is a torus equipped with a metric inherited from the Euclidean metric on $\mathbb{R}^n$. Conversely, every torus arises in such way for some integer n and lattice $\Gamma \subset \mathbb{R}^n$. The usual Laplacian $\sum_{i=1}^n \partial^2/\partial x_i^2$ on $\mathbb{R}^n$ descends to the Laplacian Δ on T. We want to examine the spectral properties of Δ in terms of Γ.

Consider the space of square integrable functions on T, denoted $L^2(T)$. It is equipped with inner product

$$\langle f, g \rangle = \int_T f \bar{g} dx,$$

where dx is the volume element on T. It turns out that L^2 spaces are more convenient for the Fourier setup.

Let $\Gamma^* \in \mathbb{R}^n$ denote the dual lattice of Γ with respect to the Euclidean inner product on $\mathbb{R}^n$. Explicitly, choose a $\mathbb{Z}$-basis $v_1, \cdots, v_n$ of Γ, and let $v_1^*, \cdots, v_n^* \in \mathbb{R}^n$ denote its dual basis with respect to the Euclidean inner product. Then Γ^* is defined to be the lattice spanned by $v_1^*, \cdots, v_n^*$. It is straightforward to check that the definition is independent of the choice of basis.

For any $\xi \in \Gamma^*$, let $e_\xi \in C^\infty(T) \hookrightarrow L^2(T)$ be defined by

$$e_\xi(x) = \frac{e^{2\pi i \xi \cdot x}}{\sqrt{\mathrm{vol}(T)}},$$

where $\mathrm{vol}(T)$ denotes the volume of T.

Theorem 2.1. *$\{e_\xi : \xi \in \Gamma^*\}$ is an orthonormal basis of $L^2(T)$.*

Proof. We first show that e_ξ, $\xi \in \Gamma^*$ are orthonormal. For $\xi, \eta \in \Gamma^*$, we have

$$\langle e_\xi, e_\eta \rangle = \frac{1}{\mathrm{vol}(T)} \int_T e^{2\pi i((\xi-\eta)\cdot x)} dx.$$

When $\xi = \eta$, the integrand is 1, thus $\langle e_\xi, e_\xi \rangle = 1$. Suppose now $\xi \neq \eta$, then we can choose a basis vector $v_k \in \Gamma$ such that $(\xi - \eta) \cdot v_k = r \in \mathbb{Z}\backslash\{0\}$. Then, for any fixed $x \in \mathbb{R}^n$ we have

$$\int_0^1 e^{2\pi i(\xi-\eta)\cdot(x+tv_k)} dt = e^{2\pi i(\xi-\eta)\cdot x} \int_0^1 e^{2\pi i r t} dt = 0,$$

so $\langle e_\xi, e_\eta \rangle = 0$ by Fubini's theorem (slicing the torus along the lines in direction v_k).

It remain to show the basis $\{e_\xi\}$ is complete. To show this we apply the Stone-Weierstrass theorem (see [5, Theorem 7.33]). One readily check that $\{e_\xi\}$ is closed under multiplication and conjugation, vanishes at no point (i.e. for any $x \in T$, there exists ξ such that $e_\xi(x) \neq 0$), and separates points (i.e. for $x, y \in T$, $x \neq y$, there exists ξ such that $e_\xi(x) \neq e_\xi(y)$). The conclusion is that the linear span of $\{e_\xi\}$ is dense in $C^0(T)$, thus is in turn dense in $L^2(T)$. □

By Theorem 2.1, a function $f \in L^2(T)$ can be uniquely written as $f = \sum_{\xi \in \Gamma^*} a_\xi e_\xi$ for some $a_\xi \in \mathbb{C}$. This is called the Fourier series of f. The coefficients a_ξ are determined by the *Fourier inversion formula*

$$a_\xi = \hat{f}(\xi) = \langle f, e_\xi \rangle = \int_T f e_{-\xi} dx.$$

Theorem 2.2. *In the notation as before, a complete basis of eigenfunctions for Δ on $L^2(T)$ is given by $\{e_\xi\}$. The spectrum of Δ on T is*

$$\sigma(\Delta) = \{-4\pi^2 |\xi|^2 : \xi \in \Gamma^*\}.$$

Proof. We compute that

$$\Delta e_\xi(x) = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} e_\xi(x) = -4\pi^2 \sum_{k=1}^n \xi_k^2 e_\xi(x) = -4\pi^2 |\xi|^2 e_\xi(x).$$

Here ξ_k denotes the k-th coordinate of ξ. The statement now follows from Theorem 2.1. □

3 Preliminary on Lie Groups

This section is a tranisitional section between our introductory case of the Laplacian acting on Tori and the more general case of the Laplacian acting on compact Lie groups. Here, we provide the necessary background on Lie theory for later sections.

3.1 Topological and Lie groups

Definition 3.1. A *topological group* G is a topological space G with a group structure where multiplication and inversion are continuous. Formally, there are continuous maps

$$\begin{aligned} m : G \times G &\to G, \\ i : G &\to G, \end{aligned}$$

and an identity element $e \in G$ which satisfy the group axioms:

$$\begin{aligned} m(m(a,b),c) &= m(a, m(b,c)) \\ m(e,a) &= a = m(a,e) \\ m(a,i(a)) &= e = m(i(a),a). \end{aligned}$$

A topological group has both topological and group theoretic properties. For instance, a topological group G is *compact* (resp. *locally compact*) if the underlying topological space G is compact (resp. locally compact). Similarly a topological group G is abelian if the group operation m is commutative on the underlying set of G.

We will commonly use $\cdot$ instead of m to denote the group operation i.e.

$$a \cdot b := m(a,b).$$

The particular example of topological groups we will be concerned with are Lie groups. Recall that an n-dimensional manifold X is a topological space which for every point $p \in X$, there exists an open set U containing p such that U is diffeomorphic to an open ball in Euclidean space $\mathbb{R}^n$. For our purposes the following definition of Lie group suffices:

Definition 3.2. A *(real) Lie group* G is a topological group where the underlying space G is a real manifold and the group operations m and i are smooth maps (i.e. infinitely differential).

Since our main focus is compact Lie groups and every compact Lie group is diffeomorphic to a closed subset of $\mathbb{R}^n$, a reader not familar with manifolds should think of a manifold as simply a subspace of Euclidean space $\mathbb{R}^n$ cut out by smooth equations (for instance a sphere). In such a space G the definition of smooth maps from $G \times G$ to G or G to G is the familiar definition of smooth for Euclidean space.

Example 3.3. The general linear groups $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$ are both Lie groups and as spaces are isomorphic to open subsets of $\mathbb{R}^{n^2}$ and $\mathbb{R}^{4n^2}$ respectively. The unitary subgroup

$$U(n) = \{g \in GL_n(\mathbb{C}) | g \cdot g^* = I\}$$

is a compact Lie group.

To fully define category of Lie (resp. topological) groups, we need to say what morphisms between Lie (resp. topological) groups are. These need to respect both the group and manifold (resp. topological) structure of G. Specifically, morphisms $\phi : G \to H$ are smooth (resp. continuous) maps which are also group homomorphisms.

Since a (real) Lie group G's underlying space is a (real) manifold, it is natural to talk about tangents, vector fields and differentials.

Definition 3.4. For a manifold M, the tangent space at a point $p \in M$ is the equivalence classes of curves $\phi : (-1,1) \to M$ with $\phi(0) = p$ under the equivalence $\phi = \phi'$ if

$$\frac{d}{dt}\phi|_0 = \frac{d}{dt}\phi'|_0.$$

Since a manifold locally is Euclidean, we may glue these together locally and form a topological space called the *tangent bundle*

$$TM = \cup_{p \in M} T_p M.$$

For subsets

For Lie groups the group operations makes tangents, vector fields and differentials very tractable.

Proposition 3.5. *The tangent bundle of a Lie group G is trivializable–it is diffeomorphic to*

$$T_eG \times G.$$

Proof. We have a smooth map

$$TG \to T_eG \times G$$

sending an equivalence class $[\phi] \in T_gG$ to $([g^{-1} \cdot \phi], g)$ which has a smooth inverse $([\phi], g) \mapsto [g \cdot \phi]$. □

This motivates us to pay special attention to the tangent space at the identity element.

The tangent space has a natural projection $pr : TM \to M$ sending a tangent vector at p to p.

Definition 3.6. A *vector field* X for a manifold M is a section of the tangent bundle i.e. a smooth map $f : M \to TM$ such that $pr \circ f$ is the identity on M. The set of vector fields of M is denoted $\mathfrak{X}(M)$. For a Lie group G, a vector field X is *left-invariant* if $X(g) = g \cdot X(e)$.

Notice that each left invariant vector field X is determined by its value $X(e)$ and that each element $[\phi]$ of T_eM determines a unique left invariant vector by defining $X(g) = [g \circ \phi]$. Thus

Proposition 3.7. *The left invariant vector fields of G are in bijection with elements of T_eG.*

For $f \in C^\infty(M)$ (i.e. a smooth function on M, there is a directional derivative along a vector field X which corresponds to at each point p taking the derivative

$$\lim_{t \to 0} \frac{f(X(p)(t)) - f(p)}{t}.$$

Lemma 3.8. *Taking the direction derivative gives a bijection between vector fields and maps $D : C^\infty(M) \to C^\infty(M)$ satisfying*

1. *D is $\mathbb{C}$-linear*

2. *$D(fg) = fD(g) + gD(f)$ (this is called the* Liebniz rule*).*

Such maps are called derivations*.*

As in the familiar Euclidean case, for general manifolds, we may compose vector fields and also multiply them by elements of $C^\infty(M)$.

Definition 3.9. Any action on $C^\infty(M)$ which is a $C^\infty(M)$-linear combination of compositions of derivations is called a *partial differential operator*. The set of partial differential operators of M forms a $C^\infty(M)$-algebra under we adding partial differential operators and composing them. We denote this algebra by $\mathrm{PDO}(M)$.

An alternate definition of partial differential operators that may be more intuitive for readers is it is the module of $C^\infty(M)$-linear combination of compositions of derivations quotiented by the ideal generated by elements which act in the same way on $C^\infty(M)$.

3.2 Lie algebra and universal enveloping algebra

Definition 3.10. A *Lie algebra* over a the complex (resp. real) numbers is a complex (resp. real)-vector space L equipped with bracket operation $[-,-] : L \times L \to L$ that

1. is bilinear,

2. $[x, x] = 0$ for all $x \in L$

3. $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$ for all $x,y,z \in L$.

A morphism between Lie algebras L and L' is a linear map $\phi : L \to L'$ such that $\phi[v,w] = [\phi(v), \phi(w)]$.

Example 3.11. Let V be a complex vector space over the field and let $\mathfrak{gl}(V)$ denote the linear endomorphisms of V with the bracket structure

$$[T,S] = T \circ S - S \circ T.$$

Then $\mathfrak{gl}(V)$ is a complex Lie algebra over k.

The important example for us will be slightly more difficult

Proposition 3.12. *Suppose M is a manifold. Then the vector fields on M form a real Lie algebra with the bracket operation given by*

$$[X,Y](f) := X \circ Y(f) - Y \circ X(f)$$

where we are explicitly identifying vector fields X, Y, and $[X,Y]$ with their associated derivation on $C^\infty(M)$.

As in the previous subsection, we pay special attention to the left-invariant vector fields:

Exercise 3.13. For a Lie group G, the left-invariant vector fields form a sub-Lie algebra of the vector fields of G.

The solution to this exercise is entirely computational based on verifying the axoims of Lie algebras and that $[X,Y]$ is in fact left-invariant when both X and Y are. From now on for a Lie group G, we let $\mathfrak{g}$ refer to the Lie algebra of left invariant vector fields on G. Importantly for us:

Example 3.14. For the Lie group $GL_n(\mathbb{C})$, the associated Lie algebra of left invariant differentials is isomorphic to $\mathfrak{gl}(\mathbb{C}^n)$.

This follows from computing the tangent space at the identity which in Proposition 3.7 we showed is isomoprhic to the space of left invariant vector fields.

Definition 3.15. For a vector space V over a field k. Then *tensor algebra* of V is defined as

$$T(V) = k \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \dots$$

where multiplication is defined by concatenating tensors (for instance $(a \otimes b) \times (c) = a \otimes b \otimes c$. For a Lie algebra $\mathfrak{g}$ over $k = \mathbb{R}$ or $\mathbb{C}$, its *enveloping algebra* is the associated algebra $U(\mathfrak{g}) = T(\mathfrak{g})/I$, where I is the ideal generated by the relations $a \otimes b - b \otimes a = [a,b]$, $a, b \in \mathfrak{g}$.

$U(\mathfrak{g})$ is an associative algebra which is universal in the sense that every Lie algebra homomorphism $\mathfrak{g} \to A$ to an associative algebra factors as $\mathfrak{g} \to U(\mathfrak{g}) \to A$ where the second map is an associative algebra map.

In particular, the natural map

$$\mathfrak{g} \mapsto \mathfrak{X}(G) \to \mathrm{PDO}(G)$$

factors through $U(\mathfrak{g})$. Hence we get a map

$$U(\mathfrak{g}) \to \mathrm{PDO}(G). \tag{2}$$

whose image consists of left-invariant partial differential operators.

In a similar vein to how rings have anti-homomoprhisms which swap the order of multiplication, Lie algebras have

Definition 3.16. A lie algebra *anti-homomorphism* is a linear map $\phi : L \to L'$ such that

$$\phi[x, y] = [\phi(y), \phi(x)].$$

Example 3.17. For any Lie algebra L, the map $\phi(v) = -v$ is an anti-automorphism since

$$\phi[v, w] = -[v, w] = [w, v] = [-w, -v].$$

This Lie algebra anti-automorphism extends to an algebra anti-automorphism $T(L) \to T(L)$ sending

$$x_1 \otimes x_2 \dots x_n \to (-1)^n x_n \otimes x_{n-1} \dots x_1.$$

This map sends the ideal I generated by the relations $a \otimes b - b \otimes a = [a, b]$ to itself and hence extends an anti-automorphism of $U(L)$. We denote this anti-automorphism by $w \mapsto w^*$.

3.3 Representations of Lie groups and Lie algebras

As in the theory of finite groups, an important tool to analyze Lie and topological groups is their (complex) representations. In the finite groups case, a representation is a map $G \to GL(V)$ for a complex vector space V. In the Lie and topological groups case, we require this map to satisfy the additional structure of the manifold.

Definition 3.18. A representation of a topological group G is a map of topological groups $\pi : G \to GL(V)$ (in other words π is continuous and a group homomorphism).
A representation of a Lie group G is a map of Lie groups $\pi : G \to GL(V)$ (in other words π is smooth and a group homomorphism).
A representation of a Lie algebra L is a Lie algebra map $\pi : L \to \mathfrak{gl}(V)$ (in other words $\pi[x, y] = \pi(x)\pi(y) - \pi(y)\pi(x)$).

As in the case of $\mathbb{R}^n$, for a locally compact topological group G, there is a unique nonzero (up to scalar multiplication) left Haar measure dg, which we can integrate over defined by the following properties:

1. It is left invariant in that for any subset S and any $h \in G$
$$\int_G 1_S dg = \int_G 1_{hS} dg$$
2. For any compact set K
$$\int_G 1_K dg < \infty$$
3. For any open set U,
$$\int 1_U dg = \inf_{K \supseteq U} \int 1_K dg$$
where K is compact.
4. For any subset X which can be written as a countable sequence of unions and intersections of open and closed sets, then
$$\int 1_X dg = \inf_{U \subseteq X} \int 1_U dg.$$

For Lie groups, the measure is much more tangible since the measure arises from a measure on Euclidean space. For compact groups, more is true about the measure: Let

$$Vol(G) = \int_G 1_G dg$$

where here we made a choice of Haar measure dg on G.

Proposition 3.19. *If G is a compact group, then in fact dg is also right invariant i.e.*

$$\int_G 1_S dg = \int_G 1_{Sh} dg.$$

A locally compact group where dg is also right invariant is called unimodular.

Proof. Define dg_h to be the measure defined by

$$\int_G f(g)\, dg_h = \int_G f(gh)\, dg.$$

Then dg_h is a left invariant measure so by the uniqueness of the Haar measure of G, dg_h is a constant multiple of dg. To verify that this constant is 1, we compute

$$\int_G 1_G dg_h = \int_G 1_G(gh) dg = Vol(G) = \int_G 1_G dg.$$

$\square$

Notice when G is not compact the above argument does not work since $\int_G 1_G dg$ will be infinite.

Remark 3.20. For compact Lie groups, condition (2) guarantees that $\int_G 1_G dg < \infty$. Thus for any complex representation $\pi : G \to GL(V)$, there exists a positive definite bi-linear form $\langle \cdot, \cdot \rangle_\pi$ on V_π given by choosing any positive definite bi-linear form $\langle \cdot, \cdot \rangle$ on V_π and defining

$$\langle v, w \rangle_\pi := \frac{1}{Vol(G)} \int_G \langle \pi(g)v, \pi(g)w \rangle dg.$$

Thus every representation of compact Lie groups is *unitary*. We will use this fact in computations in Section 4.

Another important result we will need for Section 4 is Schur's lemma: Given two representations V, W of G, define

$$Hom_G(V, W) := \{\mathbb{C}\text{-linear maps } V \text{ to } W \text{ which commutes with the action of } G\}.$$

Lemma 3.21 (Schur's Lemma)**.** *Let G be a finite dimensional Lie group and $(\pi, V_\pi), (\tau, V_\tau)$ be irreducible representations of G. Then*

1. *If $\tau \neq \pi$, then $Hom_G(V_\tau, V_\pi) = 0$.*
2. *If $\tau \cong \pi$, then $Hom_G(V_\tau, V_\pi) = \mathbb{C}$.*

Given a Lie group representation $\pi : G \to GL(V)$, it induces a map on tangent space $d\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ (called the *differential* of π) which is a Lie algebra representation. It turns out that to a certain degree the map $d\pi$ determines much of the map π and when G is in fact connected, $d\pi$ determines π. For this reason, the representation theory of Lie groups is closely related to representation theory of Lie algebras.

Notice that $\mathfrak{gl}(V)$ is a Lie algebra over $\mathbb{C}$ (since V is a complex vector space) and thus any Lie algebra homomorphism $d\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ factors through the *complexification* of $\mathfrak{g}$ which is the Lie algebra $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$.

The representation theory of complex Lie algebras is well understood and is important to applications of our results although we will not need the full strength of the theory to state our results. Since the theory is rather involved, we just state the results without proof but do include some techniques and motivation. A more thorough analysis of the representation theory of semisimple Lie algebras can be found in [2].

Lemma 3.22. *Let $\mathfrak{h}$ be an abelian complex Lie algebra (i.e. for all $a, b \in \mathfrak{h}$, $[a, b] = 0$). Then for any representation $\pi : \mathfrak{h} \to \mathfrak{gl}(V)$, the space V decomposes into 1-dimensional eigenspaces. In other words for finite dimensional V, the image matrices $\pi(\mathfrak{h})$ are simultaneously diagonalizable.*

A Lie algebra L is called *simple* if the the only subspaces $W \subseteq L$ such that $[L, W] \subseteq W$ are $W = 0$ and $W = L$. A Lie algebra L is called *semisimple* if it is a direct sum of simple Lie algebras. For a general Lie algebra L, a *Cartan subalgebra* $\mathfrak{h}$ is a maximal abelian subalgebra of L.

For a semisimple Lie algebra L and a representation $\pi : L \to \mathfrak{gl}(V)$, we may restrict π to a Cartan subalgebra $\mathfrak{h}$. Then as a corollary of Lemma 3.22, V decomposes into $\mathfrak{h}$-eigenspaces.

Definition 3.23. For a representation $\pi : L \to \mathfrak{gl}(V)$ and a Cartan subalgebra $\mathfrak{h}$ of L. The λ-*weight space* of V is the subspace of V where $\mathfrak{h}$ acts as the character λ i.e. for all $h \in \mathfrak{h}$,

$$h \cdot v = \lambda(h)v$$

Theorem 3.24 ([2] Section 20)**.** *The irreducible representations $\pi : L \to \mathfrak{gl}(V)$ of a semisimple Lie algebra are classified by their weight spaces. In particular*

1. *Each weight space is* 1*-dimensional.*
2. *There are finitely many $\mathfrak{h}$-eigenvalues and they lie on a lattice.*
3. *There is a unique "largest" $\mathfrak{h}$ eigenvalue.*

3.4 Bi-invariant Riemannian Metrics on Compact Lie Groups

For our purpose, we need to define the Laplace operator on Lie groups. The natural option is to give a Riemannian metric on G and use the Laplace-Beltrami operator on G as a Riemannian manifold. For the definition of Laplace-Beltrami operator, we refer readers to Appendix A. In this section we focus on giving G a metric. Throughout this section G denotes a compact connected Lie group.

For $g \in G$, let $\ell_g, r_g : G \to G$ denote the left, right multiplication by g, respectively. A metric on G is said to be *left* (resp. *right*) *invariant* if it is invariant under pullback by any ℓ_g (resp. r_g) for all $g \in G$. It is said to be *bi-invariant* if it is both left invariant and right invariant.

Left invariant metrics on G are easy to obtain, as shown by the following lemma.

Lemma 3.25. *Under the identification $\mathfrak{g} = T_eG$, the map*

$$\{\textit{left invariant metrics on } G\} \to \{\textit{inner products on } \mathfrak{g}\},\ \langle\cdot,\cdot\rangle \mapsto B = \langle\cdot,\cdot\rangle_e \tag{3}$$

is a bijection. Here an inner product means a positive definite symmetric bilinear form on a vector space.

Proof. For an inner product B on $\mathfrak{g}$, define $\langle\cdot,\cdot\rangle_g = \ell_g^* B$ for all $g \in G$. Then $B \mapsto \langle\cdot,\cdot\rangle$ gives an inverse to (3). □

There is a natural Lie group representation G on $\mathfrak{g}$ defined by

$$Ad\colon G \to GL(\mathfrak{g}),\ Ad_g X = r_{g^{-1}}^* \ell_g^* X.$$

Its differential is the Lie algebra representation $\mathfrak{g}$ on itself called the adjoint representation, defined by

$$ad\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}),\ ad_X Y = [X, Y].$$

An bilinear form B on $\mathfrak{g}$ is said to be *ad-invariant* if it is invariant under the adjoint representation. In other word, B is ad-invariant if $B([Z, X], Y) + B(X, [Z, Y]) = 0$ for all $X, Y, Z \in \mathfrak{g}$.

Proposition 3.26. *The map (3) restricts to a bijection between bi-invariant metrics on G and ad-invariant metrics on $\mathfrak{g}$.*

Proof. In the domain of (3) bi-invariance is the same as right invariance. Therefore we have

$$\begin{aligned}
&\langle\cdot,\cdot\rangle \text{ is bi-invariant}\\
\iff\ & r_{g^{-1}}^*\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle \text{ for all } g\in G\\
\iff\ & r_{g^{-1}}^*\ell_g^*\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle \text{ for all } g\in G\\
\iff\ & Ad_g^*\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle \text{ for all } g\in G\\
\iff\ & (Ad_g^*\langle\cdot,\cdot\rangle)_e = \langle\cdot,\cdot\rangle_e \text{ for all } g\in G \text{ (since both sides are left-invariant)}\\
\iff\ & (1\otimes ad_X^* + ad_X^*\otimes 1)B = B \text{ for all } X\in\mathfrak{g} \text{ (since } G \text{ is connected)}\\
\iff\ & B \text{ is ad-invariant.}
\end{aligned}$$

□

Proposition 3.26 allows us to identify the set of bi-invariant metrics on G to the set ad-invariant inner product on $\mathfrak{g}$. Moreover these sets are nonempty (one can see this from our discussion below, or to average a fixed left-invariant metric over its pullbacks by r_g, $g\in G$, using the Haar measure on G). The latter set can actually be written out explicitly. The rest of discussions in this section is less important for understanding our main theorem, but is nevertheless interesting to include.

Lemma 3.27. *The Lie algebra $\mathfrak{g}$ is a direct sum of an abelian Lie algebra and a semisimple Lie algebra.*

Proof. See [3, Corollary 4.25]. □

Lemma 3.28. *Suppose a Lie algebra L is the direct sum of two Lie algebras L_1, L_2, where L_2 is semisimple. Then all ad-invariant inner product on L are given by $B = B_1 + B_2$, where B_i is an ad-invariant inner product on L_i (which is zero on the other direct component), $i = 1, 2$.*

f

Proof. Clearly any $B_1 + B_2$ gives an ad-invariant inner product on L. Conversely, let B be any ad-invariant inner product on $\mathfrak{g}$, we prove it has the desired form. It suffices to show for all $X\in L_1$ that $B(X, L_2) = 0$.

For any $Y_1, Y_2 \in L_2$, by ad-invariant of B we have $0 = B([Y_1, X], Y_2)+B(X, [Y_1, Y_2]) = B(X, [Y_1, Y_2])$. It follows from semisimpleness of L_2 that $B(X, L_2) = B(X, [L_2, L_2]) = 0$, as desired. □

Definition 3.29. The *Killing form* of a Lie algebra L is the bilinear form κ on L given by $\kappa(X, Y) = \mathrm{tr}(ad_X ad_Y)$. Here $\mathrm{tr}(\cdot)$ denotes the trace operator on $\mathfrak{gl}(\mathfrak{g})$.

Lemma 3.30. *All ad-invariant inner product on a simple Lie algebra L are given by a negative scalar times the Killing form on L.*

Proof. See [2, Theorem 5.1] for a proof that the Killing form κ on L is negative definite. By

$$\kappa([Z, X], Y) + \kappa(X, [Z, Y]) = \mathrm{tr}((ad_Z ad_X ad_Y - ad_X ad_Y ad_Z) + (ad_X ad_Z ad_Y - ad_Z ad_Y ad_X)) = 0$$

we see κ is ad-invariant. Therefore any $\lambda\kappa$, $\lambda < 0$ is an ad-invariant inner product on L.

Conversely, suppose B is any ad-invariant inner product on L. By nondegeneracy of B and κ, they induces isomorphisms $\hat{B}, \hat{\kappa}\colon L \to L^*$. Now ad-invariance shows that $(\hat{\kappa})^{-1}\circ\hat{B}\colon L\to L$ is a map of representations, where L on both sides denote the adjoint representation $L \to \mathfrak{gl}(L)$. By Schur's Lemma we conclude that $(\hat{\kappa})^{-1}\circ\hat{B} = \lambda\cdot \mathrm{id}_L$ for some $\lambda\in\mathbb{C}$. Hence $B = \lambda\kappa$. By positive-definiteness of B we have $\lambda < 0$. □

By Lemma 3.27, Lemma 3.28 and Lemma 3.30 we deduce the following.

Corollary 3.31. *We can write $\mathfrak{g} = \mathfrak{t}\oplus(\oplus_{i=1}^r \mathfrak{s}_i)$ for an abelian Lie algebra $\mathfrak{t}$ and simple Lie algebras $\mathfrak{s}_i$. All ad-invariant inner product on $\mathfrak{g}$ are given by $B_0 + (\sum_{i=1}^r \lambda_i\kappa_i)$, where B_0 is an inner product on $\mathfrak{t}$, κ_i is the Killing form on $\mathfrak{s}_i$, and $\lambda_i < 0$, $i = 1, \cdots, r$.* □

4 Fourier Transform of Compact Groups

In this section we generalize the notion of Fourier series to compact Lie groups. In 4.1 we demonstrate the Fourier series connection with the representation theory of compact Lie groups and in 4.2 we show how left invariant differential operators act on Fourier series.

4.1 The Peter-Weyl Theorem

Let G be a compact connected Lie group and dg a Haar measure of G. Most of the results in the section hold for general compact topological groups but for convenience we will treat G as a Lie group throughout.

Definition 4.1. Given $f \in C^\infty(G)$ and a finite dimensional unitary irreducible representation (π, V_π) of G, define

$$\widehat{f}(\pi) := \int_G f(g)\pi(g)\, dg$$

which is an element of $End(V_\pi)$. The integrand here is a matrix and the intergal of a matrix is a matrix of integrals.

The object $\widehat{f}$ is in a sense a "function" on irreducible representation π which takes values in V_π. When G is abelian, each irreducible representation is one-dimensional and $\widehat{f}$ is literally a function.

Example 4.2. Suppose that $G = \mathbb{R}^n/\Gamma$. Then the irreducible representations are given by $Vol(G)$ multiples of functions e_η which form the orthonormal basis of $L^2(G)$ from Theorem 2.1:

$$\sqrt{Vol(G)}e_\eta(x + \Gamma) = e^{2\pi i\eta\cdot x}$$

where $\eta \in \Gamma^*$. The function $\widehat{f}$ is given by

$$\widehat{f}(\sqrt{Vol(G)}e_\gamma) = \langle f, \sqrt{Vol(G)}e_\gamma\rangle.$$

In general, $\widehat{f}$ is the analog of the Fourier series for general compact groups. We will soon see a similar decomposition of $L^2(G)$ into a direct sum of spaces associated to irreducible representations given by $\widehat{f}$. The orthonormal basis of $L^2(G)$ which arises via this decomposition consists of matrix coefficients:

Definition 4.3. For a representation (π, V_π), let $m_{v,\lambda} : G \to \mathbb{C}$ for $v \in V_\pi$ and $\lambda \in V_\pi^*$ be defined as

$$m_{v,\lambda}(g) := \lambda(\pi(g^{-1})v).$$

A *matrix coefficient of* π is a linear combination of $m_{v,\lambda}$s.

First we show that the matrix coefficients of different irreducible representations are orthogonal and that there exists orthonormal matrix coefficients for each irreducible representation π.

Lemma 4.4 (Schur Orthogonality Relations). *Let $m_{v_1,\lambda_1}, m_{v_2,\lambda_2}$ be two matrix coefficients of irreducible dimensional representations $\pi : G \to Gl(V_\pi)$ and $\rho : G \to Gl(V_\rho)$ respectively. Then*

$$\int_G m_{v_1,\lambda_1}(g)\overline{m_{v_2,\lambda_2}(g)}\, dg = \begin{cases} Vol(G)\frac{\lambda_1(v_2)\overline{\lambda_2(v_1)}}{dim(\pi)} & \text{if } \pi = \rho \\ 0 & \text{if } \pi \neq \rho \end{cases} \tag{4}$$

Proof. Throughout this proof we use the fact that if $\xi : G \to Gl(V)$ is irreducible so is its dual representation $\xi^* : G \to Gl(V^*)$ defined by $\xi^*(g)\lambda(v) = \lambda(\xi(g^{-1}(v))$. We also use the corollary of Schur's lemma that if $\xi : H \to W$ and $\xi' : H \to W'$ are irreducible representations of H then

$$Hom_H(W \otimes W'^*, \mathbb{C}) \cong \begin{cases} \mathbb{C} & \text{if } W \cong W' \\ 0 & \text{if } W \neq W'. \end{cases}$$

When $\pi \neq \rho$, the map $V_\pi \otimes V_\rho^* \to \mathbb{C}$ defined by linearly extending the map

$$(v, \lambda') \mapsto \int_G m_{\lambda,v}(g)\overline{m_{\lambda',v'}(g)}dg$$

is G-equivariant. By Schur's lemma it is 0.

Suppose $\pi = \rho$ and let $V = V_\pi = V_\rho$. Define $W := V \otimes V^*$ where V^* is the dual of V. Then W is an irreducible $G \times G$-representation where the first copy of G acts as π on V and the second acts as π^* on V^*. Note that $W^* \cong W$. The LHS and RHS of (4) are both $G \times G$-equivariant linear homomorphisms from $(v_1, \lambda_1), (v_2, \lambda_2) \in W \times W^*$ to $\mathbb{C}$. Thus the LHS of 4 is a constant multiple of the RHS of 4. It therefore suffices to verify that (4) holds in the case that $v_1 = v_2 = v$ and $\lambda_1(x) = \lambda_2(x) = \langle v, x\rangle$ where v is a unit length vector i.e. $\langle v, v\rangle = 1$. Here we are using that for compact groups, every representation is unitary (see Remark 3.20). Then

$$\int_G \langle v, g^{-1}v\rangle\overline{\langle v, g^{-1}v\rangle}\, dg = \langle v, \int_G g^{-1}\langle v, gv\rangle v\, dg\rangle.$$

Notice the map

$$w' \mapsto \int_G g^{-1}\langle v, gw'\rangle v\, dg$$

is a G equivariant linear homomorphism $V \to V$ (here we use that G is unimodular). Hence by Schur's lemma it is a scalar multiple and furthermore we can compute its trace since: Let T be the linear map $V \to V$ sending a vector s to $\langle v, s\rangle v$.

$$\begin{aligned}
&Tr(w' \mapsto g^{-1}\langle v, gw'\rangle v)\\
&= Tr(w' \mapsto g^{-1}T(gw'))\\
&= Tr(T)\\
&= \langle v, v\rangle\\
&= 1.
\end{aligned}$$

Thus

$$\begin{aligned}
Tr\left(w' \mapsto \int_G g^{-1}\langle v, gw'\rangle v\, dg\right) &= \int_G \langle v, v\rangle\, dg\\
&= Vol(G).
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_G \langle v, g^{-1}v\rangle\overline{\langle v, g^{-1}v\rangle}\, dg &= \langle v, \int_G g^{-1}\langle v, gv\rangle v\, dg\rangle.\\
&= \frac{Vol(G)}{dim(V)}\langle v, v\rangle\\
&= \frac{Vol(G)}{dim(V)}.
\end{aligned}$$

□

Now we can prove the Peter-Weyl theorem which is the decomposition of $L^2(G)$ into matrix coefficients.

Theorem 4.5 (Peter-Weyl Theorem)**.** *For each irreducible representation of G, choose an orthonormal basis for the space of its matrix coefficients. Then the union of these orthonormal bases over all irreducible representations gives an orthonormal basis for $L^2(G)$.*

Proof. Since any two metric coefficients from different irreducible representations are orthogonal by Lemma 4.4, it remains to show the union of all matrix coefficients of π, as π runs over all irreducible representations of G, is complete in $L^2(G)$. We use the same argument as in the proof of Theorem 2.1. It suffices to show this set of functions is closed under multiplication and conjugation, separates points, and vanishes at no points. For a matrix coefficient e_1 of π_1 and e_2 of π_2, one easily check that e_1e_2 is a matrix coefficient of $\pi_1 \otimes \pi_2$, that $\bar{e_1}$ is a matrix coefficient of π_1^* (the dual representation of π_1). Moreover, the constant 1 function is a matrix coefficient for the trivial representation. It remains to prove the set of matrix coefficients separates points.

Let V be a finite dimensional faithful unitary representation of G; in other words, V is complex vector space equipped with a Hermitian inner product and a Lie group embedding $G \hookrightarrow U(V)$, where $U(V)$ denotes the unitary group of V (such representation always exists, see [1, Exercise 4.7.1]). The Hermitian metric on V splits it into irreducible unitary representations $V_1 \oplus \cdots \oplus V_k$. For any $g_1, g_2 \in G$, $g_1 \neq g_2$, there is some i such that g_1, g_2 acts on V_i differently. Hence there exists a matrix coefficient of V_i that has different values on g_1, g_2. This finishes the proof. ☐

The Peter-Weyl theorem is an important foundational result in the representation theory of compact groups. Notably the Peter-Weyl theorem implies in analogy to the abelian case (or the $\mathbb{R}^n$ and $\mathbb{R}^n/\Gamma$ case), the Fourier transform has an inverse formula:

Corollary 4.6 (Fourier inversion formula)**.** *Suppose $f \in C^\infty(G)$, then*

$$f(g) = \sum_{\rho \in Irr(G)} \frac{dim(\rho)}{Vol(G)} Tr(\rho(g^{-1}\hat{f}(\rho))$$

where $Vol(G) = \int_G 1dg$ wher $Tr(\rho(g^{-1})\hat{f}(\rho))$ is the trace of $\rho(g^{-1})\hat{f}(\rho)$.

Proof. By the Peter-Weyl theorem, it is enough to show the Fourier inversion formula on a matrix coefficient $m_{\lambda,v}$ for $v \in V_\pi$ and $\lambda \in V_\pi^*$ with π irreducible. Fix any $w \in V_\rho$ with ρ irreducible and not equivalent to π. Then the map from V_π to V_ρ defined by

$$v \mapsto \widehat{m_{\lambda,v}}(\rho)w$$

is G-equivariant. By Schur's lemma, it is the 0 map. Now consider the integral

$$\frac{1}{\langle v, v\rangle} \int_G m_{v,\lambda}(g)\overline{\langle v, \pi(g^{-1}v)\rangle}.$$

By the Schur orthogonality relations, for an orthonormal basis e_i,

$$\begin{aligned} Tr(\widehat{m_{v,\lambda}}(\pi)) &= \sum_i \langle e_i, \int_G m_{v,\lambda}(g)\pi(g)e_i \, dg\rangle \\ &= \sum_i \int_G m_{v,\lambda}(g) \sum_i \langle e_i, \pi(g)e_i\rangle dg \\ &= \sum_i \frac{Vol(G)}{dim(\pi)} \lambda(e_i)\langle e_i, v\rangle \\ &= \frac{Vol(G)}{dim(\pi)} \lambda(v) \\ &= \frac{Vol(G)}{dim(\pi)} m_{v,\lambda}(1). \end{aligned}$$

This establishes the equality of the LHS and RHS of the Fourier inversion formula. ☐

4.2 Left Invariant Differential Operators Under Fourier Transform

In this subsection we use the Fourier transform to analyze the spectrum of differential operators. First, a corollary of the Fourier inversion formula is

Corollary 4.7. *For an operator D on $C^\infty(G)$ and $f \in C^\infty(G)$*

$$\widehat{D(f)}(\pi) = \lambda \widehat{f}(\pi) \qquad \text{for all } \pi$$

if and only if f is an eigenfunction of D with eigenvalue λ.

Note here that the if direction is immediate from the definition of $\widehat{f}$ (Definition 4.1).

As suggested in Section 3, the partial differential operators on Lie groups which behave nicely with respect to the group action are left-invariant differential operators. In particular, in Proposition 3.7 we showed that left-invariant vector fields on G correspond to elements of $\mathfrak{g}$ or the tangent space at the identity of G. The spectrum of these operators is computable:

Lemma 4.8. *Let X be a left-invariant vector field on G corresponding to $x \in \mathfrak{g}$. Then*

$$\widehat{X(f)}(\pi) = \widehat{f}(\pi) \circ -d\pi(x)$$

where $d\pi \colon \mathfrak{g} \to End(V_\pi)$ is the differential of π.

Proof. Using that we can swap limits and integrals when the domain of integration is compact:

$$\begin{aligned}
\int_G X(f)(g)\pi(g)\,dg &= \int_G \lim_{t\to 0} \frac{f(g\cdot\exp(tx)) - f(g)}{t}\pi(g)\,dg \\
&= \lim_{t\to 0}\int_G \frac{f(g\cdot\exp(tx)) - f(g)}{t}\pi(g)\,dg \\
&= \lim_{t\to 0}\int_G f(g)\frac{\pi(g\cdot\exp(-tx)) - \pi(g)}{t}\,dg \\
&= -\int_G f(g)X(\pi)(g)\,dg
\end{aligned}$$

where

$$X(\pi)(g) = \lim_{t\to 0}\frac{\pi(g\cdot\exp(tx)) - \pi(g)}{t}.$$

Notice that

$$X(\pi)(gx) = \pi(g)X(\pi)(x)$$

and

$$\begin{aligned}
X(\pi)(1) &= \lim_{t\to 0}\frac{\pi(\exp(x))^t - 1_{V_\pi}}{t} \\
&= \log(\pi(\exp(x))).
\end{aligned}$$

Thus

$$\begin{aligned}
\widehat{X(f)}(\pi) &= \int_G X(f)(g)\pi(g)\,dg \\
&= -\int_G f(g)X(\pi)(g)\,dg \\
&= \widehat{f}(\pi)\circ(-\log(\pi(\exp(x)))).
\end{aligned}$$

□

Recall the anit-automorphism of $U(\mathfrak{g})$ given by $w \mapsto w^*$ in example 3.17. By successively applying Lemma 4.8,

Corollary 4.9. *For any left-invariant differential operator $D \in \mathrm{PDO}(G)$ corresponding to w in $U(\mathfrak{g})$ via (2) and any $f \in C^\infty(G)$*

$$\widehat{D(f)}(\pi) = \widehat{f}(\pi) \circ d\pi(w^*).$$

Proof. For any composition of left invariant derivations $X_1 \circ X_2 \ldots \circ X_r$ corresponding to elements $x_1, x_2 \ldots x_n \in \mathfrak{g}$, by successively applying Lemma 4.8,

$$\begin{aligned}
(X_1 \circ \widehat{X_2 \ldots \circ} X_r(f))(\pi) &= (X_2 \circ \widehat{X_3 \ldots \circ} X_r(f))(\pi) \circ -d\pi(x_1) \\
&= (X_3 \circ \widehat{X_4 \ldots \circ} X_r(f))(\pi) \circ -d\pi(x_2) \circ -d\pi(x_1) \\
&= \ldots \\
&= \widehat{f}(\pi) \circ (-d\pi(x_1) \ldots \circ -d\pi(x_2) \circ -d\pi(x_1))
\end{aligned}$$

Now for arbitrary $D = \sum a_i \cdot (X_{i,1} \circ X_{i,2} \ldots \circ X_{i,i_r})$ where each $X_{i,1}$ is left invariant corresponding to $x_{i,1} \in \mathfrak{g}$ and as elements of $U(\mathfrak{g})$,

$$w = \sum a_i x_{i,1} \cdot x_{i_2} \ldots x_{i,i_r}.$$

Then

$$\begin{aligned}
\widehat{D(f)}(\pi) &= \left(\sum a_i \cdot (X_{i,1} \circ \widehat{X_{i,2} \ldots \circ} X_{i,i_r})(f)\right)(\pi) \\
&= \sum_i a_i \widehat{f}(\pi) \circ (-d\pi(x_{i,i_r}) \ldots \circ -d\pi(x_{i,2}) \circ -d\pi(x_{i,1})) \\
&= \widehat{f}(\pi) \circ \left(\sum_i a_i \cdot -d\pi(x_{i,i_r}) \ldots \circ -d\pi(x_{i,2}) \circ -d\pi(x_{i,1}) \right) \\
&= \widehat{f}(\pi) \circ d\pi(w^*).
\end{aligned}$$

□

Finally, this allows us to completely determine the spectrum of differential operators corresponding to w in the center of $U(\mathfrak{g})$.

Theorem 4.10. *For a compact Lie group G and a differential operator D on $C^\infty(G)$ corresponding to an element w in the center of $U(\mathfrak{g})$, the matrix coefficients $m_{v,\lambda}$ of π are eigenfunctions of D with eigenvalue $d\pi(w^*)$ (as a scalar in $\mathbb{C}$). Every eigenvalue of D is $d\pi(w^*)$ for some irreducible π.*

Proof. Note that any anti-automorphism preserves the center of $U(\mathfrak{g})$.

By Corollary 4.7, if for every irreducible representation $\rho \neq \pi$, $\widehat{m_{\lambda,v}}(\rho) = 0$, then $m_{\lambda,v}$ is an eigenvector with eigenvalue c if

$$\widehat{D(m_{\lambda,v})}(\pi) = c \cdot \widehat{m_{\lambda,v}}(\pi).$$

Fix any $w \in V_\rho$, then the map from V_π to V_ρ defined by

$$v \mapsto \widehat{m_{\lambda,v}}(\rho) v'$$

is G-equivariant. By Schur's lemma, it is the 0 map. Since v' was arbitrary in V_ρ, $\widehat{m_{\lambda,v}}(\rho) = 0$.
Since w^* is in the center of $U(\mathfrak{g})$, the map $d\pi(w^*) : V_\pi \to V_\pi$ is a G-equivariant homomorphism. By Schur's lemma it is a constant. By Corollary 4.9, this implies

$$\widehat{D(m_{\lambda,v})}(\pi) = d\pi(w^*) \cdot \widehat{m_{\lambda,v}}(\pi).$$

We conclude that $m_{\lambda,v}$ is an eigenfuction with eigenvalue $d\pi(w^*)$.
Now suppose that f is an eigenfunction of D. By the Peter-Weyl theorem, there exists some π such that $\widehat{f}(\pi) \neq 0$. By Corollary 4.7 and Corollary 4.9 $\lambda = d\pi(w^*)$. □

5 Laplacian on Compact Lie Groups

As before, let G denote a compact connected Lie group and $\mathfrak{g}$ denote its Lie algebra. Let B denote an ad-invariant inner product on $\mathfrak{g}$ and let G be equipped with the induced bi-invariant metric (Proposition 3.26).

5.1 Casimir Elements

Let K be a fixed nondegenerate symmetric bilinear form on $\mathfrak{g}$. It determines an identification $\hat{K}\colon \mathfrak{g} \xrightarrow{\cong} \mathfrak{g}^*$.

Definition 5.1. The *Casimir element* of $\mathfrak{g}$ with respect to the symmetric bilinear form K, denoted $c(K)$, is the image of $\mathrm{id}_{\mathfrak{g}} \in \mathfrak{g} \otimes \mathfrak{g}^*$ under the composite map $\mathfrak{g} \otimes \mathfrak{g}^* \xrightarrow{1\otimes(\hat{K})^{-1}} \mathfrak{g} \otimes \mathfrak{g} \to T(\mathfrak{g}) \to U(\mathfrak{g})$.

We care about the Casimir element because of the following proposition.

Proposition 5.2. *Under (2), the Casimir element $c(B)$ is mapped to the Laplacian Δ on G.*

Proof. See Appendix A. $\square$

We continue to give some nice properties of the Casimir element.

Proposition 5.3. *Suppose K is ad-invariant. Then the Casimir element $c(K)$ lies in the center of $U(\mathfrak{g})$.*

Proof. Let $X_1, \cdots, X_n$ be an orthonormal basis of $\mathfrak{g}$ with respect to K. Write $[X_i, X_j] = c_{ij}^k X_k$ for some structural constants $c_{ij}^k \in \mathbb{R}$. Then $c_{ij}^k = -c_{ji}^k$ by anti-communtativeness of Lie bracket and $c_{ki}^j + c_{kj}^i = 0$ by ad-invariance of K. It follows that for all j, we have

$$\begin{aligned} X_j \otimes c(K) &= \sum_i (X_j \otimes X_i \otimes X_i) = \sum_{i,k} c_{ji}^k X_k \otimes X_i + \sum_i X_i \otimes X_j \otimes X_i \\ &= \sum_{i,k} c_{ij}^k X_i \otimes X_k + \sum_i X_i \otimes X_j \otimes X_i = \sum_i X_i \otimes X_i \otimes X_j = c(K) \otimes X_j. \end{aligned}$$ $\square$

Let $\varphi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be an irreducible representation of $\mathfrak{g}$. Since $\mathfrak{gl}(V)$ is an associative algebra, the universal property of $U(\mathfrak{g})$ implies that φ extends to an (associative) algebra homomorphism $U(\mathfrak{g}) \to \mathfrak{gl}(V)$, which we still denote as φ.

Corollary 5.4. *Suppose K is ad-invariant. Let φ be an irreducible representation of $\mathfrak{g}$. Then $\varphi(c(K)) = c_\varphi(K)\mathrm{id}_V$ is some scalar $c_\varphi(K) \in \mathbb{C}$ times the identity.*

Proof. By Proposition 5.3 we know that $\varphi(c(K))$ commutes with the $\mathfrak{g}$-action on V. The statement now follows from Schur's lemma. $\square$

Remark 5.5. As in Corollary 3.31, write $\mathfrak{g} = \mathfrak{t} \oplus (\oplus_{i=1}^r \mathfrak{s}_i)$ and $B = B_0 + (\sum_{i=1}^r \lambda_i \kappa_i)$. Then under the natural inclusions $U(\mathfrak{t}), U(\mathfrak{s}_i) \hookrightarrow U(\mathfrak{g})$, one can write $c(B) = c(B_0) + \sum_{i=1}^r \lambda_i c(\kappa_i)$.

5.2 Spectral Properties of the Laplacian

Since G is connected, the differential of an irreducible representation π of G is an irreducible representation $d\pi$ of $\mathfrak{g}$. For notational convenience, we still use $c_\pi(B)$ to denote the constant $c_{d\pi}(B)$ as in Corollary 5.4.

Theorem 1.1. *For each irreducible representation π of G, every matrix coefficient of π is an eigenfunction of Δ with eigenvalue $c_\pi(B)$. In particular, the set of eigenfunctions is complete, and the spectrum of the Laplacian on G is $\sigma(\Delta) = \{c_\pi(B)\colon \pi \text{ is an irreducible representation of } G\}$.*

Proof. Notice that $c_\pi(B)^* = c_\pi(B)$. Now this follows from the Proposition 5.2, Theorem 4.10 and the Peter-Weyl Theorem 4.5. $\square$

Below are three examples. The first two recover the spectral properties of tori as we explicitly computed in Section 2. The third one is an nonabelian example.

Example 5.6. Let $G = U(1) = S^1 = \mathbb{R}/\mathbb{Z}$ be equipped with the metric inherited from the Euclidean metric on $\mathbb{R}$. Then $\mathfrak{g} = \mathbb{R}$ is an abelian Lie algebra. The inner product on $\mathfrak{g}$ corresponding to the metric on G agrees with the usual Euclidean inner product, denoted B. The Casimir element of $\mathfrak{g} = \mathbb{R}$ is $c(B) = 1 \otimes 1 \in U(\mathbb{R})$.

All irreducible representations of G are given by, for each $n \in \mathbb{Z}$,

$$\pi_n \colon \mathbb{R}/\mathbb{Z} \to GL(\mathbb{C}) = \mathbb{C}\backslash\{0\},\ x \mapsto e^{2\pi i n x}.$$

The corresponding Lie algebra representations are

$$d\pi_n \colon \mathbb{R} \to \mathfrak{gl}(\mathbb{C}) = \mathbb{C},\ v \mapsto 2\pi i n v.$$

The image of $c(B)$ under the induced map $U(\mathbb{R}) \to \mathfrak{gl}(\mathbb{C})$ is $(2\pi i n)\cdot(2\pi i n) = -4\pi^2 n^2 \in \mathfrak{gl}(\mathbb{C})$. Therefore $c_{\pi_n}(B) = -4\pi^2 n^2$.

For each π_n, a matrix coefficient is a multiple of $e^{2\pi i n x}$, and is an eigenfunction of Δ with eigenvalue $c_{\pi_n}(B) = -4\pi^2 n^2$ by Theorem 1.1. This recovers the result in Section 2.1.

Example 5.7. More generally, let $G = T = \mathbb{R}^n/\Gamma$ be a torus equipped with the metric inherited from the Euclidean metric $\mathbb{R}^n$. Then $\mathfrak{g} = \mathbb{R}^n$ is an abelian Lie algebra with corresponding inner product B being the Euclidean inner product. Let $\varepsilon_1, \cdots, \varepsilon_n$ denotes the standard basis of $\mathbb{R}^n$, then the Casimir element of $\mathfrak{g} = \mathbb{R}^n$ is $c(B) = \sum_{k=1}^n \varepsilon_k \otimes \varepsilon_k$.

All irreducible representations of T are given by, for each $\xi \in \Gamma^*$,

$$e_\xi \colon T \to GL(\mathbb{C}) = \mathbb{C}\backslash\{0\},\ x \mapsto e^{2\pi i \xi\cdot x}.$$

The corresponding Lie algebra representations are

$$de_\xi \colon \mathbb{R}^n \to \mathfrak{gl}(\mathbb{C}) = \mathbb{C},\ v \mapsto 2\pi i \xi \cdot v.$$

The image of $c(B)$ under the induced map is $\sum_{k=1}^n (2\pi i \xi \cdot \varepsilon_k)^2 = -4\pi^2|\xi|^2 \in \mathfrak{gl}(\mathbb{C})$. Therefore $c_{e_\xi}(B) = -4\pi^2|\xi|^2$.

For each e_ξ, a matrix coefficient is a multiple of $e^{2\pi i \xi\cdot x}$, and is an eigenfunction of Δ with eigenvalue $c_{e_\xi}(B) = -4\pi^2|\xi|^2$ by Theorem 1.1. This recovers Theorem 2.2 in Section 2.2.

Example 5.8. Let $G = SU(2) = \{A \in \mathbb{C}^{2\times 2} \colon AA^* = 1,\ \det(A) = 1\}$, the special unitary group of size 2. One can write

$$SU(2) = \{\left(\begin{smallmatrix} \alpha & -\beta \\ \bar\beta & \bar\alpha \end{smallmatrix}\right) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta^2| = 1\}.$$

Let $\mathbb{H} = \{a + bi + cj + dk \colon a, b, c, d \in \mathbb{R}\}$ denotes the skew field of quaternions, where $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$. Then the map

$$\left(\begin{smallmatrix} \alpha & -\beta \\ \beta & \bar\alpha \end{smallmatrix}\right) \mapsto \alpha + \beta j$$

gives a Lie group isomorphism $SU(2) \to Sp(1)$, where $Sp(1)$ is the group of unit quaternions. It is also clear that $Sp(1) = S^3$ in the natural way. Moreover, under these identifications, the left or right multiplication of $SU(2)$ on $SU(2) = Sp(1) = S^3$ preserves the round metric g on S^3 (i.e. the metric induced from $\mathbb{R}^4$). Therefore, the round metric g gives a bi-invariant metric on $SU(2)$. Below we shall always assume $SU(2)$ to be equipped with this metric and let B denote the corresponding ad-invariant inner product on the Lie algebra $\mathfrak{su}(2)$. (In fact, by Corollary 3.31 and simpleness of $\mathfrak{su}(2)$, g is up to scalar the unique bi-invariant metric on $SU(2)$.)

We want to calculate the spectrum of Δ on $S^3 = SU(2)$. By Theorem 1.1, we need to calculate $c_\pi(B)$ for all irreducible representation π of $SU(2)$.

Since $SU(2)$ is simply connected, its irreducible representations are in one-one correspondence with irreducible Lie algebra representations of $\mathfrak{su}(2)$. Below we recall the representation theory of $\mathfrak{su}(2)$. Let

$$U = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right), \ V = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right), \ W = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$$

be a basis of $\mathfrak{su}(2)$. Given an irreducible representation $\mathfrak{su}(2) \to \mathfrak{gl}(V)$, we can complexify it to an irreducible representation $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \otimes \mathbb{C} \to \mathfrak{gl}(V)$. Conversely an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ restricts to one of $\mathfrak{su}(2)$. Thus it suffices to describe irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$. Let

$$H = -iU = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \ X = -\tfrac{1}{2}(W + iV) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \ Y = \tfrac{1}{2}(W - iV) = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$$

be a (complex) basis of $\mathfrak{sl}(2,\mathbb{C})$. Then for each nonnegative integer m, there is exactly one irreducible representation $\pi_m \colon \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(V_m)$ of $\mathfrak{sl}(2,\mathbb{C})$ with dimension $m+1$. Furthermore, one can find a basis $\{v_m, v_{m-2}, v_{m-4} \cdots, v_{-m}\}$ of V_m such that

$$\pi_m(H)v_{m-2i} = (m-2i)v_{m-2i}, \ \pi_m(X)v_{m-2i} = (m-i+1)v_{m-2i+2}, \ \pi_m(Y)v_{m-2i} = (i+1)v_{m-2i-2}.$$

Here $v_{m+2} = v_{-m-2} = 0$. For a proof of this classical result one can consult [2, Section 7.2].

Since under the identification $\mathfrak{su}(2) = \mathfrak{sp}(1)$, U, V, W corresponds to i, j, k, respectively, we know they form an orthonormal basis with respect to B. Therefore $c(B) = U \otimes U + V \otimes V + W \otimes W = -(H \otimes H + 2X \otimes Y + 2Y \otimes X)$. By definition, we have

$$\begin{aligned} c_{\pi_m}(B)v_m &= -\left(\pi_m(H)\pi_m(H) + 2\pi_m(X)\pi_m(Y) + 2\pi_m(Y)\pi_m(X)\right)v_m \\ &= -\pi_m(H)(mv_m) - 2\pi_m(X)v_{m-2} - 0 = -(m^2+2m)v_m. \end{aligned}$$

Since v_m is nonzero, we conclude that $c_{\pi_m}(B) = -m(m+2)$.

By Theorem 1.1, $\sigma(\Delta) = \{-m(m+2) \colon m \in \mathbb{Z}_{\geq 0}\}$. Moreover, for each m, the space of matrix coefficients of π_m gives all eigenfunctions of Δ with eigenvalue $-m(m+2)$.

Remark 5.9. By writing out the irreducible representations of $SU(2)$ in terms of action on homogeneous polynomials one can actually show that the eigenfunctions for the eigenvalue $-m(m+2)$ are homogeneous polynomials of degree m (with respect to coordinates in $\mathbb{R}^4 \supset S^3$). These polynomials are called spherical harmonics in $\mathbb{R}^4$. Similarly one has spherical harmonics in any $\mathbb{R}^k$, but our method here only works for $k = 2, 4, 8$.

Acknowledgements

This research was carried out as part of the 18.821 project in mathematics course at the Massachusetts Institute of Technology in Fall 2020. The authors are thankful to Professor Andrei Negut and Susan Ruff for their mentoring.

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A The Laplace-Beltrami Operator

In this appendix we define the Laplace-Beltrami operator on a Riemannian manifold and prove Proposition 5.2. We assume readers have some differential geometry background.

Definition A.1. The Laplace-Beltrami operator Δ on a Riemannian manifold M is defined by $\Delta f = \mathrm{div}(\nabla f)$ for all $f \in C^\infty(M)$. Here $\mathrm{div}(X)$ denotes the divergence of a vector field X on M.

In particular, let B denotes an ad-invariant inner product on B and let G be equipped with the induced bi-invariant metric. Then Δ is defined on G. Let ∇ be the Levi-Civita connection on G.

For $X \in \mathfrak{g}$, let X^L denote the left invariant vector field on G whose value at e is X.

Lemma A.2. *For $X, Y \in \mathfrak{g}$, we have $\nabla_{X^L} Y^L = \frac{1}{2}[X, Y]^L$.*

Proof. Let $X_1, \cdots, X_n$ be an orthonormal basis of $\mathfrak{g}$. Write $[X_i, X_j] = c_{ij}^k X_k$, $c_{ij}^k \in \mathbb{R}$. Since each $\nabla_{X_i^L} X_j^L$ is left-invariant, it is of the form $b_{ij}^k X_k^L$ for some constants $b_{ij}^k \in \mathbb{R}$. Torsion-freeness of ∇ yields $b_{ij}^k - b_{ji}^k = c_{ij}^k$. Metric compatibility of ∇ yields $b_{ij}^k + b_{ik}^j = 0$. It follows that $b_{ij}^k = \frac{1}{2} c_{ij}^k$, as desired. $\square$

Proof of Proposition 5.2. Let $X_1, \cdots, X_n$ be an orthonormal basis of $\mathfrak{g}$. Then as vector fields on G, $X_1, \cdots, X_n$ are orthonormal at every point $a \in G$. For any function f on G, we compute that

$$\Delta f = \mathrm{div}(\nabla f) = \sum_{i=1}^n \langle X_i^L, \nabla_{X_i^L} \nabla f \rangle = \sum_{i=1}^n X_i^L \langle X_i^L, \nabla f \rangle - \sum_{i=1}^n \langle \nabla_{X_i^L} X_i^L, \nabla f \rangle = \sum_{i=1}^n (X_i^L (X_i^L f)),$$

where in the last step we applied Lemma A.2. The statement follows. $\square$

Work Distribution

Qiuyu wrote Section 1, Section 2, Section 3.4, Theorem 4.5, Section 5, Apprendix A. Calvin wrote Section 3 and Section 4 with the exception of Section 3.4 and Theorem 4.5.