NONNEGATIVELY CURVED $\mathfrak{sl}_2(\mathbb{R})$-CONNECTIONS ON SURFACES

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ABSTRACT. We study the space of nonnegatively curved $\mathfrak{sl}_2(\mathbb{R})$-connections on an oriented surface (possibly with boundary) with prescribed hyperbolic boundary holonomies and a nonzero (relative) euler number. When the euler number is no less than the genus of the surface, we show that the embedding of the space of nonnegatively curved connections into the space of all connections is a homotopy equivalence. When the euler number is no more than the negative of the genus of the surface, we show that the embedding of the space of flat connections into the space of nonnegatively curved connections is a homotopy equivalence.

1. INTRODUCTION

Throughout this paper, let G denote $PSL_2(\mathbb{R})$. Let $\tilde{G}$ be its universal cover, and $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ be its Lie algebra.

There is a faithful G-action on $\mathbb{R}P^1$ arises in the natural way, namely by left multiplication. An element in $\mathfrak{g}$ is said to be *nonnegative* (resp. *nonpositive*) if its associated vector field on $\mathbb{R}P^1$ does not point in the clockwise (resp. counterclockwise) direction anywhere.

Let S be an oriented surface (with or without boundary). A $\mathfrak{g}$-connection A on S with curvature F_A is said to be *nonnegatively curved* if $F_A(v_1, v_2) \in \mathfrak{g}$ is nonnegative for all oriented $(v_1, v_2) \in T_pS$, $p \in S$. Equivalently, A is nonnegatively curved if at any point in S, the infinitesimal parallel transport counterclockwisely gives rises to a nonpositive element in $\mathfrak{g}$.

For a closed surface S with genus g, let $\mathcal{A}(S)_e$ (resp. $\mathcal{A}_{\geq 0}(S)_e, \mathcal{A}_{flat}(S)_e$) denote the space of all (resp. nonnegatively curved, flat) $\mathfrak{g}$-connections on S with euler number e. More explicitly, we are considering connections on the principle G-bundle over S whose associated $\mathbb{R}P^1$-bundle has euler number e.

For an open surface S with genus g and $b > 0$ boundary components $\partial S_1, \cdots, \partial S_b$, every G-bundle over it is trivializable since $H^2(S) = 0$. We shall put constraint on the boundary holonomies to obtain nontrivial results. Explicitly, for given hyperbolic conjugacy classes C_i of G, $i = 1, \cdots, b$, let $\mathcal{A}(S, \{C_i\})_e$ (resp. $\mathcal{A}_{\geq 0}(S, \{C_i\})_e, \mathcal{A}_{flat}(S, \{C_i\})_e$) denote the space of all (resp. nonnegatively curved, flat) $\mathfrak{g}$-connections on S whose boundary holonomy along ∂S_i belongs to C_i for all i, and whose relative euler number is e (see [Gol88, Section 3] for more discussions about relative euler numbers). In general we just write $\mathcal{A}_\bullet(S, \{C_i\})_e$ to denote both closed and open cases, with the understanding that in the closed case the set $\{C_i\}$ is empty. Here $\mathcal{A}_\bullet$ denotes any of $\mathcal{A}, \mathcal{A}_{\geq 0}, \mathcal{A}_{flat}$. In the rest of the introduction section only, we will further abbreviate the notation by writing $\mathcal{A}_\bullet$ to denote $\mathcal{A}_\bullet(S, \{C_i\})_e$.

Partially funded by NSF grant DMS-1904997.

In various settings described above, we will be interested in how the space of non-negatively curved connections fits into the following sequence up to weak equivalence:

$$\mathcal{A}_{flat} \hookrightarrow \mathcal{A}_{\geq 0} \hookrightarrow \mathcal{A} \tag{1}$$

or perhaps the sequence module gauge equivalence:

$$\mathcal{A}_{flat}/\mathcal{G} \hookrightarrow \mathcal{A}_{\geq 0}/\mathcal{G} \hookrightarrow \mathcal{A}/\mathcal{G} \tag{2}$$

where $\mathcal{G}$ denotes the gauge group on the corresponding G-bundle, which for our convenience is defined to act on $\mathcal{A}$ on the left by pushforward. Note that $\mathcal{G}$ acts freely provided $e \neq 0$.

Here, our convention for weak equivalence follows [Sei19, Section 3]. All spaces of connections and spaces of paths (which will arise later) are always assumed to be smooth. Under this setting, weak equivalence is with respect to smooth maps from finite dimensional compact smooth manifolds (possibly with corners). For example, a space P with some smooth structure is said to be weakly contractible if for every smooth map $f\colon M \to P$ from a smooth manifold M, there is a smooth homotopy $M \times [0,1] \to P$ from f to a constant map. Note that between finite dimensional smooth manifolds, the notion of weak equivalence coincide with the usual homotopy equivalence (by Whitney approximation theorem and Whitehead theorem).

Under this convention, however, it does not make much sense to talk about weak homotopy type of moduli spaces $\mathcal{A}_\bullet/\mathcal{G}$. We will therefore refrain from such discussion, and whenever we do so, the purpose is only to be pedantic.

The space of all connections is easy to understand. In the closed case, $\mathcal{A}$ is contractible and $\mathcal{G} \simeq Map(S, \mathbb{S}^1) \simeq \mathbb{S}^1 \times \mathbb{Z}^{2g}$, so

$$\mathcal{A}/\mathcal{G} \simeq B\mathcal{G} \simeq \mathbb{C}P^\infty \times \mathbb{T}^{2g} = Sym^\infty(S),$$

where $\mathbb{T}^m$ is the m-dimensional torus and $Sym^\infty(S) = \varinjlim Sym^k(S)$ is the infinite symmetric product of S (for basic properties of symmetric products one may consult [BGZ04]). In the surface with boundary case, evaluation on boundary yields a weak equivalence

$$\mathcal{A} \xrightarrow{\simeq} (\mathbb{S}^1)^b \times \mathbb{Z}^{b-1} = \mathbb{T}^b \times \mathbb{Z}^{b-1},$$

Note here hyperbolicity of C_i is used. Moreover, let $\mathcal{A}' \subset \mathcal{A}$ denote the space of $\mathfrak{g}$-connections with certain fixed boundary values, and $\mathcal{G}' \subseteq \mathcal{G}$ be the group of gauge transformations that fix the boundary values of connections in $\mathcal{A}'$. Then $\mathcal{A}'$ is contractible and $\mathcal{G}' \simeq Map((S, \partial S), (\mathbb{S}^1, *)) \simeq H^1(S, \partial S) = \mathbb{Z}^{2g+b-1}$ (hyperbolicity is used again), so

$$\mathcal{A}/\mathcal{G} \simeq \mathcal{A}'/\mathcal{G}' \simeq B\mathcal{G}' \simeq \mathbb{T}^{2g+b-1} = Sym^\infty(S).$$

The space of flat connections is also well understood. The horizontal tangent spaces in the relevant G-bundle with respect to a flat connection integrate to a foliation, and by looking at parallel transport we obtain the following description:

$$\mathcal{A}_{flat}/\mathcal{G} \simeq \{\rho\colon \pi_1(S)^{op} \to G \mid \rho(\partial S_i) \in C_i,\ e(\rho) = e\}/G \triangleq \mathrm{Rep}(S, \{C_i\})_e/G,$$

where $e(\rho)$ is the euler number or relative euler number of the G-bundle given by ρ, and G acts by conjugation. The space on the right has been examined by many people including Goldman [Gol88], Hitchin [Hit87]. We refer to Mondello [Mon16] where the following summary for both closed and open cases is available:

Proposition 1.1 (Mondello). *Suppose $e \neq 0$, then* $\mathrm{Rep}(S, \{C_i\})_e/G \simeq Sym^{-\chi(S)-|e|}(S)$. *Here it is understood that $Sym^k(-) = \emptyset$ for $k < 0$.*

We want to understand how does $\mathcal{A}_{\geq 0}$ fits in (1). Here is our main theorem which answers this question partially, leaving some gaps for $g \geq 2$ cases.

Main Theorem. Let S be an oriented surface with genus g and b boundary components. Let $C_1, \cdots, C_b$ be hyperbolic conjugacy classes of G.

(a) Suppose $e \geq \max\{g, 1\}$. Then the inclusion $\mathcal{A}_{\geq 0}(S, \{C_i\})_e \hookrightarrow \mathcal{A}(S, \{C_i\})_e$ is a weak equivalence.

(b) Suppose $e \leq -\max\{g, 1\}$. Then the inclusion $\mathcal{A}_{flat}(S, \{C_i\})_e \hookrightarrow \mathcal{A}_{\geq 0}(S, \{C_i\})_e$ is a weak equivalence.

In this paper, we first introduce some basic notions and properties for the Lie group G in Section 2. In particular, we prove that the space of nonnegative path in $\tilde{G}$ with given endpoints is weakly contractible, if not empty.

In Section 3, following [Sei19, Section 4], we establish weak equivalences from spaces of nonnegative curved connections to some particular subsets of some G^m or $\tilde{G}^m$ which we call nonpositive representation spaces.

In Section 4 we give two tricks to deform the terms appearing in the expression of nonpositive representation spaces in the positive direction. Finally, in Section 5, using these tricks, we deform relevant nonpositive representation spaces and prove our main theorem. We also calculate the homotopy type of nonpositive representation spaces in the genus zero case using an independent elementary method. This can be used to give a nice way understanding the map in Proposition 1.1 in the genus zero case.

2. Preliminary

We begin with some basic properties and notions about G, $\tilde{G}$. Whenever we write elements in G in the form of a matrix, we are writing down one of its lift in $SL_2(\mathbb{R})$.

All elements in G except the identity I are classified into three types: elliptic, parabolic, and hyperbolic. An elliptic element is conjugate to exactly one of

$$r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \ \theta \in (0, \pi). \tag{3}$$

A hyperbolic element is conjugate to exactly one of

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \ \lambda \in (1, \infty).$$

A parabolic element is conjugate to exactly one of

$$\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}, \ \varepsilon = \pm 1,$$

and according to $\varepsilon = -1, 1$, this parabolic element is said to be nonnegative, nonpositive, respectively. The distinction of different type of elements is the clearest when one looks at their action on $\mathbb{R}P^1$.

Accordingly, we have the notion of elliptic, nonnegative/nonpositive parabolic, hyperbolic elements in $\tilde{G}$.

A smooth path $p(t)$ in G is said to be *nonnegative* if for any $\xi \in \mathbb{R}P^1$, $p(t)(\xi)$, treated as a path in $\mathbb{R}P^1$, does not move in the clockwise direction anywhere. It is said to be *positive* if $p(t)(\xi)$ moves in strictly counterclockwise direction everywhere.

An equivalent condition for p to be nonnegative is that $p'(t)p(t)^{-1}$ is a nonnegative element in $\mathfrak{g}$ for any t; a sufficient condition for p to be positive is that $p'(t)p(t)^{-1}$ is a positive element in $\mathfrak{g}$ for any t. Here, nonnegative elements in $\mathfrak{g}$ are defined as in the introduction (i.e. those whose associated vector field on $\mathbb{R}P^1$ point nonclockwisely), and positive elements are defined to be the elements in the interior of the solid cone of nonnegative elements (equivalently, those whose associated vector field on $\mathbb{R}P^1$ point strictly counterclockwisely). Explicitly, an element in $\mathfrak{g}$ is nonnegative (resp. positive) if and only if it can be written as

$$\left(\begin{smallmatrix} a & b-c \\ b+c & -a \end{smallmatrix}\right),\ c \geq \sqrt{a^2+b^2} \text{ (resp. } c > \sqrt{a^2+b^2}\text{)}. \tag{4}$$

By definition it is clear that nonnegative/positive paths are invariant under conjugation. Multiplication of several nonnegative paths is nonnegative, and is positive if one of the factors is. A path in $\tilde{G}$ is said to be nonnegative/positive if its image in G is. Likewise we have the notion of nonpositive/negative paths in G or $\tilde{G}$.

If there exists a nonnegative path from $\tilde{g}_0$ to $\tilde{g}_1$, then we write $\tilde{g}_0 \leq \tilde{g}_1$. Clearly, $\leq$ is a partial order on $\tilde{G}$. Moreover, for fixed $\tilde{g} \in \tilde{G}$, the space of $\tilde{g}' \in \tilde{G}$ with $\tilde{g}' \leq \tilde{g}$ is a closed subspace of $\tilde{G}$ (this will become clearer later in light of Corollary 2.4).

The lift from $g \in G$ to $\tilde{g} \in \tilde{G}$ is equivalent to a lift of its action on $\mathbb{R}P^1$ to an action on $\mathbb{R}$ (viewed as the universal cover of $\mathbb{R}/\pi\mathbb{Z} = \mathbb{R}P^1$). Any $\tilde{g} \in \tilde{G}$ has a *rotation number*, formally defined as

$$\operatorname{rot}(\tilde{g}) = \lim_{n\to\infty} \frac{\tilde{g}^n(x) - x}{\pi n} \text{ for any } x \in \mathbb{R}. \tag{5}$$

Specifically, if $\tilde{g}$ is not elliptic, then its image $g \in G$ has an eigenvector $\xi \in \mathbb{R}P^1$ which lifts to some $x \in \mathbb{R}$ and we have $\operatorname{rot}(\tilde{g}) = (\tilde{g}(x) - x)/\pi \in \mathbb{Z}$. If $\tilde{g}$ is elliptic, then it is conjugate to an element whose action on $\mathbb{R}$ is the translation by some θ, and we have $\operatorname{rot}(\tilde{g}) = \theta/\pi \notin \mathbb{Z}$.

We shall always use $\tilde{I}_k \in \tilde{G}$ to denote the lift of the identity element $I \in G$ with rotation number $k \in \mathbb{Z}$.

Some general properties of rotation numbers are: they are continuous, nondecreasing along nonnegative paths, increasing along positive paths inside the elliptic locus, and satisfy the quasimorphism property

$$|\operatorname{rot}(\tilde{g}_1\tilde{g}_2) - \operatorname{rot}(\tilde{g}_1) - \operatorname{rot}(\tilde{g}_2)| \leq 1.$$

We refer our readers to [Sei19, Section 3] and [LM97] for more details.

In general there is no canonical lift from G to $\tilde{G}$, but in the following two situations there is such a lift:
(1) The commutator of $g_0, g_1 \in G$ has a canonical lift to $\tilde{G}$: take any lift $\tilde{g}_0, \tilde{g}_1$ of g_0, g_1, then $[\tilde{g}_0, \tilde{g}_1] \in \tilde{G}$ is independent of the choice of $\tilde{g}_0, \tilde{g}_1$. Later we shall sometimes abuse the notation and write $[g_0, g_1]$ to denote this canonical lift.
(2) If $g \in G$ is not elliptic, then there is a unique lift of g with rotation number zero. Later we shall sometimes write $\tilde{g}$ to denote this preferred lift.

We end this section by proving the following result about nonnegative paths, which generalizes some discussions in Seidel [Sei19, Section 3].

Proposition 2.1. *Let $\tilde{g}_0, \tilde{g}_1 \in \tilde{G}$ be arbitrary. Then the space of nonnegative paths from $\tilde{g}_0$ to $\tilde{g}_1$ is either empty or weakly contractible.*

Before proving this proposition, it is beneficial for us to establish a new parametrization of $\tilde{G}$. For any $\theta \in \mathbb{R}$, let $\tilde{r}_\theta$ be the lift of r_θ (which is defined by (3)) with rotation number θ/π. We also define the *displacement function* for $\tilde{g} \in \tilde{G}$ to be

$$\varphi_{\tilde{g}} \colon \mathbb{R} \to \mathbb{R}, x \mapsto \tilde{g}(x) - x. \tag{6}$$

For example, $\varphi_{\tilde{r}_\theta} \equiv \theta$.

The following lemma is immediate.

Lemma 2.2. *If $\varphi_{\tilde{g}}$ is nonnegative and $\varphi_{\tilde{g}}(0) = 0$, then $\tilde{g}$ is the identity or a nonnegative parabolic element with rotation number 0. Furthermore, one can parametrize all such $\tilde{g}$ by*

$$\Phi(\lambda) = \widetilde{\begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix}}, \ \lambda \in [0, \infty). \qquad \square$$

One can easily check $\varphi_{\tilde{r}_\theta \tilde{g}}(x) = \varphi_{\tilde{g}}(x) + \theta$ and $\varphi_{\tilde{r}_\theta \tilde{g} \tilde{r}_{-\theta}}(x) = \varphi_{\tilde{g}}(x + \theta)$. Since $\varphi_{\tilde{g}}$ is π-periodic, $\inf \varphi_{\tilde{g}} := h$ is achieved by some $\theta^* \in [0, \pi)$. By checking that $\varphi_{\tilde{r}_{-h} \tilde{r}_{-\theta^*} \tilde{g} \tilde{r}_{\theta^*}}$ do satisfy the conditions of Lemma 2.2, we immediately see $\tilde{g} = \tilde{r}_{\theta^* + h} \Phi(\lambda) \tilde{r}_{-\theta^*}$ for some unique $\lambda \in \mathbb{R}_{\geq 0}$. With a little bit abuse of notation, denote it by $\tilde{g} = \Phi(\lambda, \theta^*, h)$. Furthermore, this decomposition is unique as long as $\lambda \neq 0$, since the graph of $\varphi_{\Phi(\lambda, \theta^*, h)}$ being a translation of $\varphi_{\Phi(\lambda)}$ implies $\arg\min \varphi_{\tilde{g}}$ is unique mod π. We summarize this as below.

Lemma 2.3. *$\{\Phi(\lambda, \theta, h)\}_{(\lambda, \theta) \in \mathbb{R}_{\geq 0} \times \mathbb{R}P^1/\sim, h \in \mathbb{R}}$ is a global parametrization of $\tilde{G}$, where $(\lambda_1, \theta_1) \sim (\lambda_2, \theta_2)$ if and only if $\lambda_1 = \lambda_2 = 0$ or $(\lambda_1, \theta_1) = (\lambda_2, \theta_2)$.* $\square$

Proof of Proposition 2.1. Replacing $\tilde{g}_1$ by $\tilde{g}_0^{-1} \tilde{g}_1$ if necessary, we may assume $\tilde{g}_0 = \tilde{I}_0 = \Phi(0, 0, 0)$.

We denote by $\tilde{G}_{\geq 0}$ the set $\{\tilde{g} \in \tilde{G} \mid \tilde{I}_0 \leq \tilde{g}\}$. Observe that when we fix θ and increase λ and/or h, $\varphi_{\Phi(\lambda, \theta, h)}$ is increasing and thus $\tilde{g}$ is moving in the nonnegative direction. This implies, whenever $h \geq 0$, we have a nonnegative path $t \mapsto \Phi(t\lambda, \theta, th)$, $t \in [0, 1]$ from $\Phi(0, \theta, 0) = \Phi(0, 0, 0)$ to $\Phi(\lambda, \theta, h)$, and therefore $\Phi(\lambda, \theta, h) \in \tilde{G}_{\geq 0}$. We denote this "canonical path" from $\Phi(0, 0, 0)$ to $\tilde{g} \in \tilde{G}_{\geq 0}$ by $\mathfrak{p}_{\tilde{g}}$. One can check that in fact $\mathfrak{p}_{\tilde{g}}$ depends smoothly on $\tilde{g}$.

Now for fixed $\tilde{g}_1 \in \tilde{G}_{\geq 0}$, let $\mathcal{P}$ denote the space of nonnegative paths from $\tilde{I}_0$ to $\tilde{g}_1$. We have the following deformation retract

$$H \colon [0, 1] \times \mathcal{P} \to \mathcal{P},$$

$$H_s(\tilde{p})(t) = \tilde{p}((1 - s)t) \mathfrak{p}_{\tilde{p}(1-s)^{-1} \tilde{g}_1}(t)$$

from $\mathcal{P}$ onto the one point space $\{\mathfrak{p}_{\tilde{g}_1}\}$. $\square$

The proof above also shows the following corollary.

Corollary 2.4. *Let $\tilde{g}_0, \tilde{g}_1 \in \tilde{G}$. Then $\tilde{g}_0 \leq \tilde{g}_1$ if and only if $\varphi_{\tilde{g}_0} \leq \varphi_{\tilde{g}_1}$.* $\square$

3. Nonnegatively Curved Connections and Nonnegative Paths

For a connection $A \in \Omega^1(S;\mathfrak{g})$ on the trivial G-bundle $S \times G \to S$ and a loop $\gamma\colon [0,1] \to S$, parallel transport along γ gives a map $\tau_\gamma\colon G \to G$. The element $hol_A(\gamma) := \tau_\gamma(I) \in G$ is called the *holonomy of A along γ*. In general, in an arbitrary G-bundle without a prescribed trivialization, the holonomy of a connection along a loop is defined only up to conjugacy. When we speak of a holonomy without specifying the trivialization, it is always understood that we are working on the trivial bundle $S \times G \to S$ over S.

Let $\gamma_1\gamma_2$ denote the concatenation of loops γ_1 and γ_2. Then the right G-equivariance of parallel transport yields

$$hol_A(\gamma_1\gamma_2) = hol_A(\gamma_2)hol_A(\gamma_1). \tag{7}$$

Assuming a fixed trivialization, any holonomy $hol_A(\gamma)$ has a canonical lift to $\tilde{G}$, denoted $\widetilde{hol}_A(\gamma)$. We still call this the holonomy of A along γ and this abuse of notation should not cause any confusion. Note (7) still holds with hol replaced by $\widetilde{hol}$.

As remarked in the introduction, a connection is nonnegatively curved if and only if the infinitesimal holonomy is a nonpositive element in $\mathfrak{g}$ everywhere. On a larger scale, we will soon see a correspondence between nonnegatively curved connections and nonnegative path in $\tilde{G}$ (Lemma 3.1).

Equip the cylinder $\mathbb{S}^1 \times [0,1]$ with the orientation opposite to the product orientation. Fix a basepoint $* \in \mathbb{S}^1$. Let c_s denote the loop consisting of the segment from $(*,0)$ to $(*,s)$, the loop around $\mathbb{S}^1 \times \{s\}$ with the usual (counterclockwise) orientation, and the segment from $(*,s)$ to $(*,0)$, concatenated in this order.

For the next two lemmas, we essentially follow Seidel [Sei19, Section 4]. With Proposition 2.1 in hand, some results are generalized. Let $\mathcal{A}_{\geq 0}(\mathbb{S}^1 \times [0,1])$ denote the space of nonnegatively curved connections on $\mathbb{S}^1 \times [0,1]$ and $\mathcal{A}_{\geq 0}(\mathbb{S}^1 \times [0,1], \cdot, \cdot)$ denote its subspace with prescribed boundary behaviors (orientation for $\mathbb{S}^1 \times \{0\}$ is taken to be counterclockwise, as opposed to the induced one). Here $\cdot$ denotes a connection on $\mathbb{S}^1$, an element in $\tilde{G}$, or a conjugacy class of $\tilde{G}$. For the unit disk D we similarly define $\mathcal{A}_{\geq 0}(D, \cdot)$. Here we always work on the trivial bundle.

Lemma 3.1. *For any $A \in \mathcal{A}_{\geq 0}(\mathbb{S}^1 \times [0,1])$, the path $\tilde{p}(s) = \widetilde{hol}_A(c_s)$ is nonpositive. Moreover, for a fixed $a \in \Omega^1(\mathbb{S}^1;\mathfrak{g})$ with holonomy $\tilde{g}_0$ (with respect to the basepoint $*$) and a nonpositive path $\tilde{p}\colon [0,1] \to \tilde{G}$ with $\tilde{p}(0) = \tilde{g}_0$, then the space of $A \in \mathcal{A}_{\geq 0}(\mathbb{S}^1 \times [0,1])$ with $A|_{\mathbb{S}^1 \times \{0\}} = a$, $\widetilde{hol}_A(c_s) = \tilde{p}(s)$ is weakly contractible.*

Proof. See Seidel [Sei19, Lemma 4.16]. □

Lemma 3.2. *Fix a connection $a \in \Omega^1(\mathbb{S}^1;\mathfrak{g})$ with holonomy $\tilde{g} \in \tilde{G}$. Then $\mathcal{A}_{\geq 0}(D, a)$ is weakly contractible if $\tilde{g} \leq \tilde{I}_0$ and empty otherwise.*

Proof. (See also [Sei19, Proposition 4.21])

Let $\phi_t\colon D \to D$ be a smooth family of smooth maps with $\phi_0 = id$, $\phi_t|_{\mathbb{S}^1} = id$, $det(D\phi_t) \geq 0$, and such that ϕ_1 retracts a neighborhood of 0 to 0. Then pulling-back by ϕ_t shows that $\mathcal{A}_{\geq 0}(D, a)$ is weak equivalent to its subspace consisting of connections that are trivial near 0. Equivalently, it is weak equivalent to the space of connections in $\mathcal{A}_{\geq 0}(\mathbb{S}^1 \times [0,1])$ that restrict to a on $\mathbb{S}^1 \times \{1\}$ and restrict to zero on $\mathbb{S}^1 \times [0,1/2]$. Applying a similar pullback argument again, we see this space is in turn weak equivalent to $\mathcal{A}_{\geq 0}(\mathbb{S}^1 \times [0,1], 0, a)$.

Let $\tilde{C}$ denote the conjugacy class of $\tilde{g}$. Lemma 3.1 gives a weak equivalence $\mathcal{A}_{\geq 0}(\mathbb{S}^1 \times [0,1], 0, \tilde{C}) \to \mathcal{P}$, where $\mathcal{P}$ denotes the space of nonpositve path in $\tilde{G}$ from $\tilde{I}_0$ to some point in $\tilde{C}$, which is empty if $\tilde{g} \not\leq \tilde{I}_0$.

Assume now $\tilde{g} \leq \tilde{I}_0$, then by Proposition 2.1, evaluating at the endpoint 1 yields a homotopy equivalence $\mathcal{P} \to \tilde{C}$. Composition of the two maps above yields a weak equivalence

$$\mathcal{A}_{\geq 0}(\mathbb{S}^1 \times [0,1], 0, \tilde{C}) \to \tilde{C}, \ A \mapsto \widetilde{hol}_A(c_1). \tag{8}$$

By applying a family of gauge transformations that are constant along each $\mathbb{S}^1 \times \{s\}$ and trivial along $\mathbb{S}^1 \times \{1\}$, we can retract $\mathcal{A}_{\geq 0}(\mathbb{S}^1 \times [0,1], 0, \tilde{C})$ onto its subspace $\mathcal{A}'_{\geq 0}(\mathbb{S}^1 \times [0,1], 0, \tilde{C})$, where the prime indicate the subspace of connections that are trivial when restricting to $\{*\} \times [0,1]$. Then, replacing $\mathcal{A}_{\geq 0}$ by $\mathcal{A}'_{\geq 0}$ in (8), we get a weak equivalence whose fiber $\mathcal{A}'_{\geq 0}(\mathbb{S}^1 \times [0,1], 0, \tilde{g})$ is weakly contractible.

Finally, the space $\mathcal{A}'_{\geq 0}(\mathbb{S}^1 \times [0,1], 0, \tilde{g})$ retracts onto $\mathcal{A}'_{\geq 0}(\mathbb{S}^1 \times [0,1], 0, a)$ via a family of gauge transfromations that are trivial on $\mathbb{S}^1 \times \{0\} \cup \{*\} \times [0,1]$ and $\mathcal{A}_{\geq 0}(\mathbb{S}^1 \times [0,1], 0, a)$ retracts onto $\mathcal{A}'_{\geq 0}(\mathbb{S}^1 \times [0,1], 0, a)$ via the previous family of gauge transformations. Therefore $\mathcal{A}_{\geq 0}(\mathbb{S}^1 \times [0,1], 0, a)$ is also weakly contractible and the statement follows. □

Now we begin to analyze our problem in hand.

First let us consider the closed case. Let S be a closed surface with genus $g > 0$. Choose a basepoint $* \in S$ and loops α_i, β_i at $*$ generating $\pi_1(S)$. Furthermore assume all these loops do not meet each other except at $*$. Then S can be regard as a $4g$-gon with edges $\beta_g^{-1}, \alpha_g^{-1}, \beta_g, \alpha_g, \cdots, \beta_1, \alpha_1$ with the usual identifications. We may also assume that this order of edges represents the positive orientation of S.

If h_t is a family of loops in the $4g$-gon based at $*$ that shrinks the boundary loop to some loop $\tau\tau^{-1}$ where τ is any path with $\tau((0,1])$ disjoint from the boundary loop, then, thinking as loops in S, Lemma 3.1 implies that $p(t) = hol_A(h_t)$ is a nonnegative path in G from $hol_A(h_0)$ to $hol_A(h_1) = I$, for any $A \in \mathcal{A}_{\geq 0}(S)_e$. The condition e on euler number says precisely that $p(t)$ lifts to a path in $\tilde{G}$ from

$$\widetilde{hol}_A(h_0) = [\widetilde{hol}_A(\alpha_1), \widetilde{hol}_A(\beta_1)] \cdots [\widetilde{hol}_A(\alpha_g), \widetilde{hol}_A(\beta_g)]$$

to $\tilde{I}_e$. Here, when defining $hol, \widetilde{hol}$, we made a spherical blowup of S at $\tau(1)$ and fixed a trivialization of the pullbacked bundle over the blown-up surface (and adjusted h_1 accordingly; it is assumed that all h_t avoided $\tau(1)$ for $t < 1$).

Proposition 3.3. *The map*

$$\mathcal{A}_{\geq 0}(S)_e \to \tilde{G}^{2g}, \ A \mapsto (\widetilde{hol}_A(\alpha_1), \widetilde{hol}_A(\beta_1), \cdots, \widetilde{hol}_A(\alpha_g), \widetilde{hol}_A(\beta_g))$$

is a weak equivalence onto its image, which is given by

$$\widetilde{\mathrm{Rep}}_{\leq 0}(S)_e = \{(\tilde{X}_1, \tilde{Y}_1, \cdots, \tilde{X}_g, \tilde{Y}_g) \in \tilde{G}^{2g} \mid [\tilde{X}_1, \tilde{Y}_1] \cdots [\tilde{X}_g, \tilde{Y}_g] \leq \tilde{I}_e\}.$$

Proof. Like in the proof of Lemma 3.2, we choose a family of self maps ϕ_t on S that shrinks a neighborhood of the boundary of our $4g$-gon to the boundary. Pulling-back by these maps yields a map ϕ_1^* which is a weak equivalence onto its the subspace consisting of connections that are flat in a neighborhood U of the boundary loop $\beta_g^{-1}\alpha_g^{-1}\cdots\beta_1\alpha_1$. We may assume that $S\backslash U$ is diffeomorphic to a disk. Also note that the pullback preserves holonomy along the boundary loop. Now Lemma 3.2 applying to $S\backslash U$ implies that the image of the given map is $\widetilde{\mathrm{Rep}}_{\leq 0}(S)_e$, and that the fiber above each point in the image is weakly contractible. The desired result follows. □

Next we consider the case where S has genus g with $b > 0$ boundaries $\partial S_1, \cdots \partial S_b$, and let C_i be fixed hyperbolic conjugacy class of G. Choose a basepoint $* \in S\backslash\partial S$ and loops $\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, \gamma_1, \cdots, \gamma_b$ at $*$ generating $\pi_1(S)$, such that all these paths do not meet each other except at endpoints, and that $\gamma_i = \sigma_i\tau_i\sigma_i^{-1}$ where σ_i is a path from $*$ to a point on ∂S_i and τ_i is a loop around ∂S_i. Then S can be regarded as a $(4g + 3b)$-gon with edges $\sigma_b^{-1}, \tau_b, \sigma_b, \cdots \beta_1^{-1}, \alpha_1^{-1}, \beta_1, \alpha_1$ with the usual identifications. We may also assume that this order of edges represents the positive orientation of S.

If h_t is a family of loops in the $(4g+3b)$-gon based at $*$ that shrinks the boundary loop to some loop $\tau\tau^{-1}$, then, thinking as loops in S, Lemma 3.1 implies that $\tilde{p}(t) = \widetilde{hol}_A(h_t)$ is a nonnegative path in G from $\widetilde{hol}_A(h_0)$ to $\widetilde{hol}_A(h_1) = \tilde{I}_0$, for any $A \in \mathcal{A}_{\geq 0}(S, \{C_i\})_e$. Here we are working on the trivial bundle over S. This time, the condition e on relative euler number says precisely that

$$e = -\sum_{i=1}^{b} \mathrm{rot}(\widetilde{hol}_A(\gamma_i)),$$

which is a gauge invariant integer.

Let $\pi : \tilde{G} \to G$ denote the covering map and let

$$\boldsymbol{C} = \prod_{i=1}^{b} C_i;$$

$$\tilde{\boldsymbol{C}} = \{(\tilde{Z}_1, \cdots, \tilde{Z}_b) \in \prod_{i=1}^{b} \pi^{-1}(C_i) \mid \sum_{i=1}^{b} \mathrm{rot}(\tilde{Z}_i) = 0\}.$$

Proposition 3.4. *The map*

$$\mathcal{A}_{\geq 0}(S, \{C_i\})_e \to \tilde{G}^{2g} \times \tilde{\boldsymbol{C}},\ A \mapsto (\widetilde{hol}_A(\alpha_1), \widetilde{hol}_A(\beta_1), \cdots, \widetilde{hol}_A(\alpha_g), \widetilde{hol}_A(\beta_g), \widetilde{hol}_A(\gamma_1), \cdots, \widetilde{hol}_A(\gamma_{b-1}), \widetilde{hol}_A(\gamma_b)\tilde{I}_e)$$

is a weak equivalence onto its image, which is given by

$$\widetilde{\mathrm{Rep}}_{\leq 0}(S, \{C_i\})_e = \{(\tilde{X}_1, \tilde{Y}_1, \cdots, \tilde{Z}_b) \in \tilde{G}^{2g} \times \tilde{\boldsymbol{C}} \mid [\tilde{X}_1, \tilde{Y}_1]\cdots[\tilde{X}_g, \tilde{Y}_g]\tilde{Z}_1\cdots\tilde{Z}_b \leq \tilde{I}_e\}.$$

The proof is similar to the closed case and we skip. Note that the formula here contains the closed case as a special situation where $b = 0$. Moreover, although we have excluded the case $S = \mathbb{S}^2$ in our previous discussion, the same conclusions hold for this case as well, namely that $\mathcal{A}_{\geq 0}(\mathbb{S}^2)_e$ is contractible if $e \geq 0$ and is empty otherwise. To see this, one trivialize a neighborhood of some point in $\mathbb{S}^2$ by the same pullback trick as before, then Lemma 3.2 yields the desired result. Therefore, for later use we shall simply refer to Proposition 3.4 for the result for any surface S.

Remark 3.5. One may also want to conclude that the map

$$\mathcal{A}_{\geq 0}(S,\{C_i\})_e/\mathcal{G} \to \mathrm{Rep}_{\leq 0}(S,\{C_i\})_e/G,\ [A] \mapsto [(hol_A(\alpha_1), hol_A(\beta_1), \cdots, hol_A(\gamma_b))]$$

is a weak equivalence. Here

$$\mathrm{Rep}_{\leq 0}(S,\{C_i\})_e = \{(X_1,Y_1,\cdots,Z_b) \in G^{2g} \times \boldsymbol{C} \mid [X_1,Y_1]\cdots[X_g,Y_g]\tilde{Z}_1\cdots\tilde{Z}_b \leq \tilde{I}_e\}$$

and G acts by conjugation. The quotient should be interpreted as the homotopy quotient $EG \times_G \mathrm{Rep}_{\leq 0}(S,\{C_i\})_e$ in case the conjugation action is not free.

To conclude this, one argues as follows: realize $\mathcal{G}$ as the semidirect product of $\mathcal{G}_0$ and G, where $\mathcal{G}_0$ denotes the gauge that acts as identity at $*$. Then $\mathcal{A}_{\geq 0}(S,\{C_i\})/\mathcal{G}_0 \xrightarrow{\simeq} \mathrm{Rep}_{\leq 0}(S)_e$, and mod out the extra G corresponds to mod conjugation on the right hand side.

However, we refrain from taking such perspective since we did not really define a topology on $\mathcal{A}_{\geq 0}(S,\{C_i\})$ and it does not make much sense to discuss the weak homotopy type of the quotient $\mathcal{A}_{\geq 0}(S,\{C_i\})/\mathcal{G}$.

Lastly, we point out that all results in this section remains true if we replace the pair (nonnegatively curved connections, nonnegative/nonpositive paths) by (flat connections, constant paths) or (all connections, all paths). The proofs are similar, if not easier. Note that the counterpart for Proposition 2.1 in the all connection case, namely that the space of all paths in $\tilde{G}$ with given endpoints is weakly contractible, is trivially true.

In particular we recover the following descriptions for $\mathcal{A}_{flat}$ and $\mathcal{A}$ (possibly $b = 0$):

$$\mathcal{A}_{flat}(S,\{C_i\})_e \xrightarrow{\simeq} \widetilde{\mathrm{Rep}}(S,\{C_i\})_e;$$

$$\mathcal{A}(S,\{C_i\})_e \xrightarrow{\simeq} \tilde{G}^{2g} \times \tilde{\boldsymbol{C}}.$$

Here

$$\widetilde{\mathrm{Rep}}(S,\{C_i\})_e = \{(\tilde{X}_1,\tilde{Y}_1,\cdots,\tilde{Z}_b) \in \tilde{G}^{2g} \times \tilde{\boldsymbol{C}} \mid [\tilde{X}_1,\tilde{Y}_1]\cdots[\tilde{X}_g,\tilde{Y}_g]\tilde{Z}_1\cdots\tilde{Z}_b = \tilde{I}_e\}.$$

By our construction, it is clear that in all cases, the following diagram commutes. Here vertical arrows are weak equivalences.

$$\begin{array}{ccccc} \mathcal{A}_{flat}(S,\{C_i\})_e & \hookrightarrow & \mathcal{A}_{\geq 0}(S,\{C_i\})_e & \hookrightarrow & \mathcal{A}(S,\{C_i\})_e \\ \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq \\ \widetilde{\mathrm{Rep}}(S,\{C_i\})_e & \hookrightarrow & \widetilde{\mathrm{Rep}}_{\leq 0}(S,\{C_i\})_e & \hookrightarrow & \tilde{G}^{2g} \times \tilde{\boldsymbol{C}}. \end{array} \tag{9}$$

Remark 3.6. Pedantically, one also have the corresponding commutative diagram for moduli spaces:

$$\begin{array}{ccccc} \mathcal{A}_{flat}(S,\{C_i\})_e/\mathcal{G} & \hookrightarrow & \mathcal{A}_{\geq 0}(S,\{C_i\})_e/\mathcal{G} & \hookrightarrow & \mathcal{A}(S,\{C_i\})_e/\mathcal{G} \\ \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq \\ \mathrm{Rep}(S,\{C_i\})_e/G & \hookrightarrow & \mathrm{Rep}_{\leq 0}(S,\{C_i\})_e/G & \hookrightarrow & (G^{2g} \times \boldsymbol{C})/G, \end{array}$$

where $\mathrm{Rep}(S,\{C_i\})$ is defined in the similar way as above.

4. Deformation Techniques in G

In this section we perform two tricks to deform some particular elements in G in some nice ways. More explicitly, we perform a way to deform the space of hyperbolic elements in the positive direction and a way to deform a pair of elements in G such that their commutator moves in the positive direction.

These two easy tricks will be the key for our analysis in Section 5 of nonpositive representation spaces.

4.1. Deformation of Hyperbolic Elements.

We begin by introducing a parametrization for hyperbolic elements in G.

For any hyperbolic element $g \in G$, let $\xi_+(g), \xi_-(g) \in \mathbb{R}P^1$ denote the eigenvector of the left multiplication map $g\colon \mathbb{R}^2/\{\pm 1\} \to \mathbb{R}^2/\{\pm 1\}$ with the larger, smaller absolute value of eigenvalue, respectively, and let $\lambda(g)$ denote the larger absolute value of the eigenvalue. Also, let $\xi(g)$ denote the midpoint of the (counterclockwise) arc $\widehat{\xi_-(g)\xi_+(g)}$. For example, when $\lambda > 1$, for the element

$$g = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} \in G$$

we have $\xi_-(g) = 0$, $\xi_+(g) = \pi/2$, $\xi(g) = \pi/4$ and $\lambda(g) = \lambda$.

Let $Hyp \subset G$ denote the set of hyperbolic elements. Then the map

$$\xi_- \times \xi_+ \times \lambda\colon Hyp \to ((\mathbb{R}P^1)^2 \backslash \Delta) \times (1, \infty),$$

is a diffeomorphism. Here Δ denotes the diagonal of $\mathbb{R}P^1 \times \mathbb{R}P^1$. The inverse of this map is given by

$$\Psi(\xi_-, \xi_+, \lambda) = \frac{1}{\sin(\xi_- - \xi_+)} \begin{pmatrix} \cos\xi_+ & \cos\xi_- \\ \sin\xi_+ & \sin\xi_- \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} \sin\xi_- & -\cos\xi_- \\ -\sin\xi_+ & \cos\xi_+ \end{pmatrix}. \tag{10}$$

Lemma 4.1. *Let $(\xi_-, \xi_+, \lambda) \in (\mathbb{R}P^1)^2 \times (1, \infty)$ and $a, b > 0$. Then the path*

$$p\colon (-\varepsilon, \varepsilon) \to G,$$

$$t \mapsto \Psi(\xi_- - at, \xi_+ + bt, \lambda)$$

is positive.

Proof. It suffices to check the velocity vector at $t = 0$ is positive. After conjugation and changing a, b we may assume $\xi_- = 0$, $\xi_+ = \pi/2$.

Let $p_+(t) = \Psi(0, bt, \lambda)$. By (10) we have

$$p_+(t) = \begin{pmatrix} \lambda^{-1} & (\lambda^{-1}-\lambda)\tan(bt) \\ 0 & -\lambda \end{pmatrix}.$$

Thus $p_+'(0) = \begin{pmatrix} 0 & b(\lambda^{-1}-\lambda) \\ 0 & 0 \end{pmatrix}$. Similarly for $p_-(t) = \Psi(-at, \pi/2, \lambda)$ we have $p_-'(0) = \begin{pmatrix} 0 & 0 \\ a(\lambda-\lambda^{-1}) & 0 \end{pmatrix}$. Therefore

$$p'(0)p(0)^{-1} = (\lambda - \lambda^{-1}) \begin{pmatrix} 0 & -b\lambda^{-1} \\ a\lambda & 0 \end{pmatrix},$$

which is a positive element in $\mathfrak{g}$ by (4). $\square$

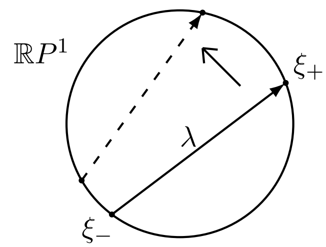

FIGURE 1. Deform $\Psi(\xi_-, \xi_+, \lambda)$ in positive direction

Lemma 4.2. *Let $C = C_\lambda$ be a hyperbolic conjugacy class consists of $g \in Hyp$ with $\lambda(g) = \lambda$. Let $\xi \in \mathbb{R}P^1$ be arbitrary. Then the path*

$$\gamma_{C,\xi}\colon (0, \pi/2) \to C,\ t \mapsto \Psi(\xi - t, \xi + t, \lambda) \tag{11}$$

is positive. Moreover, for any ε-neighborhood U of ξ (resp. $\xi + \pi/2$) in $\mathbb{R}P^1$, the element $\gamma_{C,\xi}(t)$ maps every point in $\mathbb{R}P^1 \backslash U$ clockwisely (resp. counterclockwisely) into U for sufficiently small $t > 0$ (resp. sufficiently large $t < \pi/2$).

Proof. The first statement follows from Lemma 4.1. For the second statement, after conjugation we assume $\xi = 0$. By (10) we have

$$\gamma_{C,0}(t) = \begin{pmatrix} \frac{1}{2}(\lambda+\lambda^{-1}) & \frac{\cot t}{2}(\lambda-\lambda^{-1}) \\ \frac{\tan t}{2}(\lambda-\lambda^{-1}) & \frac{1}{2}(\lambda+\lambda^{-1}) \end{pmatrix},$$

and the statement is clear. □

Corollary 4.3. *There is a homotopy $\gamma\colon [0, 1) \times Hyp \to Hyp$ satisfying:*

(a) $\gamma_0 = id_{Hyp}$.

(b) For any $Z \in Hyp$, the path $t \mapsto \gamma_t(Z)$ stays in a fixed conjugacy class and is positive.

(c) For any $Z \in Hyp$ and any ε-neighborhood U of $\xi(Z) + \pi/2$ in $\mathbb{R}P^1$, the element $\gamma_t(Z)$ maps every point in $\mathbb{R}P^1 \backslash U$ counterclockwisely into U for sufficiently large t.

Proof. For any hyperbolic conjugacy class C and $Z \in C$, let $t_0 = \frac{1}{2}(\xi_+(Z) - \xi_-(Z)) \in (0, \pi/2)$ and take

$$\gamma_t(Z) = \gamma_{C,\xi(Z)}(t\pi/2 + (1 - t)t_0).$$ □

4.2. **Deformation of Elements in a Commutator.** Recall that the commutator has a canonical lift in $\tilde{G}$.

Lemma 4.4. *Suppose $X, Y \in G$ with $\operatorname{rot}([X, Y]) \neq 0$, then both X and Y must be hyperbolic. In particular, if $\operatorname{rot}([X, Y]) < 0$, then $\xi_-(X), \xi_-(Y), \xi_+(X), \xi_+(Y) \in \mathbb{R}P^1$ are distinct and are arranged in the counterclockwise order.*

Proof. For the first statement, we proceed by contradiction and show that if X is not hyperbolic, then $[X, Y]$ must have a fixed point in $\mathbb{R}$ and hence $\operatorname{rot}([X, Y]) = 0$. The same argument will apply for Y being not hyperbolic. Let $\tilde{X}, \tilde{Y}$ be arbitrary lifts of X, Y.

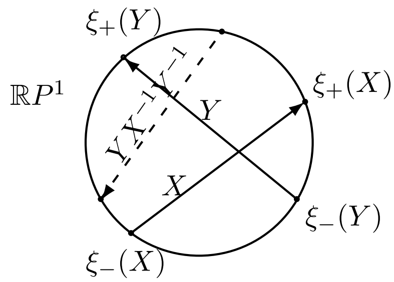

FIGURE 2. Conjugate X^{-1} by Y

If X is elliptic, we can apply simultaneous conjugation to assume $\tilde{X} = \tilde{r}_\theta$. Hence $\varphi_{\tilde{X}} \equiv \theta$. On the other hand, from (5) we know that $\mathrm{rot}(\tilde{g}) \in [\min \varphi_{\tilde{g}}/\pi, \max \varphi_{\tilde{g}}/\pi]$ for any $\tilde{g} \in \tilde{G}$. In particular, we get $\min \varphi_{\tilde{Y}\tilde{X}^{-1}\tilde{Y}^{-1}} \le -\theta \le \max \varphi_{\tilde{Y}\tilde{X}^{-1}\tilde{Y}^{-1}}$. By continuity, there exists some $x \in \mathbb{R}$ such that $\varphi_{\tilde{Y}\tilde{X}^{-1}\tilde{Y}^{-1}}(x) = -\theta$, and x is therefore fixed by $[X, Y]$.

If X is parabolic, say nonnegative parabolic, choose lift $\tilde{X}$ with rotation number 0 and we have $\min \varphi_{\tilde{X}} = 0$ (see Lemma 2.2). Suppose this minimum is attained by $x_0 \in \mathbb{R}$. Note that $\tilde{Y}\tilde{X}^{-1}\tilde{Y}^{-1}$ is a nonpositive parabolic element with rotation number 0 implies $\max \varphi_{\tilde{Y}\tilde{X}^{-1}\tilde{Y}^{-1}} = 0$. Suppose this maximum is attained by $x_1 \in \mathbb{R}$. Then we have:

$$[X, Y](x_1) \ge x_1 \text{ and } [X, Y]((\tilde{Y}\tilde{X}^{-1}\tilde{Y}^{-1})^{-1}(x_0)) \le (\tilde{Y}\tilde{X}^{-1}\tilde{Y}^{-1})^{-1}(x_0).$$

By continuity, there exists some x fixed by $[X, Y]$.

For the second statement, $\mathrm{rot}([X, Y]) < 0$ implies $[X, Y]$ moves all the points in $\mathbb{R}P^1$ clockwisely (referred to as negatively later). In particular, as $\xi_-(YX^{-1}Y^{-1}) = Y(\xi_+(X))$ and $\xi_+(YX^{-1}Y^{-1}) = Y(\xi_-(X))$ are both fixed by $YX^{-1}Y^{-1}$, X must move them negatively. That is, $Y(\xi_+(X)), Y(\xi_-(X)) \in \overset{\frown}{\xi_+(X)\xi_-(X)}$. Especially, Y moves $\xi_-(X)$ negatively and $\xi_+(X)$ positively. Hence $\xi_+(X) \in \overset{\frown}{\xi_-(Y)\xi_+(Y)}$ and $\xi_-(X) \in \overset{\frown}{\xi_+(Y)\xi_-(Y)}$, which finish our proof. □

Lemma 4.5. *Fix hyperbolic $X \in G$ and $\xi_-, \xi_+ \in \mathbb{R}P^1$ satisfying $\xi_-(X), \xi_-, \xi_+(X), \xi_+$ are distinct and are arranged in the counterclockwise order, then the path*

$$\tau_{X,\xi_-,\xi_+} : (1, \infty) \to \tilde{G}, t \mapsto [X, \Psi(\xi_-, \xi_+, t)]$$

is negative. Moreover, for sufficiently small t, $\mathrm{rot}(\tau_{X,\xi_-,\xi_+}(t)) > -1/2$.

Proof. Let Y_t denote $\Psi(\xi_-, \xi_+, t)$, then it suffices to argue that $Y_t X^{-1} Y_t^{-1}$ is moving in the negative direction. Indeed, observe that $\lambda(Y_t X^{-1} Y_t^{-1}) = \lambda(X)$ is fixed, $\xi_-(Y_t X^{-1} Y_t^{-1}) = Y_t(\xi_+(X))$ moves positively while we increase t, and $\xi_+(Y_t X^{-1} Y_t^{-1}) = Y_t(\xi_-(X))$ moves negatively while we increase t. Therefore Lemma 4.1 applies, which shows that τ_{X,ξ_-,ξ_+} is negative. The second statement is clear as $\lim_{t\to 1} Y_t = I$ implies $\lim_{t\to 1} \tau_{X,\xi_-,\xi_+}(t) = [X, I] = \tilde{I}_0$. □

Remark 4.6. In fact, we always have $\mathrm{rot}(\tau_{X,\xi_-,\xi_+}(t)) < 0$ for all t and $\mathrm{rot}(\tau_{X,\xi_-,\xi_+}(t)) = -1$ for sufficiently large t.

Let $Br\colon G \times G \to \tilde{G}$ denote the lifted commutator map.

Corollary 4.7. *There is a homotopy $\tau\colon [0,1] \times G \times G \to G \times G$ satisfying:*
(a) $\tau_0 = id_{G\times G}$, $\tau_t = \tau_1$ *for* $t \geq 1/2$, $\tau_t(X,Y) = (X,Y)$ *if* $\mathrm{rot}([X,Y]) \geq -1/2$.
(b) For any $X, Y \in G$, *the path* $t \mapsto Br \circ \tau_t(X,Y)$ *is nonnegative. It is positive on* $[0,1/2]$ *if* $\mathrm{rot}([X,Y]) < -1/2$.
(c) For any $X, Y \in G$, $\mathrm{rot}(Br \circ \tau_1(X,Y)) \geq -1/2$.

Proof. For $\mathrm{rot}([X,Y]) \leq -1/2$, because of Lemma 4.5 we may choose the unique (as rotation number is strictly increasing along positive path in the elliptic locus) $\lambda_{X,Y}$ such that $\mathrm{rot}([X, \Psi(\xi_-(Y), \xi_+(Y), \lambda)]) = -1/2$. Then $\lambda_{X,Y} \leq \lambda(Y)$, with equality if and only if $\mathrm{rot}([X,Y]) = -1/2$. Note that the uniqueness of $\lambda_{X,Y}$ implies its continuity in X, Y. Now for $t \leq 1/2$, take

$$\tau_t(X,Y) = \begin{cases} (X,Y), & \text{if } \mathrm{rot}([X,Y]) > -1/2 \\ (X, \Psi(\xi_-(Y), \xi_+(Y), (1-2t)\lambda(Y) + 2t\lambda_{X,Y})), & \text{if } \mathrm{rot}([X,Y]) \leq -1/2. \end{cases} \quad \square$$

5. Nonpositive Representation Spaces

5.1. **Proof of Main Theorem.** By the commutative diagram (9), to prove our main theorem, it suffices to prove the corresponding statement in terms of representation spaces:

Proposition 5.1. *Let S be an oriented surface with genus g and b boundary components. Let $C_1, \cdots, C_b$ be hyperbolic conjugacy classes of G.*
(a) Suppose $e \geq \max\{g,1\}$. *Then the inclusion* $\widetilde{\mathrm{Rep}}_{\leq 0}(S, \{C_i\})_e \hookrightarrow \tilde{G}^{2g} \times \tilde{\boldsymbol{C}}$ *is a homotopy equivalence.*
(b) Suppose $e \leq -\max\{g,1\}$. *Then the inclusion* $\widetilde{\mathrm{Rep}}(S, \{C_i\})_e \hookrightarrow \widetilde{\mathrm{Rep}}_{\leq 0}(S, \{C_i\})_e$ *is a homotopy equivalence.*

As a shorthand, we use U to denote a generic element $(X_1, Y_1, \cdots, X_g, Y_g, Z_1, \cdots, Z_b) \in G^{2g} \times \boldsymbol{C}$ and $\tilde{P}(U)$ to denote the product $[X_1, Y_1] \cdots [X_g, Y_g] \tilde{Z}_1 \cdots \tilde{Z}_b \in \tilde{G}$. Similarly, we use $\tilde{U}$ to denote a generic element in $\tilde{G}^{2g} \times \tilde{\boldsymbol{C}}$ and $\tilde{P}(\tilde{U})$ to denote the corresponding product. Moreover, let γ, τ be as constructed in the previous section.

Lemma 5.2. *The homotopy*

$$\Gamma\colon [0,1) \times G^{2g} \times \boldsymbol{C} \to G^{2g} \times \boldsymbol{C},\ \Gamma_t = (\tau_t)^g \times (\gamma_t)^b$$

satisfies:
(a) $\Gamma_0 = id_{G^{2g} \times \boldsymbol{C}}$.
(b) For any $U \in G^{2g} \times \boldsymbol{C}$, $\mathrm{rot}(\tilde{P}(\Gamma_t(U))) > -\max\{g,1\}$ *for sufficiently large* t.
(c) For any $U \in G^{2g} \times \boldsymbol{C}$, *the path* $t \mapsto \tilde{P}(\Gamma_t(U))$ *is nonnegative. It is positive wherever* $\mathrm{rot}(\tilde{P}(\Gamma_t(U))) \leq -\max\{g,1\}$.

Moreover, Γ lifts to a homotopy $\tilde{\Gamma}\colon [0,1) \times \tilde{G}^{2g} \times \tilde{\boldsymbol{C}} \to \tilde{G}^{2g} \times \tilde{\boldsymbol{C}}$ that satisfies the corresponding properties.

Proof. (a) is immediate. For (b), write $U = (X_1, Y_1, \cdots, Z_b)$. Then for sufficiently large t, there exists some r_i such that the displacement function φ defined by (6) satisfies

$$\varphi_{Br \circ \tau_t(X_i, Y_i)} \geq -r_i > -\pi;$$

$$\varphi_{\tilde{\gamma}_t(Z_j)} > \begin{cases} \sum_{i=1}^{g}(r_i - \pi)/b, & g > 0 \\ -\pi/b, & g = 0 \end{cases}$$

everywhere on $\mathbb{R}$. Therefore

$$\varphi_{\tilde{P}(\Gamma_t(U))} > -\max\{g, 1\}\pi$$

and (b) follows. For (c), by construction of γ, τ we see $\tilde{P}(\Gamma_t(U))$ is positive wherever $\Gamma_t(U)$ is nonconstant. Also, any maximal interval where $\Gamma_t(U)$ is constant must be of form $[t_0, 1)$ (in which case $b = 0$, $t_0 = 0$ or $1/2$). Therefore (c) follows from (b).

The statement about lifting is clear. □

Reverse all the constructions in Section 4 to make hyperbolic elements or commutators deform in the negative direction, we get

Corollary 5.3. *There is a homotopy* $\Gamma^-\colon [0,1) \times G^{2g} \times \boldsymbol{C} \to G^{2g} \times \boldsymbol{C}$ *satisfying:*

(a) $\Gamma_0^- = id_{G^{2g}\times\boldsymbol{C}}$.

(b) For any $U \in G^{2g} \times \boldsymbol{C}$, $\operatorname{rot}(\tilde{P}(\Gamma_t^-(U))) < \max\{g, 1\}$ *for sufficiently large* t.

(c) For any $U \in G^{2g} \times \boldsymbol{C}$, *the path* $t \mapsto \tilde{P}(\Gamma_t^-(U))$ *is nonpositive. It is negative wherever* $\operatorname{rot}(\tilde{P}(\Gamma_t^-(U))) \geq \max\{g, 1\}$.

Moreover, Γ^- *lifts to a homotopy* $\tilde{\Gamma}^-\colon [0,1) \times \tilde{G}^{2g} \times \tilde{\boldsymbol{C}} \to \tilde{G}^{2g} \times \tilde{\boldsymbol{C}}$ *that satisfies the corresponding properties.* □

Proof of Proposition 5.1. Let

$$\widetilde{\operatorname{Rep}}_{\overline{Par^-}}(S, \{C_i\})_e = \{\tilde{U} \in \tilde{G}^{2g} \times \tilde{\boldsymbol{C}} \mid \tilde{P}(\tilde{U}) \text{ is either } \tilde{I}_e \text{ or a nonpositive parabolic element with rotation number } e\}.$$

This space is the boundary of $\widetilde{\operatorname{Rep}}_{\leq 0}(S, \{C_i\})_e$ as a (closed) subspace of $\tilde{G}^{2g} \times \tilde{\boldsymbol{C}}$.

(a) For any $\tilde{U} \in \tilde{G}^{2g} \times \tilde{\boldsymbol{C}}$, by Corollary 5.3 we can find a minimal $t = t(\tilde{U}) \in [0, 1)$ such that $\tilde{\Gamma}_t^-(\tilde{U}) \in \widetilde{\operatorname{Rep}}_{\leq 0}(S, \{C_i\})_e$. Then $t = 0$ for $\tilde{U} \in \widetilde{\operatorname{Rep}}_{\leq 0}(S, \{C_i\})_e$, and t is the unique time such that $\tilde{\Gamma}_t^-(\tilde{U}) \in \widetilde{\operatorname{Rep}}_{\overline{Par^-}}(S, \{C_i\})_e$ for

$$\tilde{U} \in (\tilde{G}^{2g} \times \tilde{\boldsymbol{C}}) \backslash \widetilde{\operatorname{Rep}}_{\leq 0}(S, \{C_i\})_e \cup \widetilde{\operatorname{Rep}}_{\overline{Par^-}}(S, \{C_i\})_e.$$

Here, to conclude uniqueness, we used the (strict) negativity statement in (c) of Corollary 5.3.

Therefore is t continuous in $\tilde{U}$, and

$$\tilde{\Lambda}_s^-(\tilde{U}) = \tilde{\Gamma}^-_{st(\tilde{U})}(\tilde{U}),\ s \in [0, 1]$$

gives a deformation retract from $\tilde{G}^{2g} \times \tilde{\boldsymbol{C}}$ onto $\widetilde{\operatorname{Rep}}_{\leq 0}(S, \{C_i\})_e$.

(b) For any $\tilde{U} \in \widetilde{\operatorname{Rep}}_{\leq 0}(S, \{C_i\})_e$, by Lemma 5.2, we can find a maximal $t = t(\tilde{U}) \in [0, 1)$ such that $\tilde{\Gamma}_t(U) \in \widetilde{\operatorname{Rep}}_{\leq 0}(S, \{C_i\})_e$. Then t is the unique time such that $\tilde{\Gamma}_t(\tilde{U}) \in \widetilde{\operatorname{Rep}}_{\overline{Par^-}}(S, \{C_i\})_e$, and

$$\tilde{\Lambda}_s(\tilde{U}) = \tilde{\Gamma}_{st(\tilde{U})}(\tilde{U}),\ s \in [0, 1]$$

gives a deformation retract from $\widetilde{\operatorname{Rep}}_{\leq 0}(S, \{C_i\})_e$ onto $\widetilde{\operatorname{Rep}}_{\overline{Par^-}}(S, \{C_i\})_e$.

If $\chi(S) \geq 0$, it is straighforward to check that $\widetilde{\mathrm{Rep}}_{\leq 0}(S, \{C_i\})_e = \emptyset$ for all $e < 0$, and the original statement is trivial. Assume now $\chi(S) < 0$, then a result by Mondello [Mon16, Theorem 2.19(e)] implies that the inclusion

$$\mathrm{Rep}(S, \{C_i\})_e/G \hookrightarrow \mathrm{Rep}_{\overline{Par^-}}(S, \{C_i\})_e/G$$

a deformation retract. Since $e < 0$ implies that the conjugation action is free and proper (for properness see [Mon16, Lemma 2.10, Remark 2.13]), we see that

$$\widetilde{\mathrm{Rep}}_{\overline{Par^-}}(S, \{C_i\})_e \to \mathrm{Rep}_{\overline{Par^-}}(S, \{C_i\})_e/G$$

is a fiber bundle with fiber $G \times \mathbb{Z}^{2g+\max\{0,b-1\}}$. Therefore the homotopy long exact sequence and five lemma and Whitehead theorem implies that the inclusion

$$\widetilde{\mathrm{Rep}}(S, \{C_i\})_e \hookrightarrow \widetilde{\mathrm{Rep}}_{\overline{Par^-}}(S, \{C_i\})_e$$

is a homotopy equivalence. The statement follows. □

5.2. **Nonpositive Representation Spaces in Genus Zero Case.** Recall that Proposition 1.1 states that for $e < 0$, we have a homotopy equivalence

$$\mathrm{Rep}(S, \{C_i\})_e/G \simeq Sym^{-\chi(S)+e}(S). \tag{12}$$

In this section, assuming genus zero, we prove the following proposition without invoking part (b) of our main theorem.

Proposition 5.4. *Suppose S has genus zero and $b > 0$ boundary components, e is an arbitrary integer. Then the space $\mathrm{Rep}_{\leq 0}(S, \{C_i\})_e/PSO(2)$ is homotopy equivalent to $Sym^{-\chi(S)+e}(S)$.*

Since the G-action on $\mathrm{Rep}_{\leq 0}(S, \{C_i\})_e$ is free and proper for $e < 0$, a consequence is that:

Corollary 5.5. *Under the same setting, assume $e < 0$. Then the space $\mathrm{Rep}_{\leq 0}(S, \{C_i\})_e/G$ is homotopy equivalent to $Sym^{-\chi(S)+e}(S)$.*

While reading the following proof, one may want to visualize hyperbolic elements using arrows as in Figure 1.

Proof of Proposition 5.4. Let $\xi\colon Hyp \to \mathbb{R}P^1$ be as in Section 4.
Step 1: The map $\Xi = \xi^b\colon \mathrm{Rep}_{\leq 0}(S, \{C_i\})_e \to (\mathbb{R}P^1)^b$ is a homotopy equivalence onto its image.

Let $\xi_1, \cdots, \xi_b \in \mathbb{R}P^1$ be given. Let $\gamma_{C,\xi}$ be as in (11). Via the map $\gamma_{C_1,\xi_1} \times \cdots \times \gamma_{C_b,\xi_b}$, we may identify $\Xi^{-1}(\xi_1, \cdots, \xi_b)$ with a subset of $(0, \pi/2)^b$, denoted K. Then an element $(t_1, \cdots, t_b) \in K$ implies that $(t'_1, \cdots, t'_b) \in K$ for all $0 < t'_j \leq t_j$. Hence K is contractible.

Step 2: Lift $\xi_1, \cdots, \xi_b \in \mathbb{R}P^1$ to $x_1, \cdots, x_b \in \mathbb{R}$ such that $0 \leq x_{i+1} - x_i < \pi$. Then $(\xi_1, \cdots, \xi_b) \in im(\Xi)$ if and only if $x_b - x_1 > -e\pi$.

Let $(Z_1, \cdots, Z_b) \in \Xi^{-1}(\xi_1, \cdots, \xi_b)$. Then we have

$$\begin{aligned} e\pi &\geq \varphi_{\tilde{Z}_1 \cdots \tilde{Z}_b}(x_b) = \tilde{Z}_1 \cdots \tilde{Z}_b(x_b) - x_b > \tilde{Z}_1 \cdots \tilde{Z}_{b-1}(x_b) - x_b \\ &> \tilde{Z}_1 \cdots \tilde{Z}_{b-2}(x_{b-1}) - x_b > \cdots > x_1 - x_b. \end{aligned}$$

Conversely, assume $x_b - x_1 > -e\pi$. Choose $0 < \varepsilon < (x_b - x_1 + e\pi)/2$ such the closures of ε-neighborhoods of ξ_i and ξ_{i+1} are disjoint unless $\xi_i = \xi_{i+1}$. Choose small $t > 0$ satisfying (c) in Lemma 4.2 with respect to ε and each γ_{C_i,ξ_i}. Let $Z_i = \gamma_{C_i,\xi_i}(t)$. Then we have

$$\varphi_{\tilde{Z}_1\cdots\tilde{Z}_b}(x_b - \varepsilon) = \tilde{Z}_1 \cdots \tilde{Z}_b(x_b - \varepsilon) - (x_b - \varepsilon) < \tilde{Z}_1 \cdots \tilde{Z}_{b-1}(x_b + \varepsilon) - \pi - (x_b - \varepsilon)$$
$$< \tilde{Z}_1 \cdots \tilde{Z}_{b-2}(x_{b-1} + \varepsilon) - \pi - (x_b - \varepsilon) < \cdots < (x_1 + \varepsilon) - \pi - (x_b - \varepsilon) < e\pi - \pi.$$

Therefore $(Z_1, \cdots, Z_b) \in \mathrm{Rep}_{\leq 0}(S, \{C_i\})_e$ and the claim follows.

Step 3: $\mathrm{Rep}_{\leq 0}(S, \{C_i\})_e/PSO(2) \simeq \mathbb{T}^{b-1}_{b-2+e}$. Here $\mathbb{T}^m_k$ denotes the k-skeleton of the m-dimensional torus $\mathbb{T}^m$ equipped with the usual cell structure (so that the number of i-cells is $\binom{m}{i}$).

Let Ψ be as in (10). We observe that the conjugation of $\Psi(\xi_-, \xi_+, \lambda)$ by $r_\theta \in PSO(2)$ equals $\Psi(\xi_- + \theta, \xi_+ + \theta, \lambda)$. Therefore,

$$\mathrm{Rep}_{\leq 0}(S, \{C_i\})_e/PSO(2) \simeq \{(Z_1, \cdots, Z_b) \in \mathrm{Rep}_{\leq 0}(S, \{C_i\})_e \mid \xi(Z_1) = 0 \in \mathbb{R}P^1\}.$$

which is in turn, by Step 1, homotopy equivalent to the the space of $(\xi_1, \cdots, \xi_b) \in im(\Xi)$ with $\xi_1 = 0$. Let $d_i = (x_{i+1} - x_i)/\pi \in [0, 1)$ where x_i are as in Step 2. Then

$$\mathrm{Rep}_{\leq 0}(S, \{C_i\})_e/PSO(2) \simeq \{(d_1, \cdots, d_{b-1}) \in \mathbb{T}^{b-1} \mid \sum_{i=1}^{b-1} d_i > -e\}, \tag{13}$$

where we have identified $[0, 1)^{b-1}$ with $\mathbb{T}^{b-1}$ via the natural bijection.

Finally, the reader may check by himself/herself that the right hand side of (13) is homotopy equivalent to $\mathbb{T}^{b-1}_{b-2+e}$. For example, for $e \leq 0$ one can first deform its complement, cell by cell, onto $\mathbb{T}^{b-1}_{-e}$, and then deform $\mathbb{T}^{b-1}\backslash\mathbb{T}^{b-1}_{-e}$, cell by cell, onto $\mathbb{T}^{b-1}_{b-2+e}$ (here the cell structure is shifted by $(1/2, \cdots, 1/2)$).

Step 4: $Sym^k(S) \simeq \mathbb{T}^{b-1}_k$ for any $k \in \mathbb{Z}$.

Choose a homotopy equivalence $(\mathbb{S}^1)^{\vee(b-1)} \hookrightarrow S$. Identify $(\mathbb{S}^1)^{\vee(b-1)}$ as a subset of the group $\mathbb{T}^{b-1}$ in the natural way. Then for any $k \in \mathbb{Z}_{\geq 0}$ we have homotopy equivalences

$$Sym^k(S) \xleftarrow{\simeq} Sym^k((\mathbb{S}^1)^{\vee(b-1)}) \xrightarrow{\simeq} \mathbb{T}^{b-1}_k$$
$$[(\theta_1, \cdots, \theta_k)] \mapsto \sum_{i=1}^k \theta_i.$$

where a homotopy inverse of the second map is given by

$$\mathbb{T}^{b-1}_k \to Sym^k((\mathbb{S}^1)^{\vee(b-1)})$$
$$\sum_{i=1}^r \theta_i \to [(\theta_1, \cdots, \theta_r, 0, \cdots, 0)]$$

where $r \leq \min\{k, b-1\}$ and $\theta_1, \cdots, \theta_r \in (\mathbb{S}^1)^{\vee(b-1)}$ lie on different circles. □

Remark 5.6. Combined with Proposition 5.1, the constructions in the proof gives an easy realization of the map (12) in the genus zero case.

Also, by a carefully tracking of maps, one sees that the three chains of inclusions $\mathrm{Rep}_{\leq 0}(S, \{C_i\})_e/PSO(2)$ (or $\mathrm{Rep}_{\leq 0}(S, \{C_i\})_e/G$ in the case $e < 0$), $\mathbb{T}^{b-1}_{b-2+e}$, $Sym^{b-2+e}(S)$, indexed by e, are homotopic to each other via the maps we have constructed.

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