LINEAR FRACTIONAL TRANSFORMATIONS ON THE BERKOVICH PROJECTIVE LINE

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1. Introduction

Let K be an algebraically closed field which is complete which respect to a nontrivial non-Archimedean absolute value $|\cdot|$. In this paper we will first give the definition of the Berkovich projective line $\mathbb{P}^1_{\mathrm{Berk}} = \mathbb{P}^1_{\mathrm{Berk},K}$, then briefly discuss some of its properties. In particular we will classify the points of $\mathbb{P}^1_{\mathrm{Berk}}$ into four types according to the Berkovich's classification theorem. We will also give a metric ρ on the space $\mathbb{H}_{\mathrm{Berk}} = \mathbb{P}^1_{\mathrm{Berk}} \backslash \mathbb{P}^1(K)$. Occasionally we will refer the proof of some properties to [1], [3].

Let φ be a rational function defined on K, then φ induces a continuous map on $\mathbb{P}^1_{\mathrm{Berk}}$. We first remark that φ preserves the type of points. Then we restrict our attention to the situation when φ is a linear fractional transformation on $\mathbb{P}^1(K)$. This defines a group action $\mathrm{PSL}_2(K)$ on $\mathbb{P}^1_{\mathrm{Berk}}$. We will discuss some transitivity properties of this group action. We will compare some of our results with the similar ones for Möbius transformations on $\mathbb{P}^1(\mathbb{C})$.

As opposed to some other sources (e.g. [1], [2]), we will take a geometrical approach to prove most of the results, which depends heavily on the Berkovich's classification of points in $\mathbb{P}^1_{\mathrm{Berk}}$.

Throughout this paper we will give examples and counterexamples using the p-adic field $\mathbb{C}_p$ and Ω_p. All the properties we use can be found in [3].

2. The Berkovich Projective Line

2.1. The Berkovich affine Line $\mathbb{A}^1_{\mathrm{Berk}}$.

Definition 2.1. Let A be a ring. A *seminorm* on A is a map $|\cdot|\colon A \to \mathbb{R}_{\geq 0}$ satisfying:

- $|0| = 0$, $|1| = 1$;
- $|f+g| \leq |f| + |g|$ for any $f, g \in A$.

It is said to be *multiplicative* if

- $|fg| = |f||g|$ for any $f, g \in A$.

It is said to be *non-Archimedean* if

- $|f+g| \leq \max(|f|, |g|)$ for any $f, g \in A$.

Lemma 2.2. *If A is a ring containing K, then any multiplicative seminorm on A that extends $|\cdot|$ on K is non-Archimedean.*

Proof. For any $a, b \in A$, $n \in \mathbb{N}_+$, we have

$$|a+b|^n = |(a+b)^n| = \left|\sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}\right| \leq \sum_{i=0}^{n} \left|\binom{n}{i}\right| |a|^i |b|^{n-i} \leq (n+1)\max(|a|, |b|)^n.$$

Thus $|a+b| \leq (n+1)^{1/n} \max(|a|, |b|)$. Let $n \to \infty$ we get $|a+b| \leq \max(|a|, |b|)$. $\square$

Definition 2.3. The *Berkovich affine line* over K is a topological space $\mathbb{A}^1_{\text{Berk}} = \mathbb{A}^1_{\text{Berk},K} = \{$ multiplicative seminorm $|\cdot|$ on $K[X]\colon |\cdot|$ extends $|\cdot|$ on $K\}$, with the weakest topology such that $|\cdot| \mapsto |f|$ is continuous for any $f \in K[X]$.

For convenience, when we refer to an element in $\mathbb{A}^1_{\text{Berk}}$, we usually write x when we regard it as a point in the space $\mathbb{A}^1_{\text{Berk}}$, and $|\cdot|_x$ when we regard it as a multiplicative seminorm on $K[X]$.

Since K is algebraically closed, we can factor any $f \in K[X]$ into linear polynomials. Notice multiplication is continuous, so it suffices to require $|\cdot| \mapsto |X - a|$ to be continuous for any $a \in K$ in the above definition.

Proposition 2.4. *$\mathbb{A}^1_{\text{Berk}}$ is a locally compact Hausdorff space.*

Proof. See [1], Theorem C.3. □

2.2. **Berkovich's classification theorem.**

Although the definition of $\mathbb{A}^1_{\text{Berk}}$ looks rather abstract, in fact we can characterize all its elements quite concretely.

First we give some examples of points in $\mathbb{A}^1_{\text{Berk}}$.

Example 2.5. For any $a \in K$, define $|\cdot|_a\colon K[X] \to \mathbb{R}_{\geq 0}$, $f \mapsto |f(a)|$. It is straightforward to check that $|\cdot|_a$ is a multiplicative seminorm on $K[X]$, and that different a give rises to different $|\cdot|_a$. Therefore we can identify a with $|\cdot|_a$, thus obtain an embedding $K \hookrightarrow \mathbb{A}^1_{\text{Berk}}$ (as sets). In fact it is also an embedding of topological spaces.

Example 2.6. For any closed ball $B(a,r) \subset K$, define $|\cdot|_{B(a,r)}\colon K[X] \to \mathbb{R}_{\geq 0}$, $f \mapsto \sup_{x \in B(a,r)} |f(x)|$. It is easy to see that $|\cdot|_{B(a,r)}$ is a seminorm (in fact it is a norm). The fact that it is also multiplicative follows from an alternative definition of $|\cdot|_{B(a,r)}$, see the two lemmas below.

It is also easy to show that different $B(a,r)$ give rises to different $|\cdot|_{B(a,r)}$. In fact, suppose $B(a_1,r_1) \neq B(a_2,r_2)$. By Lemma 2.7, $\inf_{a \in K} |T - a|_{B(a_i,r_i)} = r_i$, $i = 1,2$. If $r_1 \neq r_2$, we already have $|\cdot|_{B(a_1,r_1)} \neq |\cdot|_{B(a_2,r_2)}$. If $r_1 = r_2 = r$, then $B(a_1,r_1)$, $B(a_2,r_2)$ are disjoint, so $|T - a_1|_{B(a_2,r_2)} = |a_2 - a_1| > r = |T - a_1|_{B(a_1,r_1)}$, thus we also have $|\cdot|_{B(a_1,r_1)} \neq |\cdot|_{B(a_2,r_2)}$.

Notice when $r = 0$ this is just the previous example.

Lemma 2.7. *For $f = \sum_{i=0}^{n} a_i (X-a)^i \in K[X]$ we have $|f|_{B(a,r)} = \max_{0 \leq i \leq n} |a_i| r^i$. In particular, if $r > 0$, then $|\cdot|_{B(a,r)}$ is a norm.*

Proof. See [3], Section 6.1.4, Proposition 1. □

Lemma 2.8. *The seminorm $|\cdot|_{B(a,r)}$ on $K[X]$ is multiplicative.*

Proof. See [3], Section 6.1.4, Proposition 2. □

As one may expect, in general Example 2.5, 2.6 do not cover all the points in $\mathbb{A}^1_{\text{Berk}}$. But it is quite remarkable that we are close to the truth. In fact we have the following powerful theorem.

Theorem 2.9 (Berkovich's Classification Theorem)**.** *Every point $x \in \mathbb{A}^1_{\text{Berk}}$ can be realized as*

$$|f|_x = \lim_{i \to \infty} |f|_{B(a_i,r_i)}$$

for some nested sequence of closed balls $\{B(a_i,r_i)\}$ in K. For such $\{B(a_i,r_i)\}$, we write $x = |\cdot|_{\{B(a_i,r_i)\}}$.
Moreover, we can classify all the points in $\mathbb{A}^1_{\text{Berk}}$ into four types according to the intersection $B = \cap_{i=1}^{\infty} B(a_i,r_i)$ of the corresponding nested sequence:

Type I: *If $B = \{a\}$ is a single point in K, then $|\cdot|_{\{B(a_i,r_i)\}} = |\cdot|_a$ is said to be of type I.*
Type II: *If $B = B(a,r)$ is a closed ball in K with $r \in |K^\times|$, then $|\cdot|_{\{B(a_i,r_i)\}} = |\cdot|_{B(a,r)}$ is said to be of type II.*
Type III: *If $B = B(a,r)$ is a closed ball in K with $r > 0$, $r \notin |K^\times|$, then $|\cdot|_{\{B(a_i,r_i)\}} = |\cdot|_{B(a,r)}$ is said to be of type III.*
Type IV: *If $B = \varnothing$, then $|\cdot|_{\{B(a_i,r_i)\}}$ is said to be of type IV.*

Proof. See [1], Theorem 2.2. □

Example 2.10. Ω_p is spherically complete with valued group $\mathbb{R}_{>0}$, hence $\mathbb{A}^1_{Berk,\Omega_p}$ contains only points of type I and II.

$\mathbb{C}_p$ is not spherically complete with valued group $p^{\mathbb{Q}}$, hence $\mathbb{A}^1_{Berk,\mathbb{C}_p}$ contains points of all four types.

Lemma 2.11. *Let $x = |\cdot|_{\{B(a_i,r_i)\}}$, $y = |\cdot|_{\{B(b_j,s_j)\}}$ be type IV points in $\mathbb{A}^1_{\mathrm{Berk}}$. Then $x = y$ if and only if for any j, there exists i such that $B(a_i,r_i) \subset B(b_j,s_j)$, and conversely (switch the position of x, y).*

Proof. See [1], Lemma 1.3. □

Warning 2.12. This is not true for points of other types. For example, consider the identity $|\cdot|_{\{B(a,r)\}} = |\cdot|_{\{B(a,r+1/i)\}}$.

2.3. The Berkovich projective Line $\mathbb{P}^1_{\mathrm{Berk}}$.

With $\mathbb{A}^1_{\mathrm{Berk}}$ at hand, the definition of $\mathbb{P}^1_{\mathrm{Berk}}$ is straightforward.

Definition 2.13. The *Berkovich projective line* over K is the one-point compactification of $\mathbb{A}^1_{\mathrm{Berk},K}$, denoted as $\mathbb{P}^1_{\mathrm{Berk}} = \mathbb{P}^1_{\mathrm{Berk},K} = \mathbb{A}^1_{\mathrm{Berk},K} \cup \{\infty\}$.

Figure 1 is a picture of the Berkovich projective line from [1]. We have adapted it a little here. (However this picture is a little deceptive under our setting: if we require larger elements (with respect to the partial order on $\mathbb{P}^1_{\mathrm{Berk}}$, see the definition below) to be higher in this picture, except the branch that lead to infinity, all other branches (whether above the Gauss point $\zeta_{Gauss} = |\cdot|_{B(0,1)}$ or not) should point downward.)

We extend the canonical embedding $K \hookrightarrow \mathbb{A}^1_{\mathrm{Berk}}$ to $\mathbb{P}^1(K) \hookrightarrow \mathbb{P}^1_{\mathrm{Berk}}$ by mapping $\infty \in \mathbb{P}^1(K)$ to $\infty \in \mathbb{P}^1_{\mathrm{Berk}}$. The extended map is still an embedding of topological spaces (see [1]). We keep the original classification for those ordinary points (i.e. points in $\mathbb{A}^1_{\mathrm{Berk}}$), and classify ∞ as a type I point.

Next we define a partial order on $\mathbb{P}^1_{\mathrm{Berk}}$. For all $x, y \in \mathbb{A}^1_{\mathrm{Berk}}$, we require that $x \le y$ if and only if $|f|_x \le |f|_y$ for all $f \in K[X]$. We also require that $x \le \infty$ for all $x \in \mathbb{P}^1_{\mathrm{Berk}}$ where equality holds if and only if $x = \infty$. By Definition 2.1 it is clear that $x \le y$ and $y \le x$ implies $x = y$. So $(\mathbb{P}^1_{\mathrm{Berk}}, \le)$ is a partially ordered set.

We shall give some useful geometrical descriptions of this partial order.

Lemma 2.14. *For points $x = |\cdot|_{B(a,r)}, y = |\cdot|_{B(b,s)}$ in $\mathbb{A}^1_{\mathrm{Berk}}$ not of type IV, $x \le y$ if and only if $B(a,r) \subset B(b,s)$ as balls in K.*

Proof. If $B(a,r) \subset B(b,s)$, then clearly

$$|f|_x = \sup_{\alpha \in B(a,r)} |f(\alpha)| \le \sup_{\alpha \in B(b,s)} |f(\alpha)|$$

for any $f \in K[X]$. Thus $x \le y$.
Conversely, suppose $x \le y$. Let $f = X - b$. By Lemma 2.7 we have

$$\max(|a-b|, r) = |f|_x \le |f|_y = \max(|b-b|, s) = s.$$

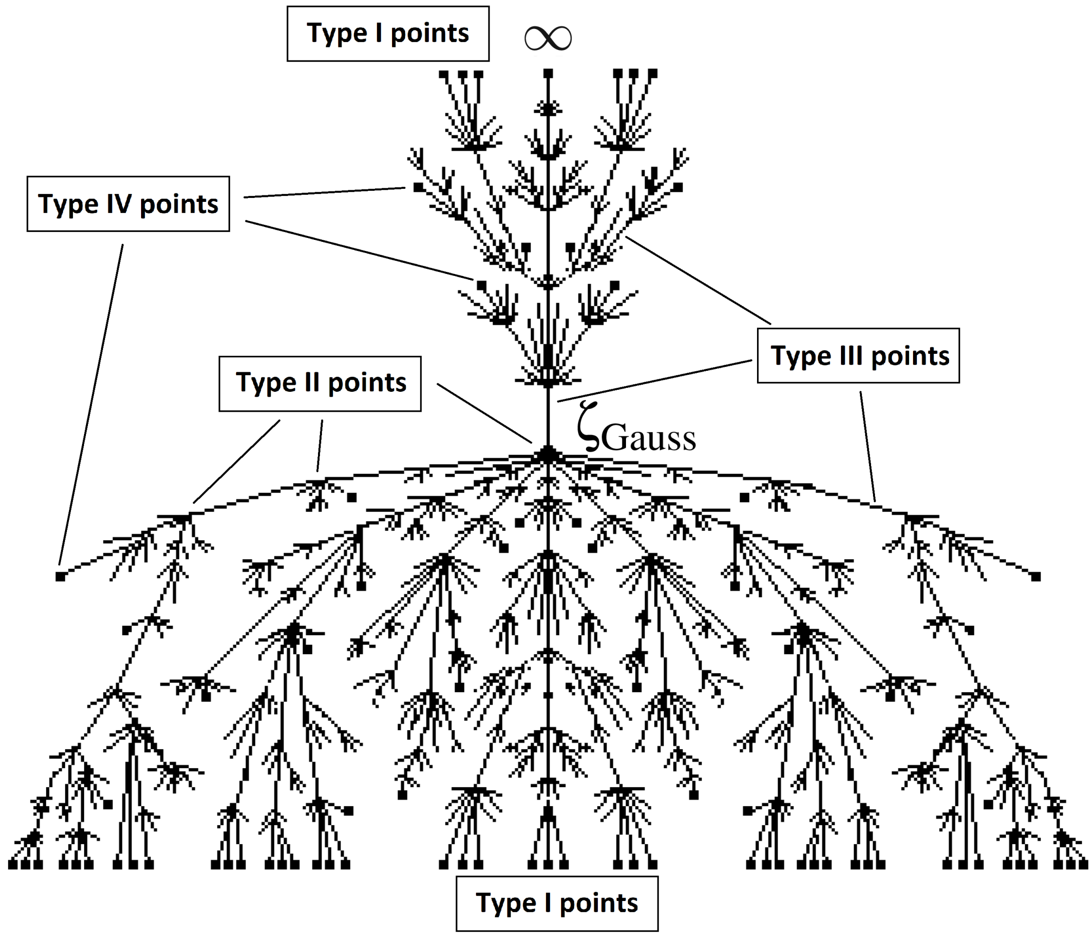

FIGURE 1. The Berkovich Projective Line

So $a \in B(b,s)$ and $r \leq s$, which implies $B(a,r) \subset B(b,s)$. $\square$

Lemma 2.15. *Type IV points are minimal in the partially ordered set* $(\mathbb{P}^1_{\text{Berk}}, \leq)$.

Proof. Write $y = |\cdot|_{\{B(a_i,r_i)\}}$, $y_i = |\cdot|_{B(a_i,r_i)}$. It is clear that $y \leq y_i$ for all i.
Suppose the statement doesn't hold. Find $x \in \mathbb{P}^1_{\text{Berk}}$ with $x < y$. Then $x \in \mathbb{A}^1_{\text{Berk}}$.
If $x = |\cdot|_{B(b,s)}$ is not of type IV, then $x \leq y \leq y_i$ for any i. By Lemma 2.14 we know $B(b,s) \subset B(a_i,r_i)$. Hence $\cap_{i=1}^{\infty} B(a_i,r_i) \supset B(b,s) \neq \varnothing$, contradiction!
Hence x is of type IV. Write $x = |\cdot|_{\{B(b_j,s_j)\}}$.
Suppose there exist i_0, j_0 such that $B(a_{i_0}, r_{i_0}) \cap B(b_{j_0}, s_{j_0}) = \varnothing$. Let $f = X - b_{j_0}$. Then we have $|f|_{B(a_i,r_i)} = |a_i - b_{j_0}| \geq |a_{i_0} - b_{j_0}|$ for all $i \geq i_0$. Hence

$$|f|_x = \lim_{i\to\infty} |f|_{B(a_i,r_i)} \geq |a_{i_0} - b_{j_0}| > s_{j_0} = |f|_{B(b_{j_0},s_{j_0})} \geq |f|_y.$$

This contradicts with $x \leq y$.
Thus for all i, j we have

$$B(a_i,r_i) \cap B(b_j,s_j) \neq \varnothing \tag{2.1}$$

Since $x \neq y$, Lemma 2.11 implies that there exists j such that $B(a_i,r_i) \not\subset B(b_j,s_j)$ for any i, or conversely.

For the former case, by (2.1) we know that $B(b_j, s_j) \subset B(a_i, r_i)$ for all i. Thus $\cap_{i=1}^{\infty} B(a_i, r_i) \neq \varnothing$. This contradicts with the fact that y is of type IV.
For the latter case, by the same argument we deduce that $\cap_{j=1}^{\infty} B(b_j, s_j) \neq \varnothing$, which contradicts with the fact that x is of type IV.
Thus the statement must hold. □

Lemma 2.16. *If $x, y \in \mathbb{A}^1_{\mathrm{Berk}}$, where $x = |\cdot|_{\{B(a_i,r_i)\}}$ is of type IV, then $x < y$ if and only if $y = |\cdot|_{B(b,s)}$ is not of type IV and that $B(b,s) \supset B(a_i, r_i)$ for some i.*

Proof. If $y = |\cdot|_{B(b,s)}$ and $B(b,s) \supset B(a_i, r_i)$ for some i, then clearly $x \leq x_i \leq y$ and $x \neq y$ holds. Here $x_i = |\cdot|_{B(a_i,r_i)}$.
Conversely, suppose $x < y$. By Lemma 2.15, y is not of type IV. Write $y = |\cdot|_{B(b,s)}$.
Since $\cap_{i=1}^{\infty} B(a_i, r_i) = \varnothing$ we know that $B(a_{i_0}, r_{i_0}) \not\supset B(b,s)$ for some i_0. We claim that $B(a_{i_0}, r_{i_0}) \subset B(b,s)$.
If not, then $B(a_{i_0}, r_{i_0}) \cap B(b,s) = \varnothing$. Consider $f = X - b$, we have

$$s = |f|_y \geq |f|_x = \lim_{i \to \infty} |f|_{B(a_i,r_i)} = |a_{i_0} - b| > s,$$

this is a contradiction. Thus the statement must hold. □

Corollary 2.17. *Type I points other than ∞ are minimal in the partially ordered set $(\mathbb{P}^1_{\mathrm{Berk}}, \leq)$.*

Proof. This follows from Lemma 2.14, 2.16. □

Proposition 2.18. *For any $x \in \mathbb{P}^1_{\mathrm{Berk}}$, the set $S_x = \{\, y \in \mathbb{P}^1_{\mathrm{Berk}} \colon x \leq y \,\}$ is totally ordered by $\leq$.*
More precisely, we have

$$S_x = \begin{cases} \{\infty\}, & x = \infty \\ \{\, |\cdot|_{B(a,s)} \colon s \geq r\} \cup \{\infty\}, & x = |\cdot|_{B(a,r)} \text{ is not of type IV} \\ \cup_{i=1}^{\infty} \{\, |\cdot|_{B(a_i,s)} \colon s \geq r_i \,\} \cup \{x, \infty\}, & x = |\cdot|_{\{B(a_i,r_i)\}} \text{ is of type IV} \end{cases}$$

Proof. By Lemma 2.14, 2.15, 2.16, everything is clear except that S_x is totally ordered when x is of type IV.
Notice that x is the least element and ∞ is the greatest element in S_x, and that $\cup_{i=1}^{n} \{\, |\cdot|_{B(a_i,s)} \colon s \geq r_i \,\} = \{\, |\cdot|_{B(a_n,s)} \colon s \geq r_n \,\}$ is totally ordered, we conclude that S_x is totally ordered. □

Corollary 2.19. *For any $x, y \in \mathbb{P}^1_{\mathrm{Berk}}$, there is a least element in $S_x \cup S_y$, which we denote as $x \vee y$. Moreover, if $x \vee y \neq x, y$, then $x \vee y$ is of type II.*

Proof. If $x \leq y$ or $y \leq x$ then the statement is clear. Now we suppose x, y are not comparable. In particular $x, y \neq \infty$.
If $x = |\cdot|_{B(a,r)}$, $y = |\cdot|_{B(b,s)}$ are not of type IV, then $x \vee y = |\cdot|_{B(a,|a-b|)}$ is of type II.
If exactly one of x, y is of type IV, suppose $x = |\cdot|_{\{B(a_i,r_i)\}}$ is of type IV, $y = |\cdot|_{B(b,s)}$. Since $x \not\leq y$, we know $B(a_i, r_i) \not\subset B(b,s)$ for some i, and that $x \vee y = x_i \vee y$ is of type II, where $x_i = |\cdot|_{B(a_i,r_i)}$.
If $x = |\cdot|_{\{B(a_i,r_i)\}}$, $y = |\cdot|_{\{B(b_j,s_j)\}}$ are both of type IV, then $B(a_i, r_i) \cap B(b_j, r_j) = \varnothing$ for some i, j, and that $x \vee y = x_i \vee y_j$ is of type II, where $x_i = |\cdot|_{B(a_i,r_i)}$, $y_j = |\cdot|_{B(b_j,s_j)}$. □

Proposition 2.20. $\mathbb{P}^1_{\mathrm{Berk}}$ *is uniquely path connected, in the sense that for any distinct $x, y \in \mathbb{P}^1_{\mathrm{Berk}}$, there exists a unique arc (the image of an injective map $\alpha \colon [0,1] \to \mathbb{P}^1_{\mathrm{Berk}}$) $[x,y]$ that*

connects x, y *(*$\alpha(0) = x$*,* $\alpha(1) = y$*).*
More precisely, for $x, y \in \mathbb{P}^1_{\mathrm{Berk}}$*,* $x \le y$ *we have*

$$[x, y] = [y, x] = \{\, z \in \mathbb{P}^1_{\mathrm{Berk}} \colon x \le z \le y \,\}.$$

And for general $x, y \in \mathbb{P}^1_{\mathrm{Berk}}$*, we have*

$$[x, y] = [x, x \vee y] \cup [x \vee y, y].$$

Proof. See [1], Lemma 2.10. □

Definition 2.21. For $x \in \mathbb{P}^1_{\mathrm{Berk}}$, a *tangent direction* at x is a path-connected component of $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{x\}$. The *tangent plane* at x is the set of all tangent directions, denoted as T_x. For $y \in \mathbb{P}^1_{\mathrm{Berk}} \backslash \{x\}$, we denote $v_x(y)$ to be the tangent direction at x that contains y.

Proposition 2.22. *Let* $x \in \mathbb{P}^1_{\mathrm{Berk}}$*, then*

$$\#T_x = \begin{cases} 1, & x \text{ is of type I, IV} \\ \#\mathbb{P}^1(k), & x \text{ is of type II} \\ 2, & x \text{ is of type III.} \end{cases}$$

Here k *is the residue field of* K*.*

Proof.

(i) $x = \infty$.
For any $y, z \neq \infty$, by Corollary 2.19 we know $y \vee z = y, z$ or a type II point, which is not ∞. Hence y, z are in the same path-connected component, $\#T_x = 1$.

Below we suppose $x \neq \infty$.
By Proposition 2.18, one of the path-connected components of $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{x\}$ is $S_x \backslash \{x\}$. We consider the rest part $M_x = \{y \in \mathbb{P}^1_{\mathrm{Berk}} \colon y < x\}$.

(ii) $x \neq \infty$ is of type I or IV.
By Lemma 2.15 and Corollary 2.17 we know $M_x = \varnothing$. Thus $\#T_x = 1$.

Below we suppose x is of type II or III. Then M_x is nonempty (for example it contains a type I point inside the ball in K that corresponds to x).
For any $y, z \in M_x$, we know that y, z are in the same path-connected component of M_x if and only if $[y, z] \subset M_x$. This is equivalent to $y \vee z < x$. Since $y \vee z \le x$ already holds, we know that y, z are in the same path-connected component if and only if $y \vee z = x$.

(iii) x is of type III.
By Corollary 2.19, we always have $y \vee z \neq x$. So M_x is connected, $\#T_x = 2$.

(iv) x is of type II.
Write $x = |\cdot|_{B(a,r)}$.
For two points $y \neq z$ in M_x, from the proof of Corollary 2.19 we can deduce that $y \vee z = y' \vee z'$ for some $y', z' \in M_x$ not of type IV.
Now the two balls $B(b, s), B(c, t)$ in K that correspond to y, z has radius strictly less than r, thus each of them has a single point image under the reduction map $\pi \colon B(a, r) \to B(a, r)/B^-(a, r) \simeq k$, where $B^-(a, r)$ is the open ball with center a and radius r. Write $\pi(B(b, s)) = \{\alpha\}$, $\pi(B(c, t)) = \{\beta\}$.
When $\alpha = \beta$, the ball that corresponds to $y \vee z$ has radius $\max(s, t, |b - c|) < r$, so $y \vee z \neq x$, thus y, z are in the same path-connected component.
When $\alpha \neq \beta$, the ball that corresponds to $y \vee z$ has radius $|b - c| = r$, so $y \vee z = x$, thus y, z are in different path-connected components.

Hence we have shown that the path-connected components of M_x are in one-one correspondence with k (clearly the reduction map is surjective). Combine with the other path-connected component $S_x\backslash\{x\}$, we know T_x is in one-one correspondence with $\mathbb{P}^1(k)$. $\square$

Although we will not make use of this fact, it is probably worth mentioning that we can describe the topology of $\mathbb{P}^1_{\text{Berk}}$ concretely using the notion of tangent planes.

Proposition 2.23. *$\cup_{x\in\mathbb{P}^1_{\text{Berk}}} T_x$ is a subbasis for the topology on $\mathbb{P}^1_{\text{Berk}}$.*

Proof. See [1], p.12. $\square$

However, it is often convenient to use an alternative definition of $\mathbb{P}^1_{\text{Berk}}$ without referring to $\mathbb{A}^1_{\text{Berk}}$. For example, this would facilitate our definition of rational maps on $\mathbb{P}^1_{\text{Berk}}$.

Let $S = \{$ multiplicative seminorm $||\cdot||$ on $K[X,Y]$: $||\cdot||$ extends $|\cdot|$ on K, and $||X||, ||Y||$ are not both zero$\}$. We define an equivalence relation on S by requiring that $||\cdot||_1 \sim ||\cdot||_2$ if and only if there exists $C > 0$, such that for any $d \geq 0$ and homogeneous $F \in K[X,Y]$ with degree d, we have $||F||_1 = C^d||F||_2$.

We say an element $||\cdot|| \in S$ is *normalized* if $\max(||X||, ||Y||) = 1$.

Lemma 2.24. *Let K_i be the set of all homogeneous polynomials in $K[X,Y]$ with degree i. Suppose $||\cdot||' \colon \cup_{i=0}^{\infty} K_i \to \mathbb{R}_{\geq 0}$ satisfies:*

- *$||\cdot||'$ agrees with $|\cdot|$ on $K_0 = K$;*
- *$||X||', ||Y||'$ are not both 0;*
- *$||f+g||' \leq ||f||' + ||g||'$ for any $f, g \in K_i$, for any i;*
- *$||fg||' = ||f||'||g||'$ for any $f, g \in \cup_{i=0}^{\infty} K_i$.*

Then there exists $||\cdot|| \in S$ that extends $||\cdot||'$.

Proof. For any $G \in K[X,Y]$, we can write $G = \sum_{i=0}^{d} G_i$ where $G_i \in K_i$. Define

$$||G|| = \max_{0\leq i\leq d} ||G_i||'.$$

Then $||\cdot||$ extends $||\cdot||'$. We check that $||\cdot||$ is a multiplicative seminorm.
Let $F = \sum_{i=0}^{d} F_i \in K[X,Y]$, $G = \sum_{i=0}^{d'} G_i \in K[X,Y]$ where $F_i, G_i \in K_i$. Add zero terms if necessary, we may assume $d = d'$. Then

$$||F+G|| = \max_{0\leq i\leq d} ||F_i + G_i||' =\leq \max_{0\leq i\leq d} (||F_i||' + ||G_i||') \leq ||F|| + ||G||.$$

So $||\cdot||$ is a seminorm. By Lemma 2.2, $||\cdot||$ is non-Archimedean.
Moreover, write $FG = \sum_{k=0}^{2d} H_k$ with $H_k \in K_k$, then for any k we have

$$||H_k|| = ||\sum_{i+j=k} F_iG_j|| \leq \max_{i+j=k} ||F_iG_j|| = \max_{i+j=k} (||F_i||||G_j||) \leq ||F||||G||.$$

Choose minimal i_0, j_0 such that $||F_{i_0}|| = ||F||$, $||G_{j_0}|| = ||G||$. Then for $k_0 = i_0 + j_0$ we have

$$||H_{k_0}|| = ||F_{i_0}G_{j_0}|| = ||F||||G||$$

since all other terms in the summation above is strictly smaller.
Hence $||FG|| = \max_{0\leq k\leq 2d} ||H_k|| = ||F||||G||$. So $||\cdot||$ is multiplicative. $\square$

Lemma 2.25. *For any $||\cdot|| \in S$, there exists a normalized $||\cdot||^* \in S$ such that $||\cdot|| \sim ||\cdot||^*$. Moreover, for any homogeneous $F \in K[X,Y]$, $||F||^*$ is independent of the choice of $||\cdot||^*$.*

Proof. First suppose $||\cdot||^*$ satisfies the requirements.
With out loss of generality we may assume $C = ||X|| \geq ||Y||$. Then $||X||^* = 1$, and $||L||^* = ||L||/C$ for any $L \in K_1$.
Now for any homogeneous $F \in K[X,Y]$ of degree d, we can write $F = F_1 \cdots F_d$ where $F_i \in K_1$ (since K is algebraically closed). Then we have

$$||F||^* = \prod_{i=1}^{d} ||F_i||^* = \prod_{i=1}^{d} \frac{||F_i||}{C} = \frac{||F||}{C^d}. \tag{2.2}$$

Hence if $||\cdot||^*$ exists, its values on homogeneous polynomials are uniquely determined.
Now the function $||\cdot||^*$ defined on $\cup_{i=0}^{\infty} K_i$ by (2.2) clearly satisfies the conditions in Lemma 2.24, so it can be extended to an element in S, which is normalized. The statement follows. ☐

Definition 2.26. The *Berkovich projective line* over K is $S/\sim$, with the weakest topology such that $z \mapsto ||F||_z^*$ is continuous for any homogeneous $F \in K[X,Y]$. Here $||\cdot||_z \in S$ is any representative of z.

Remark 2.27. Roughly speaking, Lemma 2.24 and 2.25 tell us that essentially the evaluations at homogeneous polynomials are all we need to specify an element in $\mathbb{P}^1_{\text{Berk}}$. So here we really do not care about the evaluations of non-homogeneous polynomials.

Proposition 2.28. *The Definition 2.13 and 2.26 agree.*

Proof. For the sake of this argument, denote the Berkovich projective line defined by Definition 2.13, 2.26 by A, B, respectively. Construct $\Phi\colon A \to B$ as following:
For $x \in A$, define $\Phi(x) = [||\cdot||_{\Phi(x)}]$. Where for all $F \in K[X,Y]$, $||\cdot||_{\Phi(x)} \in S$ satisfies

$$||F(X,Y)||_{\Phi(x)} = \begin{cases} |F(X,1)|_x, & x \neq \infty \\ |F(1,0)|, & x = \infty. \end{cases}$$

Then Φ is a homeomorphism that preserves $\mathbb{P}^1(K)$. For details, see [1], p.24-26. ☐

Henceforce we identify points in $\mathbb{P}^1_{\text{Berk}}$ through the homeomorphism Φ. In other words, for any $x \in \mathbb{P}^1_{\text{Berk}}$ defined though Definition 2.13, we require that

$$||F(X,Y)||_x = \begin{cases} |F(X,1)|_x, & x \neq \infty \\ |F(1,0)|, & x = \infty. \end{cases}$$

2.4. The metric ρ on the Berkovich hyperbolic space $\mathbb{H}_{\text{Berk}}$.

Definition 2.29. The *Berkovich hyperbolic space* over K is $\mathbb{H}_{\text{Berk}} = \mathbb{P}^1_{\text{Berk}} \backslash \mathbb{P}^1(K)$.

Definition 2.30. For $x \in \mathbb{A}^1_{\text{Berk}}$, write $x = |\cdot|_{\{B(a_i,r_i)\}}$. We define the *diameter* of x to be $\text{diam}(x) = \lim_{i\to\infty} r_i$. We define $\text{diam}(\infty) = \infty$.

From our previous discussion of the partial order $\leq$ on $\mathbb{P}^1_{\text{Berk}}$, it is clear that diam is monotonically increasing with respect to $\leq$.

Example 2.31. In the proof of Corollary 2.19, we pointed out that for $x = |\cdot|_{B(a,r)}$, $y = |\cdot|_{B(b,s)}$ in $\mathbb{A}^1_{\text{Berk}}$ not of type IV, we have $\text{diam}(x \vee y) = \max(r, s, |a-b|)$.
We shall make use of this result later.

Lemma 2.32. *For $x \in \mathbb{A}^1_{\text{Berk}}$, $\text{diam}(x) = 0 \iff x$ is of type I.*

Proof. Type I points obviously have diameter 0.
Conversely, suppose $x = \{B(a_i, r_i)\} \in \mathbb{A}^1_{\mathrm{Berk}}$ with $r_i \to 0$. Then $\{a_i\}$ is a Cauchy sequence in the complete space K with some limit a.
Since $|a - a_i| = \lim_{j\to\infty} |a_j - a_i| \leq r_i$, we know $a \in B(a_i, r_i)$. In particular $\cap_{i=1}^{\infty} B(a_i, r_i) \supset \{a\}$. Since $r_i \to 0$ we deduce that this intersection is exactly $\{a\}$. So x is of type I. □

Definition 2.33. The metric ρ on $\mathbb{H}_{\mathrm{Berk}}$ is defined by

$$\rho(x, y) = 2\log_v(\mathrm{diam}(x \vee y)) - \log_v(\mathrm{diam}(x)) - \log_v(\mathrm{diam}(y)),\ x, y \in \mathbb{H}_{\mathrm{Berk}},$$

where $v > 1$ is a fixed number (which we shall not specify here).
The *strong topology* on $\mathbb{H}_{\mathrm{Berk}}$ is the topology induced by ρ.

Lemma 2.34. *The metric ρ is well-defined.*

Proof. It is clear that $\rho(x, y) = \rho(y, x) \geq 0$, equality holds if and only if $\mathrm{diam}(x) = \mathrm{diam}(x\vee y) = \mathrm{diam}(y)$. By our previous discussions this can only happen when $x = y$.
It remains to check $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$. This is equivalent to

$$\log_v(\mathrm{diam}(x \vee y)) + \log_v(\mathrm{diam}(y \vee z)) \geq \log_v(\mathrm{diam}(x \vee z)) + \log_v(\mathrm{diam}(z)). \tag{2.3}$$

In fact, by Proposition 2.18 we know $x \vee y \leq y \vee z$ or $y \vee z \leq x \vee y$. Assume the first case. Then $x, z \leq y \vee z$, so $x \vee z \leq y \vee z$. Also, $y \leq x \vee y$. (2.3) now follows from monotonicity. □

Remark 2.35. For convenience, sometimes we extend ρ to $\mathbb{P}^1_{\mathrm{Berk}}$ by defining $\rho(x, x) = 0$, $\rho(x, y) = \infty$ for any $x \in \mathbb{P}^1(K)$, $y \in \mathbb{P}^1_{\mathrm{Berk}}\backslash\{x\}$.

Proposition 2.36. *The strong topology on $\mathbb{H}_{\mathrm{Berk}}$ is strictly finer than its subspace topology inherited from $\mathbb{P}^1_{\mathrm{Berk}}$.*

Proof. See [1], Lemma B.17 and p.42-44. □

3. Rational Maps on $\mathbb{P}^1_{\mathrm{Berk}}$

3.1. Definition of the induced map.

Let $\varphi \in K(X)$ be a rational function on K. We shall extend φ to a function on $\mathbb{P}^1_{\mathrm{Berk}}$. First we lift φ to a pair (F_1, F_2) of homogeneous polynomial with the same degree. Explicitly, if

$$\varphi = \frac{\sum_{i=0}^{m} a_i X^i}{\sum_{i=0}^{n} b_i X^i},\ a_m, b_n \neq 0$$

where $\sum_{i=0}^{m} a_i X^i$, $\sum_{i=0}^{n} b_i X^i$ have no common root, then we define the homogeneous lifting $F = (F_1, F_2)$ by

$$F_1 = \sum_{i=0}^{k} a_i X^i Y^{k-i},\ F_2 = \sum_{i=0}^{k} b_i X^i Y^{k-i},$$

where $k = \max(m, n)$, $a_i = 0$ for $i > m$, $b_i = 0$ for $i > n$.

Definition 3.1. The induced map on $\mathbb{P}^1_{\mathrm{Berk}}$ by φ is defined by

$$[||\cdot||_x] \mapsto [||\cdot||_{F(x)}],$$

where $||G||_{F(x)} = ||G \circ F||_x$ for any homogeneous polynomial $G \in K[X, Y]$ (by Lemma 2.24, it suffices to specify the values of $||\cdot||_{F(x)}$ at all homogeneous polynomials).
We still denote this induced map by φ, and call it a *rational map* on $\mathbb{P}^1_{\mathrm{Berk}}$.

By factoring F_1, F_2 into linear polynomials, we easily check that $||X||_{F(x)} = ||F_1(X,Y)||_x$, $||Y||_{F(x)} = ||F_2(X,Y)||_x$ cannot both be 0. All other conditions for $||\cdot||_{F(x)} \in S$ are trivially satisfied. Thus the induced map is well defined. It is also clear that the induced map agrees with the original one on $\mathbb{P}^1(K)$.

Proposition 3.2. *A nonconstant rational map φ on $\mathbb{P}^1_{\mathrm{Berk}}$ preserves the type of points.*

Proof. Since we do not need this general result here, we refer the proof to [1], Proposition 2.15. However, we will give an easy proof for the special case when φ is a linear fractional transformation in Section 4.3. □

Proposition 3.3. *Any rational map on $\mathbb{P}^1_{\mathrm{Berk}}$ is continuous.*

Proof. See [1], p.31. □

4. Linear Fractional Transformations on $\mathbb{P}^1_{\mathrm{Berk}}$

Definition 4.1. A *linear fractional transformation (LFT)* on $\mathbb{P}^1_{\mathrm{Berk}}$ is a rational map on $\mathbb{P}^1_{\mathrm{Berk}}$ induced by a linear fractional transformation $\varphi = \frac{aX+b}{cX+d} \in K(X)$, where $a, b, c, d \in K$ and $ad - bc \neq 0$.

Proposition 4.2. *Any LFT φ on $\mathbb{P}^1_{\mathrm{Berk}}$ is bicontinuous with inverse φ^{-1} (here we really mean the induced map on $\mathbb{P}^1_{\mathrm{Berk}}$ by $\varphi^{-1} \in K(X)$).*

Proof. By definition it is clear that φ^{-1} is the inverse of φ as rational maps on $\mathbb{P}^1_{\mathrm{Berk}}$. Continuity follows from Proposition 3.3. □

Thus all LFTs on $\mathbb{P}^1_{\mathrm{Berk}}$ form a group under composition that is isomorphic to $\mathrm{PSL}_2(K) = \mathrm{GL}_2(K)/\{\lambda I : \lambda \in K^\times\}$. Thus we can think LFTs as group actions of $\mathrm{PSL}_2(K)$ on $\mathbb{P}^1_{\mathrm{Berk}}$ given by

$$\mathrm{PSL}_2(K) \times \mathbb{P}^1_{\mathrm{Berk}} \to \mathbb{P}^1_{\mathrm{Berk}},\ (\varphi, x) \mapsto \varphi(x).$$

In what follows we will be interested in transitivity properties of this group action.

Corollary 4.3. *Any LFT preserves the path between two points. In other words, if $x, y \in \mathbb{P}^1_{\mathrm{Berk}}$, φ is an LFT, then $\varphi([x,y]) = [\varphi(x), \varphi(y)]$.* □

4.1. Möbius transformations on $\mathbb{P}^1(\mathbb{C})$.

We first take a detour to review some classical results.

Definition 4.4. A *Möbius transformation (MT)* is a function $\varphi\colon \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ defined by $\varphi(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

Definition 4.5. A *generalized circle* in $\mathbb{P}^1(\mathbb{C})$ is a circle in $\mathbb{C}$ or a line in $\mathbb{C}$ together with the point ∞.

For convenience we will simply call a generalized circle a circle.

Proposition 4.6. *Any MT carries circles to circles.*

Proof. See [4], Chapter 3, Theorem 14. □

We also regard MTs as group action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathbb{C}$. Below we list some classical results about transitivity properties of this group action.

Proposition 4.7. *For any tuples (x_1, x_2, x_3), (y_1, y_2, y_3) of distinct points in $\mathbb{P}^1(\mathbb{C})$, there exists a unique MT φ such that $\varphi(x_i) = y_i$, $i = 1, 2, 3$.*

Proof. Let $z_1, z_2, z_3 \in \mathbb{P}^1(\mathbb{C})$ be distinct points. Let $\alpha = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$ (which reduces to $\frac{z_2-z_3}{z-z_3}$, $\frac{z-z_1}{z-z_3}$, $\frac{z-z_1}{z_2-z_1}$ if $z_1 = \infty$, $z_2 = \infty$, $z_3 = \infty$, respectively). Then α is an MT that carries (z_1, z_2, z_3) to $(0, 1, \infty)$. Also, α^{-1} is an MT that takes $(0, 1, \infty)$ back to (z_1, z_2, z_3). The existence of φ follows from this.
Any LFT $\beta = \frac{az+b}{cz+d}$ that takes $(0, 1, \infty)$ to itself satisfies $c = 0$, $b = 0$, $\frac{a}{d} = 1$. So $\beta(z) = \frac{az}{d} = z$ is the identity map. The uniqueness of φ follows from this. $\square$

Proposition 4.8. *For any $x, y \in \mathbb{P}^1(\mathbb{C})$ and circles A, B in $\mathbb{P}^1(\mathbb{C})$ such that $x \in A$ and $y \in B$ are of the same truthness, there exists a MT φ such that $\varphi(x) = y$, $\varphi(A) = B$.*

Proof. We first find two MTs that carries x to ∞, y to ∞, respectively. Thus we only need to consider the special case when $x = y = \infty$. Under this assumption A, B are both circles (non-degenerated ones) or both lines plus the infinity point.
If A, B are circles lies in $\mathbb{C}$, let z_1, z_2 and r_1, r_2 be the centers and radii of A, B. Then $\varphi(z) = \frac{r_2}{r_1}(z - z_1) + z_2$ satisfies the requirement.
If A, B are lines plus the infinity point, by Proposition 4.7 we can let φ be an MT that carries two distinct points other than ∞ on A to two distinct points on B other than ∞ which preserves ∞. By Proposition 4.6 φ satisfies the requirement. $\square$

4.2. Geometrical descriptions.

In this section we describe the image of a ball $B(x, r)$ (as a type II or III point in $\mathbb{P}^1_{\text{Berk}}$) under LFTs. Since every LFT can be decomposed into affine maps and inversions, it suffices to give the description for these two special cases.

For simplicity, from this point on we shall drop the norm symbol $|\cdot|$ and write $\{B(x_i, r_i)\}$ for the point $|\cdot|_{\{B(x_i,r_i)\}}$ in $\mathbb{P}^1_{\text{Berk}}$ (or $B(x, r)$ for $|\cdot|_{B(x,r)}$, x for $|\cdot|_x$). It will be clear from the context whether this means a point in $\mathbb{P}^1_{\text{Berk}}$ or a nested sequence of ball (or a ball, a point) in K.

Proposition 4.9. *Let $A = aX + b$ be an affine map on $\mathbb{P}^1_{\text{Berk}}$. Then*

$$A(B(x, r)) = B(ax + b, |a|r).$$

Proof. Denote $u = B(x, r)$, $v = B(ax + b, |a|r)$.
The homogeneous lifting of A is given by $F_1 = aX + bY$, $F_2 = Y$. We have

$$\begin{aligned} ||f(X,Y)||_{A(u)} =& ||f(aX + bY, Y)||_u = |f(aX + b, 1)|_u = \sup_{z \in B(x,r)} |f(az + b, 1)| \\ =& \sup_{z \in B(ax+b,|a|r)} |f(z, 1)| = |f(X, 1)|_v = ||f(X, Y)||_v \end{aligned}$$

for any $f \in K[X, Y]$. Thus $A(u) = v$. $\square$

Proposition 4.10. *Let $I = \frac{1}{X}$ be the inversion map on $\mathbb{P}^1_{\text{Berk}}$. Then*

$$I(B(x, r)) = \begin{cases} B(\frac{1}{x}, \frac{r}{|x|^2}), & 0 \notin B(x, r) \\ B(0, \frac{1}{r}), & 0 \in B(x, r),\ r \neq 0 \\ \infty, & B(x, r) = \{0\}. \end{cases}$$

Proof. Denote $u = B(x, r)$, $v = \begin{cases} B(\frac{1}{x}, \frac{r}{|x|^2}), & 0 \notin B(x, r) \\ B(0, \frac{1}{r}), & 0 \in B(x, r),\ r \neq 0. \end{cases}$
The homogeneous lifting of I is given by $F_1 = Y$, $F_2 = X$.

(i) $0 \notin B(x,r)$.
Then $z \in B(x,r) \implies |z-x| \le r < |x|$ and $|z| = |x| \implies \frac{1}{z} \in B(\frac{1}{x}, \frac{r}{|x|^2})$.
Conversely, $\frac{1}{z} \in B(\frac{1}{x}, \frac{r}{|x|^2}) \implies z = \frac{1}{1/z} \in B(\frac{1}{1/x}, \frac{r/|x|^2}{1/|x|^2}) = B(x,r)$.
Now for any homogeneous $f \in K[X,Y]$ with degree d, we have

$$\begin{aligned}||f(X,Y)||_{I(u)} =& ||f(Y,X)||_u = |f(1,X)|_u = \sup_{z\in B(x,r)} |f(1,z)| \\ =& \sup_{z\in B(x,r)} |x|^d |f(\frac{1}{z},1)| = |x|^d \sup_{z\in B(\frac{1}{x},\frac{r}{|x|^2})} |f(z,1)| \\ =& |x|^d |f(X,1)|_v = |x|^d ||f(X,Y)||_v\end{aligned}$$

This means $[||\cdot||_{I(u)}] = [||\cdot||_v]$. Hence $I(u) = v$.

(ii) $0 \in B(x,r)$, $r \neq 0$.
Then $u = B(0,r)$.
For any homogeneous $f \in K[X,Y]$ with degree d, write

$$f(X,Y) = \sum_{i+j=d} a_{ij} X^i Y^j.$$

Apply Lemma 2.7, we have

$$\begin{aligned}||f(X,Y)||_{I(u)} =& |f(1,X)|_0 = \max |a_{ij}| r^j = r^d \max |a_{ij}| (\frac{1}{r})^i \\ =& r^d |f(X,1)|_v = r^d ||f(X,Y)||_v.\end{aligned}$$

This means $[||\cdot||_{I(u)}] = [||\cdot||_v]$. Hence $I(u) = v$.

(iii) $B(x,r) = \{0\}$.
For any $f \in K[X,Y]$ we have

$$||f(X,Y)||_{I(u)} = |f(1,X)|_u = |f(1,0)| = ||f(X,Y)||_\infty.$$

So $||\cdot||_{I(u)} = ||\cdot||_\infty$, $I(u) = \infty$. □

Theorem 4.11. *Let $\varphi = \frac{aX+b}{cX+d}$ be an LFT. Then*

$$\varphi(B(x,r)) = \begin{cases} B(\frac{ax+b}{cx+d}, \frac{|ad-bc|}{|cx+d|^2} r), & -\frac{d}{c} \notin B(x,r) \\ B(\frac{a}{c}, \frac{|ad-bc|}{|c|^2 r}), & -\frac{d}{c} \in B(x,r),\ r \neq 0 \\ \infty, & B(x,r) = \{-\frac{d}{c}\}. \end{cases}$$

Notice $c = 0$ also belongs to the first case.

Proof. If $c = 0$, then φ is an affine map, the statement follows from Proposition 4.9. Now suppose $c \neq 0$. Then $\varphi = A_2 \circ I \circ A_1$, where

$$A_1 = cX + d,\ A_2 = \frac{bc-ad}{c} X + \frac{a}{c}.$$

Apply Proposition 4.9, 4.10, we get

$$\begin{aligned}&\varphi(B(x,r)) = A_2 \circ I(B(cx+d, |c|r)) \\ =& \begin{cases} A_2(B(\frac{1}{cx+d}, \frac{|c|}{|cx+d|^2} r)) = B(\frac{ax+b}{cx+d}, \frac{|ad-bc|}{|cx+d|^2} r), & 0 \notin B(cx+d, |c|r) \\ A_2(B(0, \frac{1}{|c|r})) = B(\frac{a}{c}, \frac{|ad-bc|}{|c|^2 r}), & 0 \in B(cx+d, |c|r),\ |c|r \neq 0 \\ A_2(\infty) = \infty, & B(cx+d, |c|r) = \{0\}. \end{cases}\end{aligned}$$

Notice that the three conditions in the above equation are equivalent to the ones in the theorem. The statement follows. ☐

Corollary 4.12. *Let $x, A \in \mathbb{P}^1_{\text{Berk}}$ where x is of type I and A is of type III. Let $\varphi = \frac{aX+b}{cX+d}$ be an LFT. Then $\varphi(x) < \varphi(A)$ and $x < A$ are of the same truthness if and only if $-\frac{d}{c} \notin B_A$, here B_A is the ball in K that corresponds to A.*

Proof. Write $\varphi = A_2 \circ I \circ A_1$ as above. Affine maps preserve the partial order. Also, $A_1(-\frac{d}{c}) = 0$, $-\frac{d}{c} \in B_A \iff 0 \in B_{A_1(A)}$. Thus it suffices to prove the corollary for the special case $\varphi = I$. This follows from Proposition 4.10 and easy computations. ☐

4.3. **Invariance of types and the metric ρ under LFTs.**

Theorem 4.13. *Any LFT preserves the type of points.*

Proof. It is clear that any LFT preserves type I points.
In view of Theorem 4.11, it is also clear that any LFT preserves type II, III points, respectively.
It only remains to show that any LFT preserves type IV points.
Let $B = \{B(x_i, r_i)\}$ be a point of type IV in $\mathbb{P}^1_{\text{Berk}}$. Let $\varphi = \frac{aX+b}{cX+d}$ be an LFT.
Since $\cap_{i=1}^{\infty} B(x_i, r_i) = \varnothing$, we know that $-d/c \notin B(x_i, r_i)$ for all sufficiently large i. Since the point B in $\mathbb{P}^1_{\text{Berk}}$ only depends on $B(x_i, r_i)$ for large i, we can assume that $-d/c \notin B(x_i, r_i)$ for all i.
Now, for all $i \geq 1$, $B(x_i, r_i) \subset B(x_1, r_1)$ implies that $|x_i - x_1| \leq r_1$. On the other hand, $|x_1 + \frac{d}{c}| > r_1$. Thus $|x_i + \frac{d}{c}| = |x_1 + \frac{d}{c}|$, so $|cx_i + d| = |cx_1 + d|$ is constant. This argument make no sense to $c = 0$, but for $c = 0$ this last equality is trivially true.
Now we have $\varphi(B(x_i, r_i)) = B(\frac{ax_i+b}{cx_i+d}, \frac{|ad-bc|}{|cx_1+d|^2} r_i)$, by Theorem 4.11. The radius of these balls are decreasing with limit $\frac{|ad-bc|}{|cx_1+d|^2} r$, where $r = \text{diam}(B)$.
Note that

$$\left| \frac{ax_{i+1}+b}{cx_{i+1}+d} - \frac{ax_i+b}{cx_i+d} \right| = \frac{|ad-bc||x_{i+1}-x_i|}{|cx_1+d|^2} \leq \frac{|ad-bc|}{|cx_1+d|^2} r_i,$$

so $\{\varphi(B(x_i, r_i))\}$ is indeed a decreasing sequence of balls.
Suppose $v \in \cap_{i=1}^{\infty} \varphi(B(x_i, r_i))$, let $u = \varphi^{-1}(v) \in \mathbb{P}^1(K)$, then $v = \frac{au+b}{cu+d}$, we have

$$\frac{|ad-bc|}{|cx_1+d|^2} r_i \geq \left| v - \frac{ax_i+b}{cx_i+d} \right| = \frac{|ad-bc||u-x_i|}{|cx_1+d||cu+d|}.$$

Equivalently,

$$|u - x_i| \leq \frac{|cu+d|}{|cx_1+d|} r_i. \tag{4.1}$$

If $|cu+d| \leq |cx_1+d|$, then $u \in \cap_{i=1}^{\infty} B(x_i, r_i) = \varnothing$, contradiction.
Thus $|cu+d| > |cx_1+d| = |cx_i+d|$, so $|c||u-x_i| = |(cu+d) - (cx_i+d)| = |cu+d| > 0$. In particular $c \neq 0$.
Now (4.1) becomes $|cx_1+d| \leq |c| r_i$.
However, $|cx_1+d| = |c||x_1 + \frac{d}{c}| > |c| r_1 \geq |c| r_i$, this is a contradiction.
This means $\cap_{i=1}^{\infty} \varphi(B(x_i, r_i)) = \varnothing$, so $B' = \{\varphi(B(x_i, r_i))\}$ is a point of type IV in $\mathbb{P}^1_{\text{Berk}}$.
Now, let $F = (F_1, F_2)$ be the homogeneous lifting of φ. For any homogeneous $G \in K[X, Y]$, one can easily check

$$||G||^*_{\varphi(B)} = ||G \circ F||^*_B = \lim_{i\to\infty} ||G \circ F||^*_{B(x_i,r_i)} = \lim_{i\to\infty} ||G||^*_{\varphi(B(x_i,r_i))} = ||G||^*_{B'}.$$

Thus $\varphi(B) = B'$ is a point of type IV. The proof is complete. ☐

Note that in the proof we actually find out what the image of a type IV point is. We state this into the following corollary.

Corollary 4.14. *Let $B = \{B(x_i, r_i)\}$ be a point of type IV in $\mathbb{P}^1_{\text{Berk}}$, $\varphi = \frac{aX+b}{cX+d}$ be an LFT. Then $|cx_i + d| = \lambda \in |K^\times|$ for all $i \geq N$ for some N. Moreover,*

$$\varphi(B) = \{\varphi(B(x_i, r_i)\}_{i\geq N} = \{B(\frac{ax_i + b}{cx_i + d}, \frac{|ad - bc|}{\lambda^2} r_i)\}_{i\geq N}.$$

In particular,

$$\frac{\text{diam}(\varphi(B))}{\text{diam}(B)} = \frac{|ad - bc|}{\lambda^2} \in |K^\times|. \qquad \square$$

Theorem 4.15. *Any LFT preserves the metric ρ on $\mathbb{H}_{\text{Berk}}$.*

Proof. It suffices to prove the theorem for affine maps and the inversion map.
Let $u = B(x, r)$, $v = B(y, s)$ be two points in $\mathbb{H}_{\text{Berk}}$ not of type IV.
First we consider an affine map $A(X) = aX + b$. By Proposition 4.9, Lemma 2.14 we easily check that A preserves the order of type II or III points on $\mathbb{H}_{\text{Berk}}$. In particular it preserves $\vee$ restricted to type II or III points. So if $u \leq v$, we have

$$\rho(A(u), A(v)) = \log_v(|a|s) - \log_v(|a|r) = \log_v(s) - \log_v(r) = \rho(u, v).$$

And for general u, v not of type IV, we have

$$\begin{aligned}&\rho(A(u), A(v)) = \rho(A(u), A(u) \vee A(v)) + \rho(A(u) \vee A(v), A(v))\\ =&\rho(A(u), A(u \vee v) + \rho(A(u \vee v), A(v)) = \rho(u, u \vee v) + \rho(u \vee v, v) = \rho(u, v).\end{aligned}$$

Next we consider the inversion map $I(X) = \frac{1}{X}$.

(i) $0 \in B(x, r), B(y, s)$.
By Proposition 4.10 we know $I(u) = B(0, \frac{1}{r})$, $I(v) = B(0, \frac{1}{s})$. Thus u, v and $I(u), I(v)$ are both comparable under the partial order $\leq$. Hence

$$\rho(I(u), I(v)) = |\log_v(\frac{1}{r}) - \log_v(\frac{1}{s})| = |\log_v r - \log_v s| = \rho(u, v).$$

(ii) $0 \in B(x, r)$, $0 \notin B(y, s)$ (or $0 \notin B(x, r)$, $0 \in B(y, s)$, assume the former).
By Proposition 4.10, $I(u) = B(0, \frac{1}{r})$, $I(v) = B(\frac{1}{y}, \frac{s}{|y|^2})$.
If $B(x, r) \cap B(y, s) = \varnothing$, then $|y| > r \geq |x|$, $|y| > s$, so $|x - y| = |y| > r, s$. Thus $\rho(u, v) = 2 \log_v |y| - \log_v r - \log_v s$ by Exercise 2.31.
It is easily checked that $I(v) \leq I(u)$, thus

$$\rho(I(u), I(v)) = \log_v \frac{1}{r} - \log_v \frac{s}{|y|^2} = 2 \log_v |y| - \log_v r - \log_v s = \rho(u, v).$$

If $B(x, r) \cap B(y, s) \neq \varnothing$, then $B(y, s) \subset B(x, r)$, $\rho(u, v) = \log_v r - \log_v s$.
Also, $|\frac{1}{r}| \geq \frac{1}{r}, \frac{s}{|y|^2}$ clearly hold, thus

$$\rho(I(u), I(v)) = 2 \log_v |\frac{1}{y}| - \log_v \frac{1}{r} - \log_v \frac{s}{|y|^2} = \log_v r - \log_v s = \rho(u, v).$$

(iii) $0 \notin B(x, r), B(y, s)$.
By Proposition 4.10 we know $I(u) = B(\frac{1}{x}, \frac{r}{|x|^2})$, $I(v) = B(\frac{1}{y}, \frac{s}{|y|^2})$.
If $B(x, r) \subset B(y, s)$ (or $B(y, s) \subset B(x, r)$, assume the former), then we know $|x - y| \leq s < |y|$, so $|x| = |y|$.
Since

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x - y|}{|xy|} \leq \frac{s}{|y|^2}, \ \frac{r}{|x|^2} \leq \frac{s}{|y|^2},$$

we know that $I(u) \le I(v)$, so that

$$\rho(I(u), I(v)) = \log_v \frac{s}{|y|^2} - \log_v \frac{r}{|x|^2} = \log_v s - \log_v r = \rho(u, v).$$

If $B(x,r) \cap B(y,s) = \varnothing$, then $|x-y| > r, s$.
We now check that $|\frac{1}{x} - \frac{1}{y}| > \frac{r}{|x|^2}, \frac{s}{|y|^2}$.
By symmetry we only check the former.
If $|x| \ge |y|$, we have

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x-y|}{|xy|} > \frac{r}{|x|^2}$$

If $|x| < |y|$, we know $|x-y| = |y|$, thus we have

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{1}{|x|} > \frac{r}{|x|^2}.$$

Thus by Exercise 2.31 we see that

$$\begin{aligned}\rho(I(u), I(v)) =& 2\log_v |\frac{1}{x} - \frac{1}{y}| - \log_v \frac{r}{|x|^2} - \log_v \frac{s}{|y|^2} \\ =& 2\log_v |x-y| - \log_v r - \log_v s = \rho(u, v).\end{aligned}$$

This proves that any LFT preserves the distance between points in $\mathbb{H}_{\text{Berk}}$ not of type IV.
For any type IV point $x = \{B(a_i, r_i)\}$, we have $\rho(x, x_i) = \log_v r_i - \log_v \operatorname{diam}(x) \to 0$ as $i \to \infty$, where $x_i = B(a_i, r_i)$. Also, we can check $\rho(\varphi(x), \varphi(x_i)) \to 0$ as $i \to \infty$ by applying Corollary 4.14. Hence type II or III points are dense in $\mathbb{H}_{\text{Berk}}$ under the strong topology. Now the statement for general points in $\mathbb{H}_{\text{Berk}}$ follows from taking limit. □

Remark 4.16. Combining this result with Theorem 4.13, we know that any LFT preserves ρ in the sense of Remark 2.35.

Corollary 4.17. *Any LFT restricts on $\mathbb{H}_{\text{Berk}}$ to a continuous map with respect to the strong topology.* □

4.4. **Type I, II points under LFTs.**

Theorem 4.18. *For any tuples (x_1, x_2, x_3), (y_1, y_2, y_3) of distinct points in $\mathbb{P}^1_{\text{Berk}}$ of type I, there exists a unique LFT φ such that $\varphi(x_i) = y_i$, $i = 1, 2, 3$.*

Proof. The proof is exactly the same as the classical one. See Proposition 4.7. □

Theorem 4.19. *For any tuples (x_1, x_2, A), (y_1, y_2, B) of distinct points in $\mathbb{P}^1_{\text{Berk}}$, where x_i, y_i are of type I, $i = 1, 2$, and A, B are of type II, such that $A \in [x_1, x_2]$, $B \in [y_1, y_2]$, there exists an LFT φ such that $\varphi(x_i) = y_i$, $i = 1, 2$, and $\varphi(A) = B$.*

Proof. Let α, β be LFTs that take (x_1, x_2), (y_1, y_2) to $(0, \infty)$, respectively. Let $A' = \alpha(A)$, $B' = \beta(B)$. By Corollary 4.3, $A', B' \in [0, \infty]$. By Theorem 4.13, A', B' are of type II. Thus we can write $A' = B(0, r)$, $B' = B(0, s)$ for some $r, s \in |K|^\times$.
Find $u \in K^\times$ with $|u| = s/r$, then $\gamma = uX$ is an LFT that maps $(0, \infty, A')$ to $(0, \infty, B')$, so $\varphi = \beta^{-1} \circ \gamma \circ \alpha$ is an LFT that satisfies our requirement. □

Corollary 4.20. *For any pairs (x, A), (y, B) of points in $\mathbb{P}^1_{\text{Berk}}$, where x, y are of type I, A, B are of type II, there exists an LFT φ such that $\varphi(x) = y$, $\varphi(A) = B$.*

Proof. If $x < A$, let $x' = \infty$. If $x \not< A$, let $x' \in K$ be any point with $x' < A$. Choose y' similarly. Then apply Theorem 4.19 to (x, x', A), (y, y', B). □

Example 4.21. By the uniqueness part of Theorem 4.18, it is clear that in general we cannot require an LFT to carry four distinct points of type I to another four.
By Theorem 4.15, it is also clear that in general we cannot require an LFT to carry two distinct points of type II to another two.
The same are true under classical settings, where type II points in $\mathbb{P}^1_{\mathrm{Berk}}$ correspond to generalized circles in $\mathbb{P}^1(\mathbb{C})$.

Example 4.22. If we drop the condition that $A \in [x_1, x_2]$, $B \in [y_1, y_2]$, then Theorem 4.19 does not hold in general.
By Proposition 2.22, we can take a tangent direction at the Gauss point $\zeta_{Gauss} = B(0,1)$ that does not contain 0 or ∞. Take any A, B of type II contained in this tangent direction with different diameter, then $A, B < \zeta_{Gauss}$. We claim that there exists no LFT that carries $(0, \infty, A)$ to $(0, \infty, B)$.
If not, let φ be such an LFT.
Notice that φ is bijective by Proposition 4.2. Apply Corollary 4.3, we have

$$\{\varphi(\zeta_{Gauss})\} = \varphi([0,\infty] \cap [0,A] \cap [\infty, A]) = [0,\infty] \cap [0,B] \cap [\infty,B] = \{\zeta_{Gauss}\}.$$

Thus $\varphi(\zeta_{Gauss}) = \zeta_{Gauss}$. So we have

$$\rho(\zeta_{Gauss}, A) = -\log_v \operatorname{diam}(A) \neq -\log_v \operatorname{diam}(B) = \rho(\zeta_{Gauss}, B) = \rho(\varphi(\zeta_{Gauss}), \varphi(A)).$$

This contradicts with Theorem 4.15.

4.5. **Type III, IV points under LFTs.**

Although points of type I, II have good transitivity properties under the action of $\mathrm{PSL}_2(K)$, the same need not be true for points of type III, IV.

Example 4.23. In view of Theorem 4.11, in $\mathbb{P}^1_{\mathrm{Berk},\mathbb{C}_p}$, any LFT carries a ball of radius r to a ball of radius cr or c/r for some $c \in |\mathbb{C}_p^\times| = p^{\mathbb{Q}}$.
In particular, there exists no LFT that carries the type III point $B(0, \sqrt{2})$ to the type III point $B(0, \sqrt{3})$ in $\mathbb{P}^1_{\mathrm{Berk},\mathbb{C}_p}$.

Example 4.24. For any $r > 0$, we can find a type IV point in $\mathbb{P}^1_{\mathrm{Berk},\mathbb{C}_p}$ with diameter r (see [3], Section 3.4). In particular we can find type IV points A, B with $\operatorname{diam}(A) = \sqrt{2}$, $\operatorname{diam}(B) = \sqrt{3}$. By Corollary 4.14, there exists no LFT that carries A to B.

For type III points in $\mathbb{P}^1_{\mathrm{Berk}}$, the essential obstruction to transitivity is only the deficiency of valued group. In fact by similar argument as Theorem 4.19 we obtain the following:

Theorem 4.25. *Let (x_1, x_2, A), (y_1, y_2, B) be two tuples of distinct points in $\mathbb{P}^1_{\mathrm{Berk}}$, where x_i, y_i are of type I, $i = 1, 2$, and A, B are of type III, such that $A \in [x_1, x_2]$, $B \in [y_1, y_2]$. Suppose*
(1) $\operatorname{diam}(B)/\operatorname{diam}(A) \in |K^\times|$, *and that $x_1 < A$, $y_1 < B$ or $x_2 < A$, $y_2 < B$; or*
(2) $\operatorname{diam}(A) \cdot \operatorname{diam}(B) \in |K^\times|$, *and that $x_1 < A$, $y_2 < B$ or $x_2 < A$, $y_1 < B$.*
Then there exists an LFT φ such that $\varphi(x_i) = y_i$, $i = 1, 2$, and $\varphi(A) = B$.

Remark 4.26. By Proposition 2.22 we already know that exactly one of x_1, x_2 (resp. y_1, y_2) is smaller than A (resp. B).

Proof of Theorem 4.25.

(1) Without loss of generality we assume $x_1 < A$, $y_1 < B$.
Compose with $\alpha = \frac{1}{X - x_2}$, if necessary, we may assume $x_2 = \infty$ (from Theorem 4.11 and Corollary 4.12 we know that composition with α won't affect our other conditions).

Compose with $\beta = X - x_1$ if necessary, we may assume $x_1 = 0$. Similarly we may assume $y_2 = \infty$, $y_1 = 0$.
Now $A = B(0, r)$, $B = B(0, s)$ where $s/r = \text{diam}(B)/\text{diam}(A) \in |K^\times|$. Find $u \in K^\times$ with $|u| = s/r$, then $\varphi = uX$ satisfies our requirement.

(2) Without loss of generality we assume $x_2 < A$, $y_1 < B$.
The LFT $\gamma = \frac{1}{X - x_2}$ takes (x_1, x_2, A) to (x_1', x_2', A') where $A' \in [x_1', x_2']$, $x_1' < A'$, and $\text{diam}(A') \cdot \text{diam}(A) \in K^\times$. Now the statement follows from (1). ☐

Theorem 4.27. *Let (x, A), (y, B) be two pairs of points in $\mathbb{P}^1_{\text{Berk}}$ where x, y are of type I and A, B are of type III. Then there exists an LFT that carries (x, A) to (y, B) if and only if*

(1) $x < A$, $y < B$ or $x \not< A$, $y \not< B$, and that $\text{diam}(A)/\text{diam}(B) \in |K^\times|$*; or*
(2) $x < A$, $y \not< B$ or $x \not< A$, $y < B$, and that $\text{diam}(A) \cdot \text{diam}(B) \in |K^\times|$.

Proof. Sufficiency follows from Theorem 4.25. Now we check (1) or (2) is neccessary.
Let $\varphi = \frac{aX+b}{cX+d}$ be an LFT that carries (x, A) to (y, B). Let B_A be the ball in K that corresponds to A.
Apply Theorem 4.11 and Corollary 4.12, we know that:
If $-\frac{d}{c} \notin B_A$, then $\text{diam}(A)/\text{diam}(B) \in |K^\times|$, and $y < B \iff x < A$. This is (1).
If $-\frac{d}{c} \in B_A$, then $\text{diam}(A) \cdot \text{diam}(B) \in |K^\times|$, and $y < B \iff x \not< A$. This is (2). ☐

Corollary 4.28. *Let $A, B \in \mathbb{P}^1_{\text{Berk}}$ be of type III, then there exists an LFT φ that carries A to B if and only if* $\text{diam}(A)/\text{diam}(B) \in |K^\times|$ *or* $\text{diam}(A) \cdot \text{diam}(B) \in |K^\times|$. ☐

However, it is quite interesting that for type IV points, the deficiency of valued group is not the only obstruction to transitivity.

Theorem 4.29 (Stability under small perturbations)**.** *Equip K^4 with the topology induced by the sup norm. For any type IV point $A \in \mathbb{P}^1_{\text{Berk}}$ and LFT $\varphi = \frac{aX+b}{cX+d}$, there exists a neighborhood U of (a, b, c, d) in K^4 such that for every $(a', b', c', d') \in U$, $\alpha = \frac{a'X+b'}{c'X+d'}$ is an LFT, and that $\alpha(A) = \varphi(A)$.*

Proof. Write $A = \{B(x_i, r_i)\}$. By Corollary 4.14, delete some initial terms if necessary, we may assume that $|cx_i + d| = \lambda$ for all i, $|x_i| = \lambda_1$ for all i (take $\varphi = \frac{1}{X}$ in the corollary), and that

$$B = \varphi(A) = \{B(y_i, s_i)\} = \{B(\frac{ax_i + b}{cx_i + d}, \frac{|ad - bc|}{\lambda_1^2} r_i)\}$$

is a point of type IV.
Apply Corollary 4.14 again, we may also assume $|y_i| = \lambda_2$ for all i.
Denote $s = \text{diam}(B)$, $S = |ad - bc|$, $M = ||(a, b, c, d)|| = \max(|a|, |b|, |c|, |d|)$.
Then we have $\lambda, \lambda_1, \lambda_2, s, S, M > 0$. Now let

$$\delta_0 = \min\left(\frac{\lambda s}{(\lambda_1 + 1)(\lambda_2 + 1)}, \frac{\lambda}{\lambda_1 + 1}, \frac{S}{M}, \sqrt{S}\right).$$

Then for any $t' = t + \delta_t \in K$ with $|\delta_t| < \delta_0$, $t = a, b, c, d$, we have

- $|(a'd' - b'c') - (ad - bc)| = |\delta_a\delta_d + a\delta_d + d\delta_a - \delta_b\delta_c - b\delta_c - c\delta_b| < \max(\delta_0^2, M\delta_0) \leq S = |ad - bc|$.
  Thus $|a'd' - b'c'| = |ad - bc| > 0$.
- $|(c'x_i + d') - (cx_i + d)| = |\delta_c x_i + \delta_d| < \max(\lambda_1\delta_0, \delta_0) < \lambda = |cx_i + d|$.
  Thus $|c'x_i + d'| = |cx_i + d| = \lambda$.

- $|(a'x_i + b' - c'x_iy_i - d'y_i) - (ax_i + b - cx_iy_i - dy_i)| = |\delta_a x_i + \delta_b - \delta_c x_i y_i - \delta_d y_i| < \max(\lambda_1\delta_0, \delta_0, \lambda_1\lambda_2\delta_0, \lambda_2\delta_0) < \lambda s$.
  Thus $|a'x_i + b' - c'x_iy_i - d'y_i| \le \max(\lambda s, |ax_i + b - cx_iy_i - dy_i|) \le \lambda s_i$.

Hence $\alpha = \frac{a'X+b'}{c'X+d'}$ is an LFT, and that

$$\left|\frac{a'x_i + b'}{c'x_i + d'} - y_i\right| = \frac{|a'x_i + b' - c'x_iy_i - d'y_i|}{|c'x_i + d'|} \le s_i, \ \frac{|a'd' - b'c'|}{|c'x_i + d'|} = \frac{|ad - bc|}{|cx_i + d|} = s_i.$$

This means $\alpha(B(x_i, r_i)) = \varphi(B(x_i, r_i))$, hence $\alpha(A) = \varphi(A)$.
Now take U to be the open ball in K^4 with center (a, b, c, d) and radius δ_0 completes the proof. □

Remark 4.30. The same is true if we replace A by any point of type II or III, as one can check by applying Theorem 4.11. Obviously this is not true for points of type I.

Example 4.31. We claim that for any $r > 0$, there exist $A, B \in \mathbb{P}^1_{\mathrm{Berk},\mathbb{C}_p}$ of type IV with $\mathrm{diam}(A) = \mathrm{diam}(B) = r$ such that no LFT carries A to B.
Fix an $A \in \mathbb{P}^1_{\mathrm{Berk},\mathbb{C}_p}$ of type IV with radius r. Let

$$\mathcal{F} = \{\, \varphi(A) \colon \varphi \text{ is an LFT} \,\}.$$

For any $B \in \mathcal{F}$, Theorem 4.29 tells us that there exists an open set $U_B \in \mathbb{C}_p^4$ such that any LFT $\varphi = \frac{aX+b}{cX+d}$ with $(a, b, c, d) \in U_B$ satisfies $\varphi(A) = B$.
Since $\mathbb{C}_p$ is separable (see [3], Section 3.1), we know $\mathbb{C}_p^4$ is separable. Now $\{\, U_B \colon B \in \mathcal{F} \,\}$ is a collection of disjoint open subset in $\mathbb{C}_p^4$, so $\mathcal{F}$ is countable. But the set of type IV points in $\mathbb{C}_p$ with radius r is uncountable (see [3], Section 3.4), our claim follows.

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