Points of Continuity of Real Functions on the Real Line

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1 Introduction

Given a function \( f : \mathbb{R} \to \mathbb{R} \), we ask the following question: what can be the set of points of continuity, say \( A \), of \( f \)? In basic analysis courses, one might encounter many \( f \) with strange behavior. For example, if \( f \) is the Dirichlet function given by

\[
f(x) = \begin{cases} 
1, & \text{if } x \in \mathbb{Q} \\
0, & \text{otherwise}
\end{cases}
\]

then \( A = \mathbb{R} \); if \( f \) is the Riemann function given by

\[
f(x) = \begin{cases} 
1 \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ for some coprime } p, q \in \mathbb{Z}, q > 0 \\
0, & \text{otherwise}
\end{cases}
\]

then \( A = \mathbb{R} \setminus \mathbb{Q} \); moreover, if \( f \) is given by

\[
f(x) = \begin{cases} 
x, & \text{if } x \in \mathbb{Q} \\
0, & \text{otherwise}
\end{cases}
\]

then \( A = \{0\} \).

At a glance, \( A \) seems to be rather arbitrary. What are the restrictions on \( A \)? How to define a function \( f \) for a given \( A \) that satisfies some specific properties? This article gives the necessary and sufficient condition for a set \( A \) to be the set of points of continuity of a real function \( f : \mathbb{R} \to \mathbb{R} \).

2 The amplitude function

For a function \( f : \mathbb{R} \to \mathbb{R} \), we define its amplitude function \( \omega : \mathbb{R} \to [0, \infty) \) by

\[
\omega(x) = \lim_{\delta \to 0} \sup \{|f(x_1) - f(x_2)| | x_1, x_2 \in (x - \delta, x + \delta)\}
\]

We shall prove a property about the function \( \omega \).

Lemma. For any \( t > 0 \), the set \( S = \omega^{-1}((0, t)) \) is open in \( \mathbb{R} \).

Proof. Let \( x \in S \). Then \( \omega(x) = c < d < t \) for some \( c, d \geq 0 \).

Choose \( \delta > 0 \) such that \( \sup \{|f(x_1) - f(x_2)| | x_1, x_2 \in (x - 2\delta, x + 2\delta)\} < d \).

Then for any \( y \in (x - \delta, x + \delta) \), \( x_1, x_2 \in (y - \delta, y + \delta) \), we have \( |f(x_1) - f(x_2)| < d \). Thus we readily have \( \omega(y) \leq d < t \), \( y \in S \). This means \( (x - \delta, x + \delta) \subset S \). The lemma follows.

3 The main theorem

Now we state our necessary and sufficient condition:

Theorem. A set \( A \subset \mathbb{R} \) is the set of points of continuity for some function \( f : \mathbb{R} \to \mathbb{R} \) if and only if \( A \) is the countable intersection of some open sets \( A_1, A_2, \cdots \) in \( \mathbb{R} \).
Proof. We first begin with necessity. If $f : \mathbb{R} \to \mathbb{R}$ is a function with $A$ being its set of points of continuity, denote the amplitude function of $f$ by $\omega$. By definition of $\omega$ we know that $A = \omega^{-1}(0)$. Thus

$$A = \omega^{-1}(0) = \bigcap_{n=1}^{\infty} \omega^{-1}\left((0, \frac{1}{n})\right)$$

is a countable intersection of open sets $\omega^{-1}\left([0, \frac{1}{n})\right)$, $n = 1, 2, \cdots$, by the above Lemma.

Now we prove sufficiency. Suppose $A = \bigcap_{n=1}^{\infty} A_n$, where $A_n$ are open in $\mathbb{R}$. Define $B_n = \bigcap_{k=1}^{n} A_n$, $n \geq 1$, $B_0 = \mathbb{R}$. Then $B_0 \supset B_1 \supset \cdots$ are a series of open sets, and that $\bigcap_{n=1}^{\infty} B_n = A$.

Now we define

$$f(x) = \begin{cases} 
0, & \text{if } x \in A \\
\frac{1}{n}, & \text{if } x \in (B_{n-1} \setminus B_n) \cap \mathbb{Q} \text{ for some } n \in \mathbb{N}_+ \\
\frac{2}{2n-1}, & \text{if } x \in (B_{n-1} \setminus B_n) \setminus \mathbb{Q} \text{ for some } n \in \mathbb{N}_+. 
\end{cases}$$

We claim that the set of points of continuity of $f$ is exactly $A$.

In fact, for any $x \in A$, $\epsilon > 0$, find $N > \frac{1}{\epsilon}$, then for any $y \in B_N$, we have $|f(y) - f(x)| \leq \frac{1}{N} < \epsilon$. Since $\epsilon$ is arbitrary, we conclude that $f$ is continuous at $x$.

For any $x \notin A$, say $x \in B_{n-1} \setminus B_n$ and any neighborhood $U$ of $x$, there exist some $y \in U$ such that $x, y$ are not both rational or irrational. By definition we easily know that $f(x) \neq f(y)$, and thus $|f(y) - f(x)| \geq \max\{\sup\{|f(y) - \frac{1}{n} | f(y) \neq \frac{1}{n}\}, \sup\{|f(y) - \frac{2}{2n-1} | f(y) \neq \frac{2}{2n-1}\}\} = \frac{1}{n(2n+1)}$. We conclude that $f$ is discontinuous at $x$. \qed