Points of Continuity of Real Functions on the Real Line

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1 Introduction

Given a function $f : \mathbb{R} \to \mathbb{R}$, we ask the following question: what can be the set of points of continuity, say A, of f? In basic analysis courses, one might encounter many f with strange behavior. For example, if f is the Dirichlet function given by

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise} \end{cases}$$

then $A = \mathbb{R}$; if f is the Riemann function given by

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ for some coprime } p, q \in \mathbb{Z}, q > 0 \\ 0, & \text{otherwise} \end{cases}$$

then $A = \mathbb{R} \backslash \mathbb{Q}$; moreover, if f is given by

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise} \end{cases}$$

then $A = \{0\}$.
At a glance, A seems to be rather arbitrary. What are the restrictions on A? How to define a function f for a given A that satisfies some specific properties? This article gives the necessary and sufficient condition for a set A to be the set of points of continuity of a real function $f : \mathbb{R} \to \mathbb{R}$.

2 The amplitude function

For a function $f : \mathbb{R} \to \mathbb{R}$, we define its amplitude function $\omega : \mathbb{R} \to [0, \infty]$ by $\omega(x) = \lim_{\delta \to 0} \sup\{\, |f(x_1) - f(x_2)| \mid x_1, x_2 \in (x - \delta, x + \delta)\,\}$. We shall prove a property about the function ω.

Lemma. *For any $t > 0$, the set $S = \omega^{-1}([0, t))$ is open in $\mathbb{R}$.*

Proof. Let $x \in S$. Then $\omega(x) = c < d < t$ for some $c, d \geq 0$.
Choose $\delta > 0$ such that $\sup\{\, |f(x_1) - f(x_2)| \mid x_1, x_2 \in (x - 2\delta, x + 2\delta)\,\} < d$.
Then for any $y \in (x - \delta, x + \delta)$, $x_1, x_2 \in (y - \delta, y + \delta)$, we have $|f(x_1) - f(x_2)| < d$. Thus we readily have $\omega(y) \leq d < t$, $y \in S$. This means $(x - \delta, x + \delta) \subset S$. The lemma follows. ☐

3 The main theorem

Now we state our necessary and sufficient condition:

Theorem. *A set $A \subset \mathbb{R}$ is the set of points of continuity for some function $f : \mathbb{R} \to \mathbb{R}$ if and only if A is the countable intersection of some open sets $A_1, A_2, \cdots$ in $\mathbb{R}$.*

Proof. We first begin with necessity.
If $f : \mathbb{R} \to \mathbb{R}$ is a function with A being its set of points of continuity, denote the amplitude function of f by ω. By definition of ω we know that $A = \omega^{-1}(0)$. Thus

$$A = \omega^{-1}(0) = \bigcap_{n=1}^{\infty} \omega^{-1}([0, \frac{1}{n}))$$

is a countable intersection of open sets $\omega^{-1}([0, \frac{1}{n}))$, $n = 1, 2, \cdots$, by the above Lemma.
Now we prove sufficiency. Suppose $A = \cap_{n=1}^{\infty}$ where A_n are open in $\mathbb{R}$.
Define $B_n = \cap_{k=1}^{n} A_n$, $n \geq 1$, $B_0 = \mathbb{R}$. Then $B_0 \supset B_1 \supset \cdots$ are a series of open sets, and that $\cap_{n=1}^{\infty} B_n = A$.
Now we define

$$f(x) = \begin{cases} 0, \text{ if } x \in A \\ \frac{1}{n}, \text{ if } x \in (B_{n-1} \backslash B_n) \cap \mathbb{Q} \text{ for some } n \in \mathbb{N}_+ \\ \frac{2}{2n-1}, \text{ if } x \in (B_{n-1} \backslash B_n) \backslash \mathbb{Q} \text{ for some } n \in \mathbb{N}_+ \end{cases}$$

We claim that the set of points of continuity of f is exactly A.
In fact, for any $x \in A$, $\epsilon > 0$, find $N > \frac{1}{\epsilon}$, then for any $y \in B_N$, we have $|f(y) - f(x)| \leq \frac{1}{N} < \epsilon$. Since ϵ is arbitrary, we conclude that f is continuous at x.
For any $x \notin A$, say $x \in B_{n-1} \backslash B_n$ and any neighborhood U of x, there exist some $y \in U$ such that x, y are not both rational or irrational. By definition we easily know that $f(x) \neq f(y)$, and thus $|f(y) - f(x)| \geq \max\{ \sup\{ \left| f(y) - \frac{1}{n} \right| \mid f(y) \neq \frac{1}{n} \}, \sup\{ \left| f(y) - \frac{2}{2n-1} \right| \mid f(y) \neq \frac{2}{2n-1} \} \} = \frac{1}{n(2n+1)}$. We conclude that f is discontinuous at x. $\square$