Equivariant de Rham Theory

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Preface

This is my notes summarizing some basics of equivariant cohomology theory.

The singular cohomology theory assigns algebraic invariants to topological spaces. Since its emergence in the twentieth century, it has become an indispensable tool in studying geometry of topological spaces. In the differentiable category, Georges de Rham (1903-1990) uses differential forms on a manifold to define its de Rham cohomology, which is isomorphic to its singular cohomology (by de Rham theorem). In comparison, singular cohomology is defined for any topological spaces and any coefficient ring, while de Rham cohomology is only defined for smooth manifolds and real coefficient but is usually easier to compute and is hence preferred by differential geometers.

Suppose now we are working with G-spaces, i.e. topological spaces equipped with G-action. One has an analogue of singular cohomology that captures the additional G-structure, called G-equivariant cohomology (also known as Borel cohomology). When restricting to smooth manifolds, with the aid of differential forms one can define the G-equivariant de Rham cohomology, which agrees with the usual G-equivariant cohomology (by equivariant de Rham theorem).

In these notes, I will summarize the basics of equivariant cohomology theory, with emphasis in the differentiable point of view. Section 1 is a review of some preliminaries. Section 2 gives the definition of equivariant cohomology for general G-spaces and proves some of its basic properties. In Section 3, under some motivations developed in Section 1, we define the equivariant de Rham cohomology for G-manifolds. Section 4 is a continuation of Section 3, which introduces two most important models for computing equivariant de Rham cohomology of G-manifolds. In the end of Section 4, a proof of the equivariant de Rham theorem will be provided. Section 5 and Section 6 are some applications. The reader is assumed to be familiar with basic differential geometry and algebraic topology.

These notes emerge from the notes I made for a reading course in equivariant de Rham theory and Chern-Weil theory I took in Spring 2020 at MIT. In these notes, many proofs are omitted. Hard ones are provided with suitable references and easy ones are left as exercises for readers. I would like to thank Professor Victor Guillemin and Professor Zuoqin Wang for their valuable guidance.

1 Preliminary

Convention 1.1. Throughout these notes, G denotes a Lie group with Lie algebra $\mathfrak{g}$. Although not necessary everywhere, for simplicity, *all Lie group appears in these notes are assumed to be compact.* Sometimes but not always, we will assume a Lie group to be connected. Usually, compactness is used to average over the group, while connectedness is used to integrate local identities (given by Lie derivatives) into global ones (given by pulling-back by group elements). The coefficient ring is taken to be $\mathbb{R}$ throughout.

1.1 The Super- Language

The prefix "super-" can be added to many mathematical concept (it originated in physics, however). Briefly speaking, the word "super" means that the object is equipped with a $\mathbb{Z}/2\mathbb{Z}$-grading. Here is a list of some basic syntax of the super- language. We refer to [16, Section 3] and [7, Section 2.2] for further reading.

- A **super vector space** is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space. Alternatively, it is a direct sum of vector spaces $V = V^0 \oplus V^1$. The elements in V^0 are said to be **even** while the ones in V^1 are said to be **odd**.
- An (associative) **superalgebra** is a super vector space $A = A^0 \oplus A^1$ equipped with an associative multiplication $A \times A \to A$ such that $|ab| = |a||b|$ for any homogeneous $a, b \in A$. Here $|\cdot|(= 0, 1)$ denotes the degree of a homogeneous element.
- The **supercommutator** of a superalgebra A is the bilinear bracket $[\cdot, \cdot]$ with $[a, b] = ab - (-1)^{|a||b|}ba$ for homogeneous elements $a, b \in A$. A superalgebra is said to be **supercommutative** if its supercommutator is identically zero. A superalgebra is said to be **unital** if there is a multiplicative unit 1 (necessarily with degree 0).
- Morphisms between super vector spaces, superalgebras are defined to be degree-preserving maps that preserve the linear, algebra (possibly with unit) structure, respectively.

Convention 1.2. From now on, whenever we speak of a superalgebra without further qualification, it is understood that we are referring to a unital supercommutative associative superalgebra.

- Let A be a superalgebra. A **superalgebra over** A (or a **A-superalgebra**) is a superalgebra B equipped with a superalgebra morphism $A \to B$. Morphisms between A-superalgebras are superalgebra morphisms that respect the maps from A.

Before we continue, we shall add a remark to the use of sign. A slang for the sign in the super-language is that:

> It costs a sign to move one (odd) symbol past another (odd symbol).

Keep this in mind, we proceed with a couple more basic constructions in the super- world.

- For V a super vector space, let $End(V)$ denote the space of linear maps (instead of morphisms!) from V to itself. Then $End(V)$ is a (not necessarily supercommutative) superalgebra by defining $End(V)^i := \{f \in End(V) \colon fV^j \subset V^{i+j},\ j = 0, 1\}$.

- If A is a superalgebra, let $Der(A)$ denote the sub super vector space of $End(A)$ consists of homogeneous maps D satisfying the super Leibniz rule:

$$D(ab) = D(a)b + (-1)^{|a||D|}aD(b), \text{ for } a, b \text{ homogeneous} \tag{1}$$

  and all linear combinations of such D. Maps in $Der(A)$ are called **superderivations** of A.

- If A, B are two super vector spaces, then $A \otimes B$ is a super vector space in the obvious way. If A, B are superalgebras, then $A \otimes B$ is a superalgebra with unit $1 \otimes 1$ and $(a \otimes b)(a' \otimes b') = (-1)^{|a'||b|}aa' \otimes bb'$ for homogeneous a, b, a', b'.

- A super vector space L is called a **Lie superalgebra** if it is equipped a bilinear bracket $[\cdot, \cdot]$ which is super anti-commutative and satisfies the super Jacobi identity. More explicitly, for any homogeneous elements $a, b, c \in L$ we have

$$[a, b] = -(-1)^{|a||b|}[b, a]$$

  and

$$[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]].$$

  Note that a Lie superalgebra is not necessarily associative or unital or supercommutative.

Any $\mathbb{Z}$-grading descends to a $\mathbb{Z}/2\mathbb{Z}$-grading in the obvious way. All definitions above make sense with "$\mathbb{Z}/2\mathbb{Z}$" replaced by "$\mathbb{Z}$" (but see the remark below about $End(\cdot)$). Since we will work entirely with $\mathbb{Z}$-grading objects, we make the following convention.

Convention 1.3. From now on, whenever we speak of a super concept, it is understood that the object is equipped with a $\mathbb{Z}$-grading instead of simply a $\mathbb{Z}/2\mathbb{Z}$-grading. In particular, $|\cdot|$ values in $\mathbb{Z}$ instead of $\mathbb{Z}/2\mathbb{Z}$.

Remark 1.4. Let V be a super ($\mathbb{Z}$-graded!) vector space. It is not necessarily true that $End(V)$ is a direct sum of $End(V)^i$, $i \in \mathbb{Z}$, thus $End(V)$ is not super in this way. We shall instead define the (not necessarily supercommutative) superalgebra $End_0(V) = \oplus End(V)^i$ and define $Der(V)$ as a sub super vector space of $End_0(V)$ in the same way as above.

Example 1.5. If M is a manifold, then the de Rham complex $\Omega(M)$ is a superalgebra. Moreover, it is equipped with a differential d with degree 1, which acts on $\Omega(M)$ as a superderivation. Such superalgebra is also called a **differential graded algebra**.

We end this section with a few properties which are left as exercises for readers.

Exercise 1.6. If A, B are two super vector spaces, then

$$A \otimes B \to B \otimes A, \ a \otimes b \mapsto (-1)^{|a||b|}b \otimes a \text{ for homogeneous } a, b$$

defines a super vector space isomorphism. It is a superalgebra isomorhpism if A, B are superalgebras. In this sense we say the tensor product is supercommutative.

Exercise 1.7. Let A, B be two super vector spaces, then the linear map $i \colon End_0(A) \otimes End_0(B) \to End_0(A \otimes B)$ defined by $(i(f \otimes g))(a \otimes b) = (-1)^{|g||a|}f(a) \otimes g(b)$ for homogeneous $f \in End_0(A), g \in End_0(B), a \in A, b \in B$ is an injective superalgebra morphism.
From now on, we will omit the map i and write $f \otimes g$ as an element in $End_0(A \otimes B)$ for any $f \in End_0(A), g \in End_0(B)$.

Exercise 1.8. Let A be a superalgebra. Show that $Der(A)$ is a Lie superalgebra under its supercommutator.

Exercise 1.9. Let A, B be superalgebras.

(1) Suppose D acts on A, B as superderivations, respectively, then the diagonal action $D = D \otimes 1 + 1 \otimes D$ is a superderivation on $A \otimes B$.

(2) Suppose $L \to Der(A)$, $L \to Der(B)$ are two Lie superalgebra morphisms, then taking diagonal actions defines a Lie superalgebra morphism $L \to Der(A \otimes B)$.

1.2 Differential Geometry of G-Manifolds

In this section we review some basic constructions on a G-manifold, i.e. a manifold equipped with a G-action. In particular we list some definitions and basic properties for (smooth) principal G-bundles. The reader should feel familiar with these materials, otherwise we recommend [5] and [15, Part II] for further reading.

Let M be a manifold equipped with a right G-action. For any $X \in \mathfrak{g}$, let $\underline{X}$ denote the associated vector field of A on M:

$$\underline{X}_p = \frac{d}{dt}\bigg|_{t=0} (p \cdot exp(tX)).$$

Any left action can be converted into a right action in the usual way. If M is equipped with a left G-action, then $\underline{X}$ denotes the associated vector field of X on M with respect to the corresponding right G-action. More explicitly,

$$\underline{X}_p = -\frac{d}{dt}\bigg|_{t=0} (exp(tX) \cdot p).$$

Then, $X \mapsto \underline{X}$ defines a Lie algebra morphism $\mathfrak{g} \to \mathfrak{X}(M)$. As a shorthand, for a G-manifold M, we will use $\iota_X = \iota_{\underline{X}}$, $\mathcal{L}_X = \mathcal{L}_{\underline{X}}$ to denote the contraction, Lie differentiation by $\underline{X}$ on M, respectively. Let R denotes the right multiplication G-action on M (which is the inverse left multiplication if the G acts on the left). Then ι_X, $\mathcal{L}_X$ and the exterior derivative d acts by superderivations on the exterior superalgebra $\Omega(M)$ with degree $-1, 0, 1$, respectively, and G acts on $\Omega(M)$ by superalgebra automorphism via ρ defined by $\rho_g = R_g^*$. We have the usual differential geometry formulas, formulated in terms of supercommutators:

$$[\iota_X, \iota_Y] = 0,$$

$$[\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]},$$
$$[d, \iota_X] = \mathcal{L}_X,$$
$$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]},$$
$$[d, \mathcal{L}_X] = 0,$$
$$[d, d] = 0$$

as well as

$$\mathcal{L}_X = \frac{d}{dt}\bigg|_{t=0} \rho_{exp(tX)}, \tag{2}$$

$$\rho_g \mathcal{L}_X \rho_{g^{-1}} = \mathcal{L}_{Ad_g X}, \tag{3}$$

$$\rho_g \iota_X \rho_{g^{-1}} = \iota_{Ad_g X}, \tag{4}$$

$$\rho_g d \rho_{g^{-1}} = d, \tag{5}$$

for any $X, Y \in \mathfrak{g}$, $g \in G$.

Exercise 1.10. Let $\tilde{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where $\mathfrak{g}_{-1} = \{\iota_X : X \in \mathfrak{g}\}, \mathfrak{g}_0 = \{\mathcal{L}_X : X \in \mathfrak{g}\}$ and $\mathfrak{g}_1 = \mathbb{R}d$. Then the first six identities above define a Lie superalgebra structure on $\tilde{\mathfrak{g}}$. Also, $\mathfrak{g}_0 \cong \mathfrak{g}$ as Lie algebras.

Therefore, the information carried by superderivations $\iota_\bullet, \mathcal{L}_\bullet, d$ can be succinctly summarized as saying

$$\tilde{\mathfrak{g}} \to Der(\Omega(M)) \tag{6}$$

is a morphism of Lie superalgebras (cf. Exercise 1.8).

Next, we restrict our attention to the situation where the G-action is free.

Definition 1.11. *A* ***(smooth) principal*** G***-bundle*** *is a smooth fiber bundle $\pi\colon P \to B$ such that G acts on P freely and smoothly on the right and π is the projection onto the orbit space.*

Definition 1.12. *Suppose $\pi\colon P \to B$ is a principal G-bundle. A differential form α on P is said to be*

- ***horizontal*** *if $\iota_X \alpha = 0$ for any $X \in \mathfrak{g}$;*
- ***invariant*** *if $\rho_g \alpha = 0$ for any $g \in G$;*
- ***basic*** *if it is both horizontal and invariant.*

The space of horizontal, invariant, basic forms on P are denoted $\Omega_{hor}(P), \Omega(P)^G, \Omega_{bas}(P)$, respectively.

If α is invariant, then $\mathcal{L}_X \alpha = 0$ for all $X \in \mathfrak{g}$. Assuming G is connected, $\mathcal{L}_X \alpha = 0$ for all $X \in \mathfrak{g}$ implies α is invariant.

Proposition 1.13. *Suppose $\pi\colon P \to B$ is a principal G-bundle. Then the pullback by π gives a superalgebra isomorphism $\pi^*\colon \Omega(B) \xrightarrow{\cong} \Omega_{bas}(P)$.*

Proof. See [5, Corollary 6.13]. □

Definition 1.14. *A* ***connection*** *on a principal G-bundle $\pi : P \to B$ is a 1-form $\omega \in \Omega^1(P; \mathfrak{g}) = \Omega^1(P) \otimes \mathfrak{g}$ such that*

(i) *$\iota_X \omega = X$ for any $X \in \mathfrak{g}$.*

(ii) *$\rho_g \omega = Ad_{g^{-1}} \circ \omega$ for any $g \in G$.*

Proposition 1.15. *Every principal G-bundle has a connection.*

Proof. See [5, Corollary 6.7]. □

Definition 1.16. *The* ***curvature*** *of a connection* $\omega \in \Omega^1(P;\mathfrak{g})$ *on a principal* G*-bundle* $\pi\colon P \to B$ *is*

$$F_\omega := d\omega + \tfrac{1}{2}[\omega,\omega] \in \Omega^2(P;\mathfrak{g}).$$

Here $[\cdot,\cdot]$ *is defined to be the wedge product on forms followed by the Lie bracket on* $\mathfrak{g}$*.*

Proposition 1.17. *(i)* F_ω *is horizontal:* $\iota_X F_\omega = 0$ *for any* $X \in \mathfrak{g}$*;*

(ii) F_ω *is Ad-equivariant:* $\rho_g F_\omega = Ad_{g^{-1}} \circ F_\omega$ *for any* $g \in G$*;*

(iii) (The Bianchi identity) $dF_\omega = [\omega, F_\omega]$*.*

Proof. See [5, Theorem 7.2]. □

1.3 Spectral Sequences

Spectral sequence is an important tool in algebraic topology for computing cohomology. In this section we state without proof the existence of spectral sequence for a filtered cochain complex and some of its properties. As an important example we deduce the spectral sequence for a double complex. For missing details one may consult [2, Chapter III].

Let (C^*, d) be a cochain complex of abelian groups where $C^n = 0$ for all $n < 0$. Let

$$C^* = F^0C^* \supset F^1C^* \supset F^2C^* \supset \cdots$$

be a decreasing filtration of C^* with $\cap F^sC^* = 0$. Set $F^sC^* = C^*$ for $s < 0$. One can form the associated graded complexes of $\{F^sC^*\}$, which is by definition

$$gr^sC^* = F^sC^*/F^{s+1}C^*, \ s \in \mathbb{Z}.$$

Define

$$E_0^{s,t} = gr^sC^{s+t} = F^sC^{s+t}/F^{s+1}C^{s+t}.$$

The differential d induces a map

$$d_0\colon E_0^{s,t} \to E_0^{s,t+1}$$

with bidegree $(0,1)$, making E_0 into a bigraded complex. Let E_1 denote its cohomology, or explicitly

$$E_1^{s,t} = H^{s,t}(E_0, d_0) = H^{s+t}(gr^sC^*).$$

Note that

$$0 \to F^{s+1}C^* \to F^sC^* \to gr^sC^* \to 0$$

is a short exact sequence of cochain complexes, one obtain a long exact sequence with boundary homomorphisms ∂. Then we define $d_1\colon E_1^{s,t} \to E_1^{s+1,t}$ to be the composition

$$H^{s+t}(gr^sC^*) \xrightarrow{\partial} H^{s+t+1}(F^{s+1}C^*) \to H^{s+t+1}(gr^{s+1}C^*),$$

which is a differential with of bidegree $(1,0)$, making E_1 into a bigraded complex. Now define $E_2 = H(E_1, d_1)$, which is also bigraded. The theorem below shows that this process continues, and E_r "converges" to $H^*(C^*)$ as $r \to \infty$.

The filtration $\{F^sC^*\}$ is said to be **first quadrant** if $E_0^{s,t} = 0$ whenever $s < 0$ or $t < 0$.

Theorem 1.18. *There exists bigraded complexes (E_r, d_r), $r \geq 0$, where d_r is of bidegree $(r, -r+1)$ such that $E_{r+1} = H(E_r, d_r)$. Suppose $\{F^s C^*\}$ is first quadrant, then there is a filtration $\{F^s H^*(C^*)\}_{s \geq 0}$ of $H^*(C^*)$ that is exhaustive in the sense that $F^0 H^*(C^*) = H^*(C^*)$ and $\cap_s F^s H^*(C^*) = 0$, satisfying that for each s, t, $E_r^{s,t}$ stabilizes to some $E_\infty^{s,t}$ for sufficiently large r, and that $E_\infty^{s,t} \cong gr^s H^{s+t}(C^*)$. Moreover, all constructions above are natural in the initial data $\{F^s C^*\}$, and that (E_0, d_0), (E_1, d_1) are constructed as in the discussion above.*

The sequence $\{(E_r, d_r)\}_{r \geq 0}$ are known as the **spectral sequence** of the filtered complex C^* (with filtration $\{F^s C^*\}$).

Corollary 1.19. *Let $\{F^s C^*\}, \{F^s D^*\}$ be first quadrant filtrations of C^*, D^* and $f \colon C^* \to D^*$ be a chain map respecting these filtrations. Suppose that f induces an isomorphism $E_r(f) \colon E_r(C^*) \xrightarrow{\cong} E_r(D^*)$ for some r, then $H^*(C^*) \cong H^*(D^*)$.*

Proof. By naturality $E_r(f)$ is a chain map, therefore $E_{r+1}(f)$ is also an isomorphism. Continue, it follows that $E_\infty(f)$ is an isomorphism. Therefore f induces isomorphism in each $gr^s H^*$. Since we are working with coefficient $\mathbb{R}$, this implies that f induces isomorphism in H^* (for general coefficient ring, one uses five lemma and induction to prove f induces isomorphism in each $F^s H^*$). $\square$

Example 1.20 (Spectral Sequence of a Double Complex)**.** A **double complex** (of abelian groups) is a bigraded abelian group $C^{*,*}$ equipped with differential d with bidegree $(0, 1)$ and differential δ with bidegree $(1, 0)$ satisfying $d\delta + \delta d = 0$. It is said to be **first quadrant** if $C^{s,t} = 0$ whenever $s < 0$ or $t < 0$.

For any double complex $C^{*,*}$, we define its **total complex** to be the graded cochain complex (C^*, D) where

$$C^n = \bigoplus_{p+q=n} C^{p,q}$$

and $D = d + \delta$ is a differential of degree 1.

Let $\{F^s C^*\}$ be a decreasing filtration of C^* given by

$$F^s C^n = \bigoplus_{\substack{p+q=n \\ p \geq s}} C^{p,q}.$$

Then the associated graded complex is given by $gr^s C^n = C^{s,n-s}$. Thus $E_0^{s,t} = C^{s,t}$, d_0 agrees with the differential d on $C^{*,*}$, $E_1^{s,t} = H^{s,t}(C^{*,*}, d)$ is the vertical cohomology of the double complex $C^{*,*}$, and d_1 is induced by the differential δ. In particular, the filtration $\{F_s C^*\}$ is of first quadrant if and only if $C^{*,*}$ is of first quadrant as a double complex. In such case, there is a spectral sequence with (E_0, d_0), (E_1, d_1) given as above that converges to the cohomology of the total complex (C^*, D).

2 Equivariant Cohomology in Topology

In this section we define the G-equivariant cohomology of a G-space. See also [7, Section 1].

Remark 2.1. We shall make a technical assumption throughout these notes that all topological spaces we are considering are paracompact. Then, every fiber bundle is a Hurewicz fibration, and every vector bundle admits a bundle metric. This is not a too restrictive condition for us since every manifold is paracompact. Moreover, every CW complex is paracompact, and so will the classifying spaces we construct in Section 2.3. Alternatively, we could instead make the weaker assumption that all fiber bundles we are considering admits a numerable trivializing cover. See [9] for more details.

2.1 Equivariant Cohomology of G-Spaces

For any topological space X, singular cohomology theory associate to it a ring $H^*(X)$, and the $\mathbb{Z}$-grading on $H^*(X)$ makes it a (unital, supercommutative, associative) superalgebra. For any map $f\colon X \to Y$ between two spaces, we have an induced map

$$f^* = H^*(f)\colon H^*(Y) \to H^*(X). \tag{7}$$

The induced map is a superalgebra morphism which is functorial in the sense that $(g \circ f)^* = f^* g^*$ for maps $f\colon X \to Y$, $g\colon Y \to Z$, and that $(id_X)^* = id_{H^*(X)}$. In other words,

$$H^*\colon \mathbf{Top} \to s\mathbf{Alg}$$

is a contravariant functor from the category of topological spaces to the category of superalgebras.

Moreover, the functor H^* satisfies the homotopy invariance property: if $f, g\colon X \to Y$ are homotopic, then $f^* = g^*$. In other words it descends to a contravariant functor

$$H^*\colon \mathrm{Ho}\mathbf{Top} \to s\mathbf{Alg}$$

where Ho**Top** is the homotopy category of topological spaces, whose objects are the same as **Top** and whose morphisms are homotopy classes of maps between spaces.

As a corollary of homotopy invariance property, (7) is an isomorphism whenever f is a homotopy equivalence. In fact, a stronger statement is available. One say a map between topological spaces is a weak equivalence if it induces isomorphism in all homotopy groups.

Theorem 2.2 (Weak equivalence property)**.** *The map* (7) *is an isomorphism whenever f is a weak equivalence.*

Proof. See [8, Theorem 4.21]. □

Suppose now that X is a G-space. We want to define an analogue of singular cohomology for X that captures the additional G-structure, denoted $H^*_G(X)$.

A candidate for $H^*_G(X)$ is the cohomology ring of the orbit space X/G. However, when the G-action is not free, the space X/G may behave badly. For example, X/G may not be a smooth manifold even if X is. On the other hand, X/G usually behave nicely if the G-action is free. For example, if G acts freely on a manifold X, then the projection map $X \to X/G$ is a principal G-bundle. Therefore it is reasonable to define $H^*_G(X) = H^*(X/G)$ whenever the action is free.

We also expect our definition to satisfy the homotopy invariance property and the weak equivalence property. Therefore we may repair a non-free action as follows: Let X be any G-space (say G acts on the left). Let EG be a weakly contractible space (i.e. the map from EG to the one point space $*$ is a weak equivalence) on which G acts freely on the right. Then $EG \times X$ has the same homotopy type as X, and is equipped with a left G-action

$$g \cdot (e, x) = (eg^{-1}, gx),\ g \in G.$$

The quotient $(EG \times X)/G$, also denoted $EG \times_G X$, or simply X_G, is called the **homotopy quotient** of X by G.

Definition 2.3. *The* ***G-equivariant cohomology*** *of a G-space X is*

$$H_G^*(X) := H^*(X_G) = H^*((EG \times X)/G). \tag{8}$$

A G-equivariant map (referred to as G-map later) $X \to Y$ between G-spaces induces a map $X_G \to Y_G$, which in turn induces a map $H_G^*(Y) = H^*(Y_G) \to H^*(X_G) = H_G^*(X)$. In this way

$$H_G^* \colon G\mathbf{Top} \to s\mathbf{Alg}$$

defines a contravariant functor from the category of G-spaces to the category of superalgebras.

We haven't shown that the definition of equivariant cohomology is legitimate. In Section 2.2 we show that H_G^* is independent of the choice of EG. We also show that H_G^* satisfies the homotopy invariance property as well as the weak equivalence property. The existence of EG is deferred until Section 2.3.

2.2 Properties of Equivariant Cohomology

Strictly speaking, different choices of EG give rises to different homotopy quotient X_G, and Definition 2.3 potentially depends on this choice. We first show that H_G^* arises from different choices of EG are naturally identified.

Suppose E, E' are two weakly contractible spaces on which G acts freely on the right and X is a G-space. Then the G-map $pr_1 \colon E \times E' \to E$ by projection induces a map between fiber sequences $G \to E \times E' \times X \to (E \times E') \times_G X$ and $G \to E \times X \to E \times_G X$, which in turn induces a map between homotopy long exact sequences, and we have the commutatitive diagram

$$\begin{tikzcd}
\pi_{k+1}(G) \arrow[r] \arrow[d] & \pi_{k+1}(E \times E' \times X) \arrow[r] \arrow[d] & \pi_{k+1}((E \times E') \times_G X) \arrow[r] \arrow[d] & \pi_k(G) \arrow[r] \arrow[d] & \pi_k(E \times E' \times X) \arrow[d] \\
\pi_{k+1}(G) \arrow[r] & \pi_{k+1}(E \times X) \arrow[r] & \pi_{k+1}(E \times_G X) \arrow[r] & \pi_k(G) \arrow[r] & \pi_k(E \times X).
\end{tikzcd}$$

where all horizontal maps except the middle one are isomorphisms. Therefore the five lemma implies that the middle map is also an isomorphism. Now Theorem 2.2 says that $(E \times E') \times_G X \to E \times_G X$ induces isomorphisms in H^*. Similarly $(E \times E') \times_G X \to E' \times_G X$ induces isomorphism in H^*. Therefore $H^*(E \times_G X)$ and $H^*(E' \times_G X)$ are naturally identified, as desired.

Proposition 2.4. *(1) The functor H_G^* has the G-homotopy invariance property. In other words, if $f, g \colon X \to Y$ are G-homotopic G-maps between G-spaces, then $f^* = g^* \colon H_G^*(Y) \to H_G^*(X)$.*

(2) The functor H_G^* has the weak equivalence property. In other words, if a G-map $f \colon X \to Y$ is a weak equivalence, then $f^* \colon H_G^*(Y) \to H_G^*(X)$ is an isomorphism.

Proof. (1) A G-homotopy $X \times I \to Y$ between f, g induces a homotopy

$$EG \times_G X \times I \to EG \times_G Y, \ ([(e,x)], t) \mapsto [(e, h(x,t))]$$

between $f_*, g_* \colon EG \times_G X \to EG \times_G Y$.
(2) The G-map $id_{EG} \times f \colon EG \times X \to EG \times Y$ is also a weak equivalence. It induces a map between homotopy long exact sequences of fiber sequences $G \to EG \times X \to X_G$ and $G \to EG \times Y \to Y_G$, and the five lemma shows that $f_* \colon X_G \to Y_G$ is a weak equivalence. Then Theorem 2.2 implies the desired result. ∎

Proposition 2.5. *Suppose X is a G-space on which G acts freely. Then $H^*_G(X) = H^*(X/G)$ naturally.*

Proof. The G-map $pr_1 \colon EG \times X \to X$ induces a map between the homotopy long exact sequence of fiber sequences $G \to EG \times X \to X_G$ and $G \to X \to X/G$. Then the five lemma implies the desired result. □

Let BG denote the quotient EG/G, which is determined up to weak homotopy.

Proposition 2.6. *H^*_G is contravariant functor from the G-homotopy category of G-spaces to the category of $H^*(BG)$-superalgebras.*

Proof. It remains to show that each $H^*_G(X)$ can be naturally endowed with a $H^*(BG)$-superalgebra structure. To see this, consider the equivariant cohomology of the one point G-space $*$. By definition, $H^*_G(*) = H^*(*_G) = H^*(EG/G) = H^*(BG)$. Therefore the trivial G-map $X \to *$ induces a superalgebra map $H^*(BG) = H^*_G(*) \to H^*_G(X)$, which is clearly natural in X. □

2.3 Grassmannian Model for Classifying Spaces

In this section we give an explicit construction of a weakly contractible space EG equipped with a free right G-action. The principal G-bundle $EG \to BG$ given by the orbit projection is called the **universal G-bundle**. Its base BG is called the **classifying space** for G. Both these objects are determined up to weak homotopy. See Section 5.1 for a justification of their names.

We have assumed that G is compact. Since there is a Lie group embbedding $G \hookrightarrow O(n)$ for some integer n [3, Exercise 4.7.1], to construct EG, we may assume without loss of generality that $G = O(n)$.

Let $k \geq n$ be an integer. The Stifel variety $V_n(\mathbb{R}^k)$ is defined to be the subspace of $\mathbb{R}^{k\times n} = (\mathbb{R}^k)^n$ consists of orthonormal n-frames in $\mathbb{R}^k$. The Lie group $O(n)$ acts on $V_n(\mathbb{R}^k)$ freely on the right by right matrix multiplication. The orbit space, denoted $Gr_n(\mathbb{R}^k)$, is called the n-th Grassmannian of $\mathbb{R}^k$. It can be identified with the space of n-planes in $\mathbb{R}^k$ passing the origin.

The embedding $\mathbb{R}^k \hookrightarrow \mathbb{R}^{k+1}$ by adding 0 to the last coordinate induces an $O(n)$-equivariant embedding $V_n(\mathbb{R}^k) \hookrightarrow V_n(\mathbb{R}^{k+1})$ which descends to $Gr_n(\mathbb{R}^k) \hookrightarrow Gr_n(\mathbb{R}^{k+1})$. The direct limits

$$V_n(\mathbb{R}^\infty) = \varinjlim V_n(\mathbb{R}^k), \ Gr_n(\mathbb{R}^\infty) = \varinjlim Gr_n(\mathbb{R}^k)$$

are known as the infinite Stifel variety and the infinite Grassmannian, respectively. The projection $V_n(\mathbb{R}^\infty) \to Gr_n(\mathbb{R}^\infty)$ is a principal $O(n)$-bundle.

We check that $V_n(\mathbb{R}^\infty)$ is weakly contractible. It will follow that $V_n(\mathbb{R}^\infty) \to Gr_n(\mathbb{R}^\infty)$ is a universal $O(n)$-bundle and $BO(n) = Gr_n(\mathbb{R}^\infty)$ is a classifying space for $G = O(n)$ (more generally, $V_n(\mathbb{R}^\infty)/G$ is a classifying space for any G that embeds in $O(n)$). This is known as the Grassmannian model for classifying spaces.

Note $V_n(\mathbb{R}^\infty)$ can be also regarded as the space of orthonormal n-frames in $\mathbb{R}^\infty = \varinjlim \mathbb{R}^k$. The elements in $\mathbb{R}^\infty$ can be written as a tuple $(u_1, u_2, \cdots)^t$ where only finitely many u_i are nonzero. Any elements in $V_n(\mathbb{R}^\infty)$ can be written as an infinite matrix

$$U = (u_{ij})_{i\geq 1, 1\leq j\leq n}, \tag{9}$$

where only finitely many u_{ij} are nonzero. Under this identification, the group $O(n)$ acts by right matrix multiplication.

Below is a "swindle" argument which shows that $V_n(\mathbb{R}^\infty)$ is contractible, hence weakly contractible.

For any real matrix $V = (v_{ij})_{i\geq 1, 1\leq j\leq n}$ with full rank, let $GS(V) \in V_n(\mathbb{R}^\infty)$ denote the matrix obtained by applying the Gram-Schmidt orthonormalization to the column vectors of V. For any U as in (9), define

$$h_1(U,t) = GS(((1-t)u_{ij} + tu_{i,(j-n)})_{i,j}),\ t \in [0,1].$$

Here it is understood that $u_{ij} = 0$ for $j \leq 0$. Then h_1 retract $V_n(\mathbb{R}^\infty)$ onto its subspace, denoted $V_n'(\mathbb{R}^\infty)$, consists of matrices where the upper $n \times n$ matrix is zero. For any $U \in V_n'(\mathbb{R}^\infty)$ as in (9), define

$$h_2(U,t) = (\sin t\delta_{ij} + \cos tu_{ij})_{i,j},\ t \in [0,\pi/2].$$

Then h_1 followed by h_2 retracts $V_n(\mathbb{R}^\infty)$ onto the point $(\delta_{ij})_{i,j} \in V_n(\mathbb{R}^\infty)$. Hence $V_n(\mathbb{R}^\infty)$ is contractible.

Exercise 2.7. Check that h_1, h_2 above are well-defined. More precisely, for h_1, check that GS always takes in matrices with full rank; for h_2, check that its image lies in $V_n(\mathbb{R}^\infty)$.

3 Equivariant Cohomology in Differential Geometry

In this section we first give the definition of G^* modules and G^* algebras. Then we define their G-equivariant cohomology. The structure and materials of this section are mostly in parallel with those of Section 2. Motivations for some definitions in Section 3.1 are from Section 1.2. Our treatment essentially follows [7, Section 2].

3.1 Equivariant Cohomology of G^* Modules

Motivated by (2)∼(6), we make the following definition.

Definition 3.1. *A* **G^* *module*** *is a super vector space A equipped with a Lie superalgebra morphism $\tilde{\mathfrak{g}} \to Der(A)$ and a G-representation $\rho\colon G \to Aut(A)$ such that (2)∼(5) holds. A G^* module is a* **G^* *algebra*** *if the underlying super vector space is a superalgebra.*

Usually, we just write $\rho_g(a)$ as $g \cdot a$ for $g \in G$, $a \in A$.

Example 3.2. Suppose M is a G-manifold. Then $\Omega(M)$ is a G^* algebra.

Let $\Omega_c(M)$ denote the compactly supported de Rham complex. Then $\Omega_c(M)$ is a G^* module. It is not a G^* algebra if M is not compact since it is not unital.

In the rest of these notes, all (essentially two) examples of G^* modules we will encounter are G^* algebras. Therefore we will not put emphasize on the development of G^* modules.

Several comments are in order.

- Strictly speaking, in order that (2) makes sense, it is required that A processes some kind of topology. This will be automatic in all the cases we will encounter. See also [7, pp.17].
- A super vector space map (resp. superalgebra map) between G^* modules (resp. G^* algebras) is a morphism if it respects the G^* module structure. This makes the space of G^* modules into a category and the space of G^* algebras its subcategory.

- Suppose A, B are G^* modules (resp. G^* algebras), then the diagonal representation of G on $A \otimes B$ given by $g \cdot (a \otimes b) = (g \cdot a) \otimes (g \cdot b)$ and the diagonal map $\tilde{\mathfrak{g}} \to Der(A \otimes B)$ defined by Exercise 1.9 makes $A \otimes B$ a G^* module (resp. G^* algebra).
- Any G^* module A is automatically a cochain complex with differential d. We denote its cohomology by $H(A)$, which is a super vector space. If A is a G^* algebra, then $H(A)$ is a superalgebra.

Exercise 3.3. Let A, B be two G^* modules (resp. G^* algebras). Then the map $A \otimes B \to B \otimes A$ defined in Exercise 1.6 is a G^* module (resp. G^* algebra) isomorphism.

Motivated by Definition 1.14, we say a G^* algebra A is **regular** if there exists an element $\omega \in A^1 \otimes \mathfrak{g}$ (called a **connection**) such that,

(i) $\iota_X \omega = X$ for any $X \in \mathfrak{g}$;

(ii) $g \cdot \omega = Ad_{g^{-1}} \circ \omega$ for any $g \in G$.

A typical example of regular G^* algebra is $\Omega(M)$ where M is a manifold equipped with a free G-action.

A G^* module A is said to be **acyclic** if it is acyclic as a cochain complex, i.e. if

$$H^*(A) = \mathbb{R} = \begin{cases} \mathbb{R}, & * = 0 \\ 0, & * \neq 0. \end{cases}$$

A typical example of acyclic G^* algebra is $\Omega(M)$ where M is a contractible G-manifold.

For a G^* module A, let A_{hor}, A^G, A_{bas} denote its sub super vector space of horizontal, invariant, basic elements, as defined by the same formulae in Definition 1.12.

Exercise 3.4. (A_{bas}, d) is a cochain complex.

As a shorthand we will write $H_{bas}(A)$ to denote $H(A_{bas})$. For a typical example, take $A = \Omega(P)$ where $\pi \colon P \to B$ is a principal G-bundle, then $A_{bas} \simeq \Omega(B)$ via π^*, by Proposition 1.13. Therefore in this case

$$H_{bas}(A) = H(\Omega(B)) = H^*(B). \tag{10}$$

Having these notions for G^* modules and G^* algebras, we give the definition of G-equivariant cohomology of a G^* module and of a G-manifold. Let E denote any acyclic regular G^* algebra.

Definition 3.5. *The **G-equivariant cohomology** of a G^* module A is the super vector space*

$$H_G(A) = H_{bas}(E \otimes A)$$

which is a superalgebra if A is a G^* algebra.

Then a morphism $A \to B$ of G^* modules (resp. G^* algebras) induces $H_G(A) \to H_G(B)$. In this way H_G defines a covariant functor

$$H_G \colon G^*\mathbf{Mod} \to s\mathbf{Vect}$$

and a covariant functor

$$H_G\colon G^*\mathbf{Alg} \to s\mathbf{Alg}.$$

We haven't show that the functor H_G does not depend on the choice of E, nor have we show the existence of E. These will be done in the Section 3.2 and Section 3.3, respectively.

Definition 3.6. *The* **G*-equivariant de Rham cohomology*** *of a G-manifold M is the superalgebra*

$$H^*_{G,dR}(M) := H_G(\Omega(M)) = H((E \otimes \Omega(M))_{bas}). \tag{11}$$

Since the de Rham functor Ω is a contravariant functor

$$\Omega\colon G\mathbf{Mfd} \to G^*\mathbf{Alg},$$

we see that $H^*_{G,dR}$ defines a contravariant functor

$$H^*_{G,dR}\colon G\mathbf{Mfd} \to s\mathbf{Alg}.$$

One is encouraged to compare Definition 2.3 and Definition 3.6. In comparison, E is a "realization" of de Rham complex for EG, where acyclicity of E corresponds to weakly contractibility of EG, and regularity of E corresponds to (local) freeness of G-action on EG. The tensor product in the smooth world corresponds to the multiplication in the topological world. Restricting to basic elements corresponds to quotient by G (cf. Proposition 1.13). Hence $(E \otimes \Omega(M))_{bas}$ is a "realization" of de Rham complex for M_G. In fact, in Section 4.6 we will prove the following theorem.

Theorem 3.7. *Restricting to the category of G-manifolds, we have $H^*_{G,dR} = H^*_G$ as contravariant functors to the category of superalgebras.*

In particular, $H^*_{G,dR}(*) = H^*_G(*) = H^*(BG)$. One can slightly improve the theorem as follows.

Theorem 3.8 (Equivariant de Rham Theorem)**.** *Restricting to the category of G-manifolds, we have $H^*_{G,dR} = H^*_G$ as contravariant functors to the category of $H^*(BG)$-superalgebras.*

Therefore, we may simply write H^*_G for both the equivariant cohomology functor and the equivariant de Rham cohomology functor. But we refrain from doing this until Theorem 3.8 is proved.

Exercise 3.9. (1) Suppose A is a regular G^* algebra and $A \to B$ is a G^* algebra morphism, show that B is also a regular G^* algebra.
(Hint: consider the image of a connection of A under the map $A \to B$.)

(2) Let A, B be G^* algebras. If A or B is regular, then so is $A \otimes B$.

3.2 Properties of Equivariant de Rham Cohomology

We first show that functors H_G on G^* modules arise from different choices of acyclic regular G^* algebra E are naturally identified.

Proposition 3.10. *If A is a G^* module, B is a regular G^* algebra, and E is an acyclic regular G^* algebra, then the inclusion $B \otimes A \hookrightarrow E \otimes B \otimes A$ induces isomorphism in H_{bas}.*

Proof. See [7, Theorem 4.3.1]. ∎

Corollary 3.11. *If A is a regular G^* algebra, then $H_{bas}(A) = H_G(A)$ naturally in A.*

Proof. Take A, B in Proposition 3.10 to be $\mathbb{R}, A$, respectively. $\square$

By Proposition 3.10, if E, E' are two acyclic regular G^* algebra, then $H_{bas}(E \otimes A) = H_{bas}(E' \otimes E \otimes A) = H_{bas}(E \otimes E' \otimes A) = H_{bas}(E' \otimes A)$ (cf. Exercise 3.3). In this way, H_G arises from different choices of E are naturally identified.

Proposition 3.12. *Suppose X is a G-manifold on which G acts freely. Then $H^*_{G,dR}(X) = H^*(X/G)$ naturally.*

Proof. Since $\Omega(X)$ is a regular G^* algebra, by Corollary 3.11 and Proposition 1.13, we have a natural identification

$$H^*_{G,dR}(X) = H_G(\Omega(X)) = H_{bas}(\Omega(X)) = H(\Omega(X/G)) = H^*(X/G). \quad \square$$

Proposition 3.13. *H_G is a covariant functor from the category of G^* algebras to the category of $H_G(\mathbb{R})$-superalgebras.*

Proof. For any G^* algebra A, the inclusion $\mathbb{R} \hookrightarrow A$ induces a map $H_G(\mathbb{R}) \to H_G(A)$ natural in A. $\square$

Proposition 3.14. *$H^*_{G,dR}$ is a contravariant functor from the category of G-manifolds to the category of $H^*_{G,dR}(*)$-superalgebras.*

Proof. The trivial G-space $*$ is final in G**Mfd**. Therefore for a G-manifold M there is a map $H^*_{G,dR}(*) \to H^*_{G,dR}(M)$ natural in M. $\square$

The natural map $H_G(\mathbb{R}) \to H_G(A)$ (resp. $H^*_{G,dR}(*) \to H^*_{G,dR}(M)$) is called the **Chern-Weil** map for the G^* algebra A (resp. G-manifold M). See Section 5 for more details.

3.3 Infinite Dimensional Manifolds

An **infinite dimensional manifold** is a topological space M equipped with a filtration $M_0 \subset M_1 \subset M_2 \subset \cdots \subset M$ such that $M_i \subset M_{i+1}$ are embedding of smooth manifolds and $M = \varinjlim M_i$ as topological spaces.

Exercise 3.15. Let $M \hookrightarrow N$ be an embedded submanifold. Show that the restriction map $\Omega(N) \to \Omega(M)$ is surjective.

Suppose M is an infinite dimensional manifold defined as above. Then $\{\Omega(M_i)\}$ together with restriction maps forms an inverse system of differential graded algebras (cf. Example 1.5) where all restriction maps are surjective. The **de Rham complex** of M is defined to be the differential graded algebra

$$\Omega(M) := \varprojlim \Omega(M_i).$$

Any form $\theta \in \Omega(M)$ can be identified with a compatible sequence of forms $\theta_i \in \Omega(M_i)$.

Recall that in Section 2.3 we constructed a contractible space EG equipped with a free G-action using the Grassmannian model. Explicitly, choose an integer n such that $G \hookrightarrow O(n)$, then

$$EG = V_n(\mathbb{R}^\infty) = \varinjlim V_n(\mathbb{R}^k).$$

Define
$$E = \Omega(V_n(\mathbb{R}^\infty)) = \varprojlim \Omega(V_n(\mathbb{R}^k)).$$
We check that E is acyclic and regular.

Exercise 3.16. Let $H \hookrightarrow G$ be an embedding of (compact) Lie groups. Check that any regular G^* algebra is also a regular H^* algebra.
(Hint: Construct a connection by first choosing an element satisfying (i) in Definition 1.14, then doing an averaging.)

Regularity of E: By exercise 3.16, it suffices to check that E is regular in the case $G = O(n)$. Let $U = (u_{ij})_{i\geq 1, 1\leq j\leq n}$ be coordinates of $EG = V_n(\mathbb{R}^\infty)$, as in Section 2.3. Then $\omega = U^t dU$ is a matrix valued 1-form on EG. We check that ω is actually $\mathfrak{o}(n)$-valued, and is a connection of $E = \Omega(EG)$.

To see ω values in $\mathfrak{o}(n)$, we check that the (i,j)-coordinate plus the (j,i)-coordinate of ω equals
$$\sum_{k=1}^{\infty}(u_{ki}du_{kj} + u_{kj}du_{ki}) = d(\sum_{k=1}^{\infty} u_{ki}u_{kj}) = d\delta_{ij} = 0.$$

To see ω is a connection, we check that
(i) For any $X \in \mathfrak{g}$, we have $\underline{X}_U = UX$ (as a tangent vector at $U \in EG$), therefore
$$\iota_X \omega = U^t(UX) = X.$$

(ii) For any $g \in G$, we have
$$g \cdot \omega = (Ug)^t d(Ug) = g^t U^t (dU) g = g^{-1} U^t (dU) g = Ad_{g^{-1}}\omega.$$

Acyclicity of E: We begin with a lemma.

Lemma 3.17. *Suppose $\{A_i\}_{i\geq 0}$ is an inverse system of cochain complexes (resp. differential graded algebras) with inverse limit A where all restriction maps are surjective. Suppose that for each k, the inverse system $H^k(A_i)$ stabilizes, i.e. $H^k(A_{i+1}) \xrightarrow{\cong} H^k(A_i)$ for sufficiently large i, then the stable cohomology $\varprojlim H^*(A_i)$ equals $H^*(A)$ as super vector spaces (resp. superalgebras).*

Proof. Let d denote the differentials of each A_i and $r\colon A_{i+1} \to A_i$, $r_i\colon A \to A_i$ denote the restriction maps.

First we assume $\{A_i\}_{i\geq 0}$ is an inverse system of cochain complexes. Since the all restriction maps $r\colon A_{i+1} \to A_i$ are surjective, an element $a \in A$ is identified with a sequence of elements $a_i \in A_i$ that are compatible with restriction maps.

Fix k, let i be an integer such that $H^k(A_*)$ and $H^{k-1}(A_*)$ stabilize for $* \geq i$. We check that r_i induces isomorphism in H^k. This will imply the desired result.

Surjectivity: Let $a_i \in A_i^k$ be a cocycle. We find a cocycle $a_{i+1} \in A_{i+1}^k$ with $ra_{i+1} = a_i$. Then inductively we can find a sequence of cocyles $a_j \in A_j^k$, $j \geq i$, compatible with restriction maps. They patch to a cocycle $a \in A^k$, proving that r_i induces a surjection in H^k.

Since $r\colon A_{i+1} \to A_i$ induces isomorphism in H^k, we can find a cocycle $a'_{i+1} \in A_{i+1}^k$ such that $ra'_{i+1} = a_i + db_i$ for some $b_i \in A_i^{k-1}$. Since $r\colon A_{i+1} \to A_i$ is surjective, we can find $b_{i+1} \in A_{i+1}^{k-1}$ with $rb_{i+1} = b_i$. Then $a_{i+1} := a'_{i+1} - db_{i+1} \in A_{i+1}^k$ is a cocycle satisfying $ra_{i+1} = a_i$, as desired.

Injectivity: Let $a \in A^k$ be a cocycle such that $a_i = db_i \in A_i^k$ for some $b_i \in A_i^{k-1}$. We prove that there exists $b_{i+1} \in A_{i+1}^{k-1}$ such that $a_{i+1} = db_{i+1}$ and $b_i = rb_{i+1}$. Then inductively we can find a compatible sequence of elements $b_j \in A_j^{k-1}$, $j \geq i$, such that $db_j = a_j$. Thus b_j patches to $b \in A^{k-1}$ with $db = a$, proving that a is a coboundary.

Since $r\colon A_{i+1} \to A_i$ induces isomorphism in H^k, we can find $b'_{i+1} \in A_{i+1}^{k-1}$ such that $db'_{i+1} = a_{i+1}$. Then $b_i - rb'_{i+1} \in A_i^{k-1}$ is a cocycle. By the same argument as in the surjectivity part, we can find a cocycle $b''_{i+1} \in A_{i+1}^{k-1}$ such that $rb''_{i+1} = b_i - rb'_{i+1}$. Then $b_{i+1} := b'_{i+1} + b''_{i+1} \in A_{i+1}^{k-1}$ satisfies $db_{i+1} = a_{i+1}$ and $rb_{i+1} = b_i$.

Finally, if $\{A_i\}_{i\geq 0}$ is an inverse system of differential graded algebras, then the underlying super vector space structure of $\varprojlim H^*(A_i)$ agrees with the inverse limit of A_i as cochain complexes. The restriction maps $H^*(A) \to H^*(A_i)$ induces a superalgebra morphism $H^*(A) \to \varprojlim H^*(A_i)$, which is an isomorphism of super vector spaces, thus also an isomorphism of superalgebras. $\square$

Here is a well-known property of Stiefel varieties.

Proposition 3.18. *$V_n(\mathbb{R}^k)$ is $(k-n-1)$-connected, i.e. its q-th homotopy group is trivial for $q \leq k-n-1$.*

Proof. Recall that $V_n(\mathbb{R}^k)$ can be regarded as a tuple of orthonormal vectors $(v_1, \cdots, v_n)$ in $\mathbb{R}^k$. The projection onto the first vector gives a map $V_n(\mathbb{R}^k) \to S^{k-1}$ which is a fiber bundle with fiber $V_{n-1}(\mathbb{R}^{k-1})$. The long exact sequence for this fiber sequence shows $V_{n-1}(\mathbb{R}^{k-1}) \to V_n(\mathbb{R}^k)$ induces isomorphism in π_q for $q \leq k-2$. Inductively one sees that $\pi_q(V_n(\mathbb{R}^k)) = \pi_q(V_0(\mathbb{R}^{k-n})) = \pi_q(*)$ is trivial for $q \leq k-n-1$. $\square$

As a corollary, the Hurewicz theorem [8, Theorem 4.32] implies that

$$H^*(V_n(\mathbb{R}^k)) = \begin{cases} 0, & 1 \leq * \leq k-n-1 \\ \mathbb{R}, & * = 0. \end{cases}$$

Now Lemma 3.17 applying to $\{\Omega(V_n(\mathbb{R}^k))\}_{k\geq 0}$ shows that $H^*(\Omega(EG)) = \mathbb{R}$, as desired.

4 The Weil Model and the Cartan Model

In this section, we introduce two models that make the computation of equivariant cohomology simpler. In the end of this section, using the tools we have developed by then, we provide a proof of the equivariant de Rham theorem. This section essentially follows [7, Section 2, 3, 4, 6] and [15, Part III].

4.1 The Weil Algebra

Roughly speaking, the Weil algebra $W(\mathfrak{g})$ for G is an acyclic regular G^* algebra that is initial (up to chain homotopy) among all regular G^* algebras.

We begin with some motivation from Section 1.2. Suppose $P \to B$ is a principal G-bundle with a chosen connection $\omega \in \Omega^1(P; \mathfrak{g})$, then the de Rham complex $\Omega(P)$ is a regular G^* algebra with connection ω. We look for a G^* algebra $W(\mathfrak{g})$ together with a map $W(\mathfrak{g}) \to \Omega(P)$.

The connection 1-form $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ can be regarded as a linear map $\mathfrak{g}^* \to \Omega^1(P)$, which extends to a superalgebra map $f_1\colon \wedge(\mathfrak{g}^*) \to \Omega(P)$. Here $\wedge(V)$ denotes the exterior algebra of a vector space V.

The curvature 2-form $F_\omega \in \Omega^2(P) \otimes \mathfrak{g}$ can similarly be regarded as a linear map $\mathfrak{g}^* \to \Omega^2(P)$ which extends to a superalgebra map $f_2\colon S(\mathfrak{g}^*) \to \Omega(P)$. Here $S(V)$ denotes the symmetric algebra of a vector space V, but with grading twice as usual, so that $S^k(V) = 0$ for odd k. (Note we are using different notational convention with [7] and [15].)

The maps f_1 and f_2 combine to a superalgebra map

$$f = f_1 \otimes f_2\colon \wedge(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*) \to \Omega(P).$$

This encourage us to define $W = W(\mathfrak{g}) = \wedge(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*)$ as a superalgebra. The group G acts on W via the coadjoint action on $\mathfrak{g}^*$. Then the Ad-equivariance of ω and F_ω implies that the map f is G-equivariant. Let $\mathcal{L}$ on W be the infinitesimal of this G-action (cf. (2)). Then f commutes with $\mathcal{L}_X$ for any $X \in \mathfrak{g}$.

We still need to specify the action of ι_X and d on W to make it a G^* algebra such that f is a G^* algebra morphism. Since ι_X and d are superderivations, it suffices to define them on a set of algebra generators, for example $\wedge^1(\mathfrak{g}^*) \cup S^2(\mathfrak{g}^*)$.

Let $\alpha \in \mathfrak{g}^*$ be arbitrary. By $\iota_X\omega = X$ and $\iota_X F_\omega = 0$, we must define

$$\iota_X(\alpha \otimes 1) = \langle X, \alpha\rangle$$

and

$$\iota_X(1 \otimes \alpha) = 0.$$

By $d\omega = -\frac{1}{2}[\omega, \omega] + F_\omega$ and $d^2 = 0$, we must define

$$d(\alpha \otimes 1) = \delta\alpha \otimes 1 + 1 \otimes \alpha,$$

extend it to $\wedge(\mathfrak{g}^*) \to \Omega(P)$, and define

$$d(1 \otimes \alpha) = -d(\delta\alpha \otimes 1).$$

Here

$$\delta\colon \wedge^1(\mathfrak{g}^*) \to \wedge^2(\mathfrak{g}^*) \tag{12}$$

is the operator defined by

$$(\delta\alpha)(X, Y) := -\alpha([X, Y]) \text{ for all } X, Y \in \mathfrak{g}.$$

Proposition 4.1. *The operators $\iota_X, \mathcal{L}_X, d$ and the G-action defined above make W a G^* algebra. Moreover, the map $f\colon W \to \Omega(P)$ is a G^* algebra morphism.*

Proof. The interested reader may check this himself/herself using the coordinate expressions given Section 4.4. See also [15, Section 19.3] for part of the computations. □

Proposition 4.2. *The Weil algebra W is acyclic and regular.*

Proof. For regularity, suppose that the element

$$\theta \in W^1 \otimes \mathfrak{g} = \wedge^1(\mathfrak{g}^*) \otimes 1 \cong \mathfrak{g}^* \otimes \mathfrak{g} \cong Hom(\mathfrak{g}, \mathfrak{g}) \tag{13}$$

corresponds to $id_{\mathfrak{g}} \in Hom(\mathfrak{g}, \mathfrak{g})$ under the usual identification. Then θ is the unique connection of W.

For acyclicity, construct a homotopy operator using coordinate expression. See [15, Theorem 19.2]. ☐

Proposition 4.3. *The Weil algebra W satisfies $H_G(W) = H_{bas}(W) = W_{bas} = S(\mathfrak{g}^*)^G$.*

Proof. By definition, we compute that $W_{bas} = (W_{hor})^G = S(\mathfrak{g}^*)^G$, which contains only even elements. Hence $d = 0$ on W_{bas} and $H_{bas}(W) = W_{bas}$. The equality $H_G(W) = H_{bas}(W)$ is by Corollary 3.11. ☐

The conditions in Definition 1.14 are convex. This allows us to interpolate maps $f: W \to \Omega(P)$ arise from different choices of ω. A more precise and general theorem states that

Theorem 4.4. *For any regular G^* algebra A, there exists a unique G^* algebra morphism $W \to A$ up to chain homotopy.*

Proof. Let θ be the connection element in W and take any connection $\omega \in A^1 \otimes \mathfrak{g}$ of A. Then $\theta \mapsto \omega$ restricted to components extends to a G^* algebra morphism $W \to A$. Conversely, any G^* algebra morphism $W \to A$ tensored with $\mathfrak{g}$ maps θ to some connection of A. Then one uses the convexity of the space of connections of A to conclude the proof. See [7, Theorem 3.3.1] for details. ☐

Strictly speaking, the "chain homotopy" above refers to chain homotopy between G^* algebras, which is a notion we have not yet defined. There is nothing mysterious, but keeping track of the G^* structure takes a little work. Interested readers may consult [7, Section 2.3.3] to find an explicit definition. We just point out that chain homotopic maps between G^* modules induce isomorphic maps in G-equivariant cohomology. Moreover, if $f, g: A \to B$ are chain homotopic map between G^* modules and C is another G^* module, then $f \otimes 1, g \otimes 1: A \otimes C \to B \otimes C$ are chain homotopic. Therefore, as a corollary of Theorem 4.4, we have

Corollary 4.5. *Let A be a regular G^* algebra and B be a G^* module. Then all G^* algebra morphism $W \to A$ induce the same map $H_G(W \otimes B) \to H_G(A \otimes B)$.* ☐

In particular, take $B = \mathbb{R}$, we have

Corollary 4.6. *Let A be a regular G^* algebra. There is a map*

$$S(\mathfrak{g}^*)^G \to H_{bas}(A)$$

defined naturally in A, which is the induced map in H_{bas} by any G^* algebra morphism $W \to A$. ☐

This map is known as the **Chern-Weil map** for the regular G^* algebra A. See Section 5 for more details.

4.2 The Weil Model

Take $E = W(\mathfrak{g})$ in Definition 3.5, we obtain the **Weil model** for a G^* module A, which is by definition the cochain complex $((W(\mathfrak{g}) \otimes A)_{bas}, d)$. Its cohomology computes the G-equivariant cohomology of A. A G^* module morphism $A \to B$ induces a map between Weil modules in the usual way, which in turn induces a map in equivariant cohomology.

Example 4.7. Let $*$ denote the one point G space. Then $\Omega(*) = \mathbb{R}$ is the trivial G^* algebra. Using Weil model and Proposition 4.3 we compute the G-equivariant de Rham cohomology of a point to be

$$H^*_{G,dR}(*) = H_G(\mathbb{R}) = H(W_{bas}) = S(\mathfrak{g}^*)^G.$$

In particular, take $G = S^1$, then

$$H^*_{S^1,dR}(*) = S(\mathbb{R})^{S^1} = \mathbb{R}[u]$$

is a polynomial ring in one indeterminant u with $|u| = 2$.

On the other hand, the topological G-equivariant cohomology of a point is $H^*_G(*) = H^*(BG)$. Using the Grassmaniann model for BS^1 one sees that $BS^1 = Gr_2(\mathbb{R}^\infty) = \mathbb{CP}^\infty$, thus we also have

$$H^*_G(*) = H^*(\mathbb{CP}^\infty) = \mathbb{R}[u].$$

If A is a G^* algebra, then a G^* algebra map $W(\mathfrak{g}) \to W(\mathfrak{g}) \otimes A$, for example the inclusion into first component, equip $H_G(A)$ with a $S(\mathfrak{g}^*)^G$-superalgebra structure. The choice of this G^* algebra map is irrelevant (cf. Corollary 4.6).

4.3 The Cartan Model

The Cartan Model for a G^* module is another model that computes its G-equivariant cohomology. Although it is less intuitive than the Weil model, it is actually more suitable for computations for various reasons we will see.

Exercise 4.8. Check that G-action on a G^* module A restricts to a G-action on A_{hor}.

Let G act on $S(\mathfrak{g}^*)$ via the coadjoint action. Let A be any G^* module. Then the diagonal action defines a G-action on $S(\mathfrak{g}^*) \otimes A$.

Theorem 4.9. *(Weil-Cartan Isomorphism) Let A be a G^* module (resp. G^* algebra). Define*

$$F\colon (W(\mathfrak{g}) \otimes A)_{hor} \to S(\mathfrak{g}^*) \otimes A$$

to be the restriction of the projection map

$$W(\mathfrak{g}) \otimes A = \wedge(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*) \otimes A \to \wedge^0(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*) \otimes A = \mathbb{R} \otimes S(\mathfrak{g}^*) \otimes A = S(\mathfrak{g}^*) \otimes A$$

to $(W(\mathfrak{g}) \otimes A)_{hor}$. Then F is a G-equivariant super vector space (resp. superalgebra) isomorphism. Therefore, it induces a super vector space (resp. superalgebra) isomorphism

$$F\colon (W(\mathfrak{g}) \otimes A)_{bas} \to (S(\mathfrak{g}^*) \otimes A)^G.$$

Proof. See [15, Theorem 21.1]. □

The map F carries the differential d on $(W(\mathfrak{g})\otimes A)_{bas}$ a differential d_G on $C_G(A) := (S(\mathfrak{g}^*)\otimes A)^G$. The cochain complex $(C_G(A), d_G)$ is called the **Cartan model** for A. Its cohomology computes the G-equivariant cohomology for A. See Proposition 4.12 for an explicit expression for d_G. A morphism $f\colon A \to B$ induces morphism $1\otimes f$ between their Cartan models, which in turn induces a map in equivariant cohomology.

If M is a G-manifold, then elements in $\Omega_G(M) := C_G(\Omega(M)) = (S(\mathfrak{g}^*)\otimes\Omega(M))^G$ are called G-equivariant differential forms on M. With respect to the differential d_G, we also have the notions of closed and exact forms on M. The cohomology of $\Omega_G(M)$ computes the G-equivariant cohomology of M.

Example 4.10. If $A = \mathbb{R}$ is the trivial G^* algebra, then $C_G(A) = S(\mathfrak{g}^*)^G$ consists of only even elements. Therefore we find again that

$$H^*_{G,dR}(*) = H_G(\mathbb{R}) = H(S(\mathfrak{g}^*)^G, d_G) = S(\mathfrak{g}^*)^G.$$

The inclusion $S(\mathfrak{g}^*)^G \hookrightarrow (S(\mathfrak{g}^*)\otimes A)^G$ into first component equips $H_G(A)$ with a $S(\mathfrak{g}^*)^G$-superalgebra structure.

4.4 Coordinate Expression

Fix a basis $X_1, \cdots, X_n$ for $\mathfrak{g}$ and $\xi^1, \cdots, \xi^n$ its dual basis for $\mathfrak{g}^*$. Let $\theta^i = \xi^i\otimes 1$ and $u^i = 1\otimes\xi^i$ be elements in $W(\mathfrak{g})$. They freely generate $W(\mathfrak{g})$ as a superalgebra. As a shorthand, write $\iota_i, \mathcal{L}_i$ for $\iota_{X_i}, \mathcal{L}_{X_i}$, respectively.

Write $[X_i, X_j] = c_{ij}^k X_k$. The coefficients $c_{ij}^k \in \mathbb{R}$ are known as the structural constants for $\mathfrak{g}$. They are skew-symmetric in i, j and satisfy the Jacobi's identity. Note the Einstein summation principle is in use, here and thereafter.

Exercise 4.11. (1) The coadjoint action ad^* is given by

$$ad^*_{X_i}\xi^k = -c_{ij}^k\xi^j.$$

(2) The operator δ as in (12) is given by

$$\delta\xi^k = -\tfrac{1}{2}c_{ij}^k\xi^i\wedge\xi^j.$$

(3) Let ω be a connection on a principal G-bundle $P\to B$ with curvature F_ω. In coordinate expression, write $\omega = \omega^i\otimes X_i$ and $F_\omega = F_\omega^i\otimes X_i$. The structural equations $d\omega = -\frac{1}{2}[\omega,\omega] + F_\omega$ and $dF_\omega = [\omega, F_\omega]$ can be written as

$$d\omega^k = c_{ij}^k\omega^i\wedge\omega^j,\ dF_\omega^k = c_{ij}^k\omega^i\wedge\omega^j.$$

In the notation of Section 4.1, the connection $\theta\in W\otimes\mathfrak{g}$ can be written as $\theta = \theta^i\otimes X_i$. The map $f\colon W\to\Omega(P)$ maps each θ^i to ω^i and each u^i to F_ω^i. Since the elements θ^i, u^i freely generate W, the operators $\iota_i, \mathcal{L}_i, d$ are uniquely determined by their action on these elements. Explicitly, one can compute that

$$\iota_i\theta^k = \delta_i^k;$$

$$\iota_i u^k = 0;$$
$$\mathcal{L}_i \theta^k = -c_{ij}^k \theta^j;$$
$$\mathcal{L}_i u^k = -c_{ij}^k u^j;$$
$$d\theta^k = u^k - \tfrac{1}{2} c_{ij}^k \theta^i \theta^j;$$
$$du^k = -c_{ij}^k \theta^i u^j.$$

Using these equations, the reader can directly check most unproven statements in Section 4.1.

Proposition 4.12. *The differential d_G on the Cartan model $C_G(A) = (S(\mathfrak{g}^*) \otimes A)^G$ given by*

$$d_G = 1 \otimes d - u^i \otimes \iota_i.$$

Here u^i acts on $S(\mathfrak{g}^*)$ by multiplication on the left.

Proof. See [15, Section 21.2]. □

4.5 Spectral Sequences for Cartan Models as Double Complexes

We begin with two lemmas concerning the induced G-action on the cohomology for a G-cochain complex.

Lemma 4.13. *Let (C, d) be a cochain complex equipped with a G-action that commutes with d. Then $C^G \hookrightarrow C$ induces an isomorphism $H(C^G) \xrightarrow{\cong} H(C)^G$.*

Proof. Let μ be the Haar measure on G. For any $a \in C$, let

$$\bar{a} = \int_G (g \cdot a) d\mu$$

denote the average of a over G, which is a G-invariant element. Clearly $\overline{da} = d\bar{a}$.

Injectivity: Let c be a G-invariant coboundary of (C, d). Then $c = db$ for some $b \in C$. Therefore we have $c = \bar{c} = \overline{db} = d\bar{b}$.

Surjectivity: Let c be a cocycle in (C, d) with $[c] \in H(C)^G$. Then $[g \cdot c] = g \cdot [c] = [c]$ for all $g \in G$, and we find that $[c] = [\bar{c}]$ is in the image of $H(C^G) \to H(C)^G$. □

Lemma 4.14. *If A is a G^* module and G is connected, then the induced G-action on $H(A)$ is trivial.*

Proof. Let $c \in A$ be a cocycle. It suffices to prove $\mathcal{L}_X c$ is a coboundary for any $X \in \mathfrak{g}$. For this we compute that $\mathcal{L}_X c = [\iota_X, d]c = d(\iota_X c)$. □

Now we come back to the main construction. The Cartan model $C_G(A)$ for a G^* module A can be given a bigrading such that $(C_G(A), d_G)$ is the total complex.

Exercise 4.15. The operators $d = 1 \otimes d$ and $\delta = -u^i \otimes \iota_i$ both act on $C_G(A)$.

Define $C^{p,q} = (S^{2p}(\mathfrak{g}^*) \otimes A^{q-p})^G$, then $(C^{*,*}, d, \delta)$ is a first quadrant double complex. By Example 1.20, the spectral sequence for this double complex has E_1 term given by

$$E_1^{p,q} = H^{p,q}(C^{*,*}, d). \tag{14}$$

Apply Lemma 4.13 to the cochain complex $(S(\mathfrak{g}^*) \otimes A, d)$, we see that $H(C^{*,*}, d) = (H(S(\mathfrak{g}^*) \otimes A, d))^G = (S(\mathfrak{g}^*) \otimes H(A))^G$. Now (14) becomes

$$E_1^{p,q} = (S^{2p}(\mathfrak{g}^*) \otimes H^{q-p}(A))^G. \tag{15}$$

In the case G acts on $H(A)$ trivially, for example when G is connected (cf. Lemma 4.14), this becomes

$$E_1^{p,q} = S^{2p}(\mathfrak{g}^*)^G \otimes H^{q-p}(A). \tag{16}$$

Theorem 4.16. *Suppose a G^* module (resp. G^* algebra) morphism $A \to B$ induces isomorphism in cohomology, then it also induces isomorphism in G-equivariant cohomology.*

Proof. By (15), the map $C_G(A) \to C_G(B)$ induces an isomorphism in the E_1 term of spectral sequences of corresponding double complexes. By Corollary 1.19, it also induces isomorphism in cohomology of total complexes. □

4.6 A Proof of Equivariant de Rham Theorem

In this section we prove Theorem 3.8. By naturality, it suffices to prove Theorem 3.7. Our proof essentially follows [7, Section 2.5], with some details filled in.

Choose integer n and an embedding $G \hookrightarrow O(n)$ of Lie groups. Let

$$EG = V_n(\mathbb{R}^\infty) = \varinjlim V_n(\mathbb{R}^k),$$

$$E = \Omega(V_n(\mathbb{R}^\infty)) = \varprojlim \Omega(V_n(\mathbb{R}^k))$$

be defined as in Section 2.3, Section 3.3, respectively.

We will justify that

$$\begin{aligned}
H_G^*(M) &= H^*(V_n(\mathbb{R}^\infty) \times_G M) & (17)\\
&= \varprojlim H^*(V_n(\mathbb{R}^k) \times_G M) & (18)\\
&= \varprojlim H_{bas}(\Omega(V_n(\mathbb{R}^k) \times M)) & (19)\\
&= H_{bas}(\Omega(V_n(\mathbb{R}^\infty) \otimes M)) & (20)\\
&= H_{bas}(\Omega(V_n(\mathbb{R}^\infty)) \otimes \Omega(M)) & (21)\\
&= H_{G,dR}^*(M). & (22)
\end{aligned}$$

By carefully tracking the maps one will see that all these identifications are natural. This will finish the proof for the equivariant de Rham theorem.

(17)(22): These are just definitions.

(19): For $k \geq n$, G acts on $V_n(\mathbb{R}^k)$ freely. Therefore $V_n(\mathbb{R}^k) \times M \to V_n(\mathbb{R}^k) \times_G M$ is a principal G-bundle and the equality follows from (10).

(18): We have seen in Section 3.3 that for each q, $\pi_q(V_n(\mathbb{R}^k))$ stabilizes to $\pi_q(V_n(\mathbb{R}^\infty))$ as $k \to \infty$. Since $\pi_q(X \times Y) = \pi_q(X) \times \pi_q(Y)$ for any spaces X, Y, we see that $\pi_q(V_n(\mathbb{R}^k) \times M)$ stabilizes to $\pi_q(V_n(\mathbb{R}^\infty) \times M)$. Now the homotopy long exact sequence for corresponding fiber sequences and the five lemma imply that $\pi_q(V_n(\mathbb{R}^k) \times_G M)$ stabilizes to $\pi_q(V_n(\mathbb{R}^\infty) \times_G M)$. Therefore $H^q(V_n(\mathbb{R}^k) \times M)$ stabilizes to $H^q(V_n(\mathbb{R}^\infty) \times M)$ [14, Theorem 7.5.9].

(20): We should first explain the notation: $V_n(\mathbb{R}^\infty) \times M$ is an infinitely dimensional manifold defined by the filtration $\{V_n(\mathbb{R}^k) \times M\}$ and $\Omega(V_n(\mathbb{R}^\infty) \times M)$ is its de Rham complex (cf. Section 3.3).

Clearly $\Omega_{bas}(V_n(\mathbb{R}^\infty) \times M) = \varprojlim \Omega_{bas}(V_n(\mathbb{R}^k) \times M)$. Since for each q, $H^q(\Omega_{bas}(V_n(\mathbb{R}^k) \times M)) = H^q(V_n(\mathbb{R}^k) \times_G M)$ stabilizes as $k \to \infty$, as explained above, Lemma 3.17 justifies the desired equality.

(21): The projection maps from $V_n(\mathbb{R}^\infty) \times M$ onto the two factors give rises to an inclusion

$$\Omega(V_n(\mathbb{R}^\infty)) \otimes \Omega(M) \hookrightarrow \Omega(V_n(\mathbb{R}^\infty) \times M). \tag{23}$$

Since $\Omega(V_n(\mathbb{R}^\infty))$ is regular, so are $\Omega(V_n(\mathbb{R}^\infty)) \otimes \Omega(M)$ and $\Omega(V_n(\mathbb{R}^\infty) \times M)$ (cf. Exercise 3.9). Therefore Corollary 3.11 shows that we may replace H_{bas} by H_G on both sides of (21). By Theorem 4.16, it suffices to show that (23) induces isomorphism in ordinary cohomology. By acyclicity of $\Omega(V_n(\mathbb{R}^\infty))$ and Künneth theorem, $\Omega(M) \hookrightarrow \Omega(V_n(\mathbb{R}^\infty)) \otimes \Omega(M)$ induces isomorphism in cohomology, so it remains to show

$$\Omega(M) \hookrightarrow \Omega(V_n(\mathbb{R}^\infty) \times M)$$

induces isomorphsm in cohomology. Since $H^q(V_n(\mathbb{R}^k) \otimes M)$ stabilizes to $H^n(M)$ as $k \to \infty$, Lemma 3.17 implies that $H^*(\Omega(V_n(\mathbb{R}^\infty) \times M)) = H^*(\Omega(M))$. The equality is justified.

5 Characteristic Classes

In this section, we develop the Chern-Weil theory for characteristic classes. Instead of working directly with connection and curvature forms in a principal bundle, our definition of the Chern-Weil map is on the algebraic level, using the language of equivariant de Rham cohomology we have developed. See also [7, Section 8]. The reader may consult [11, Chapter XII][13, Section 5, 6] for a classical development of Chern-Weil theory. Our presentation will blend with some topological perspectives. The reader may consult [12] for more details in topological development.

5.1 Characteristic Classes in Topology

In this section we define the notion of characteristic classes for (topological) principal G-bundles. We begin with some review of algebraic topology for classifying spaces.

Proposition 5.1. *Let $P \to B$ be any principal G-bundle. There is a map $f: B \to BG$ such that $P \to B$ is isomorphic to $f^*EG \to B$ as a principal G-bundle. The map f is unique up to homotopy.*

Proof. See [9, Section 10, 11]. □

In other words, the universal bundle $EG \to BG$ is the final object in the homotopy category of principal G-bundles (objects are principal G-bundles up to isomorphism that preserves base, morphisms are maps between bases up to homotopy; note that homotopic maps pullbacks to isomorphic bundles [9, Theorem 9.9]). The map f (or rather, the homotopy class of it) in the Proposition is called the **classifying map** for $P \to B$.

Let $Bun_G(B)$ denote the set of principal G-bundles over B up to isomorphism. Then Bun_G defines a contravariant functor $Bun_G \colon \mathrm{Ho}\mathbf{Top} \to \mathbf{Set}$.

Definition 5.2. *A* ***characteristic class*** *for (topological) principal G-bundles is a natural transformation*

$$c\colon Bun_G \to H^*. \tag{24}$$

Since $EG \to BG$ is final, any characteristic class is uniquely determined by its evaluation at $EG \to BG$. Therefore we obtain the following alternative definition for characteristic classes.

Definition 5.3. *A* ***characteristic class*** *for principal G-bundles is a cohomology class $c \in H^*(BG)$.*

Explicitly, given a natural transformation c as in (24), the corresponding $c \in H^*(BG)$ is given by the evaluation of c at the universal bundle $EG \to BG$. Conversely, given $c \in H^*(BG)$, the corresponding natural transformation c in (24) is given by $c(P \to B) = f^*c$, where f is the classifying map for $P \to B$. Alternatively, the G-map $P \to *$ induces a map

$$H^*(BG) = H^*_G(*) \to H^*_G(P) = H^*(B) \tag{25}$$

which maps c to $c(P \to B)$. We call this the (topological) **Chern-Weil** map for $P \to B$. Evaluation of c on G-bundle maps are defined to be the pullback map between bases. (To justify the alternative description, one may want to use the homotopy long exaxt sequence.)

5.2 Characteristic Classes for smooth Principal G-Bundles

We switch our attention back to differentiable category. The domain of functors Bun_G and H^* are restricted to the subcategory **Mfd**.

Definition 5.4. *A* ***characteristic class*** *for (smooth) principal bundles is a natural transformation between $Bun_G \to H^*$.*

Then, any characteristic class for topological principal G-bundles restricts to a characteristic class for smooth principal G-bundles. However, since there is no classifying object in **Mfd** like BG in **Top**, it is not immediate that the restriction is injective or surjective. Nevertheless, we have the following theorem due to H. Cartan.

Theorem 5.5. *A characteristic class for smooth principal G-bundles is the same as a characteristic class for topological principal G-bundles restricted to* **Mfd***.*

Proof. Using the Grassmaniann model as in Section 2.3, $EG \to BG$ can be realized as a principal G-bundle of infinite dimensional manifolds

$$EG = \varinjlim E_iG \to \varinjlim B_iG = BG$$

with $\pi_q(E_iG) = 0$ for $q \leq i$ (relabeling if necessary). Then B_iG classifies smooth principal G-bundles whose base space has dimension at most i (see [9, Section 10, 11]). Therefore, a characteristic class for smooth principal G-bundles can be identified as a compatible sequence of $c(E_iG \to B_iG) \in H^*(B_iG)$, which is the same as an element $c' \in \varprojlim H^*(B_iG) = H^*(BG)$ (the last equality is justified in the same way as for (18)). See also [4, Section 8]. $\square$

Let A be a G^* algebra. Recall that the inclusion $\mathbb{R} \to A$ induces a superalgebra map $S(\mathfrak{g}^*)^G = H_G(\mathbb{R}) \to H_G(A)$, making $H_G(A)$ a $S(\mathfrak{g}^*)^G$-superalgebra. This map is called the **Chern-Weil map** for A.

What we will be most interested in is the case $A = \Omega(P)$ where $P \to B$ is a (smooth) principal G-bundle, the Chern-Weil map for A is the induced map in $H^*_{G,dR}$ by $P \to *$, which is given by

$$S(\mathfrak{g}^*)^G = H^*_G(*) \to H^*_G(P) = H^*(B). \tag{26}$$

This is called the **Chern-Weil map** for the (smooth) principal G-bundle $P \to B$, which agrees with the map (25) under identification $S(\mathfrak{g}^*)^G = H^*_G(*) = H^*(BG)$.

Theorem 5.5 enables us to redefine characteristic classes as follows.

Definition 5.6. *A **characteristic class** for (smooth) principal G-bundles is an element of $S(\mathfrak{g}^*)^G$.*

Given an element $c \in S(\mathfrak{g}^*)^G$, the corresponding natural transformation $c\colon Bun_G \to H^*$ is just given by assigning each $P \to B$ the image of c under the Chern-Weil map (26), and each bundle map the pullback map in H^* between bases.

Since $S(\mathfrak{g}^*)^G$ is a superalgebra with only even elements, it is a graded commutative algebra over $\mathbb{R}$ in the usual sense.

The symmetric algebra $S(\mathfrak{g}^*)$ can be identified as the polynomial ring over $\mathfrak{g}$ (more precisely, over a set of basis $X_1, \cdots, X_n$ in $\mathfrak{g}$). Under this identification, G-invariant elements corresponds to Ad-invariant polynomials. Therefore, people often call $S(\mathfrak{g}^*)^G$ the (commutative) ring of invariant polynomials (with respect to G).

5.3 Chern Classes, Pontrjagin Classes, and the Euler Class

Roughly speaking, Chern Classes, Pontrjagin Classes, and Euler Classes are some well-chosen generators of the graded algebra of characteristic classes for different kinds of vector spaces. More precisely and less mysteriously, under the identification pointed out in the previous section, (a part of) Chern classes, (a part of) Pontrjagin classes, ((a part of) Pontrjagin classes + Euler class) freely generates $S(\mathfrak{g}^*)^G$ for $G = U(n), O(n), SO(2n)$, respectively. We will state the structural theorems for these $S(\mathfrak{g}^*)^G$ and refer readers to [11, Section XII.2] for proofs. But first of all, we establish a correspondence between characteristic classes of vector bundles and those of principal bundles.

All spaces in this section are smooth manifolds. However, if we replace the smooth Chern-Weil map (26) by the topological Chern-Weil map (25) in our discussions below, they make sense for general topological spaces as well.

5.3.1 Characteristic Classes for Vector Bundles

We first consider complex vector bundles. Let $Vect_n^{\mathbb{C}}\colon \mathbf{Mfd} \to \mathbf{Set}$ denote the contravariant functor sending a space to the isomorphism classes of complex n-bundles over it. Let $Vect_n^{\mathbb{C},Her}$ denote the functor sending a space to the isomorphism classes of complex n-bundles over it that are equipped with Hermitian metrics.

Exercise 5.7. The forgetful map induces a natural isomorphism $Vect_n^{\mathbb{C},Her} \xrightarrow{\cong} Vect_n^{\mathbb{C}}$.
(Hint: The existence of partition of unity equipped any vector bundle a Hermitian metric, which gives us an inverse of the given natural transformation. One need to check this inverse is well-defined. The space of Hermitian metrics over a fixed space is convex, which allows us to interpolate,

thus identify different choices naturally. One may want to make use of Gram-Schmidt orthonormalization.)

Suppose $E \to B$ is an n-dimensional complex vector bundle equipped with a Hermitian metric. Its associated unitary frame bundle $\mathcal{F}(E)$ is a principal $U(n)$-bundle over B. Conversely, let $P \to B$ be any principal $U(n)$-bundle, then $P \times_{U(n)} \mathbb{C}^n = (P \times \mathbb{C}^n)/U(n)$ is a complex n-bundle. These two constructions are inverses of each other: they give an inverse pair of natural isomorphisms between contravariant functors $Bun_{U(n)}$ and $Vect_n^{\mathbb{C},Her}$. Combine this with Exercise 5.7 we obtain a natural isomorphism

$$Vect_n^{\mathbb{C}} \xrightarrow{\cong} Bun_{U(n)},$$

which is given by choosing any metric and take the associated unitary frame bundle.

A **characteristic class** for complex n-bundles is a natural transformation $c\colon Vect_n^{\mathbb{C}} \to H^*$. Under the identification described above, this is the same as a characteristic class for principal $U(n)$-bundles.

Similarly, a **characteristic class** for real n-bundles is a natural transformation $c\colon Vect_n^{\mathbb{R}} \to H^*$, which is the same as a characteristic class for principal $O(n)$-bundles. A **characteristic class** for oriented real n-bundles is a natural transformation $c\colon Vect_n^{\mathbb{R},ori} \to H^*$, which is the same as a characteristic class for principal $SO(n)$-bundles.

Remark 5.8. The same identifications hold for topological bundles as well. Note we are under the assumption in Remark 2.1 so that partition of unity always exists.

5.3.2 Chern Classes

The Lie algebra $\mathfrak{u}(n)$ of $U(n)$ consists of $n \times n$ complex matrices X with $X + X^* = 0$. Let $c_k = c_k^{(n)}(X)$, $k = 0, 1, \cdots$ be polynomials in $\mathfrak{u}(n)$ defined by

$$\det\left(I - \frac{t}{2\pi i}X\right) = \sum c_k(X)t^k. \tag{27}$$

In particular, $c_0(X) = 1$, $c_1(X) = \frac{i}{2\pi}\mathrm{tr}(X)$, $c_n(X) = (\frac{i}{2\pi})^n \det(X)$, and $c_k(X) = 0$ for $k > n$. Clearly all c_k are Ad-invariant. Each c_k has polynomial degree k, thus is of degree $2k$ as an element in $S(\mathfrak{u}(n)^*)^{U(n)} \otimes \mathbb{C}$. The strange coefficient $\frac{1}{2\pi i}$ is taken so that c_1 satisfies a certain topological normalization which we will not come across.

Proposition 5.9. *All c_k are of real coefficient. Moreover, the ring of invariant polynomials, $S(\mathfrak{u}(n)^*)^{U(n)}$, is freely generated by $c_1, \cdots, c_n$ as an algebra over $\mathbb{R}$.* ☐

The element $c_k \in S^{2k}(\mathfrak{u}(n)^*)^{U(n)}$ is called the **k-th Chern class** for complex n-bundles. The element $c = c^{(n)} = \sum c_k = 1 + c_1 + \cdots + c_n$ is called the **total Chern class** for complex n-bundles.

Suppose $E \to B$ is a complex n-bundle. Equipped it with a metric and let $P = \mathcal{F}(E) \to B$ be the associated unitary frame bundle. The image of c_k under the Chern-Weil map (26), denoted $c_k(E)$, is called the **k-th Chern class** of $E \to B$. The image of c, denoted $c(E)$, is called the **total Chern class** of $E \to B$. As explained earlier, these element are independent of the choice of metric, and are natural in $E \to B$. Since the Chern-Weil map is degree-preserving, $c_k(E)$ is a cohomology class in $H^{2k}(B)$.

5.3.3 Pontrjagin Classes

The Lie algebra $\mathfrak{o}(n)$ of $O(n)$ consists of $n \times n$ real matrices X with $X + X^t = 0$. Let $p_k = p_k^{(n)}$, $k = 0, 1, \cdots$ be polynomials in $\mathfrak{o}(n)$ defined by

$$\det\left(I - \frac{t}{2\pi}X\right) = \sum p_k(X)t^{2k}. \tag{28}$$

Note this is well-defined, since $\det(I - \frac{t}{2\pi}X) = \det(I + \frac{t}{2\pi}X^t) = \det(I + \frac{t}{2\pi}X)$ implies that the coefficient of t^j on the left hand side is zero for odd j. By definition, we have $p_0(X) = 1$, $p_k(X) = 0$ for $k > n/2$. Each p_k is Ad-invariant, with polynomial degree $2k$.

Proposition 5.10. *The ring of invariant polynomials, $S(\mathfrak{o}(n)^*)^{O(n)}$, is freely generated by $p_1, \cdots, p_{[n/2]}$ as an algebra over $\mathbb{R}$.* □

The element $p_k \in S^{4k}(\mathfrak{o}(n)^*)^{O(n)}$ is called the **k-th Pontrjagin class** for real n-bundles.

Suppose $E \to B$ is a real n-bundle. Equip it with a metric and let $P = \mathcal{F}(E) \to B$ be the associated orthogonal frame bundle. The image of p_k under the Chern-Weil map (26), denoted $p_k(E)$, is called the **k-th Pontrjagin class** of $E \to B$, which is a cohomology class in $H^{4k}(B)$. We also have the notion of total Pontrjagin class.

5.3.4 The Euler Class

The group $SO(n)$ is the identity component of $O(n)$. Therefore the Lie algebra $\mathfrak{so}(n)$ of $SO(n)$ is just $\mathfrak{o}(n)$.

If $n = 2m$ is even, there is a polynomial Pf in $\mathfrak{so}(n)$ defined by

$$Pf(X) = \frac{1}{2^m m!} \sum_{\sigma \in S_{2m}} sgn(\sigma) X_{\sigma(1)\sigma(2)} X_{\sigma(3)\sigma(4)} \cdots X_{\sigma(2m-1)} X_{\sigma(2m)},$$

called the **Pfaffian**. Define

$$e = e^{(n)} = \frac{1}{(2\pi)^m} Pf.$$

Exercise 5.11. Show that

(1) $Pf(AXA^{-1}) = \det(A)Pf(X)$ for $A \in O(n)$. Therefore $Pf \in S(\mathfrak{so}(n))^{SO(n)}$.

(2) $Pf(X)^2 = \det(X)$. Therefore $e^2 = p_m$.

Proposition 5.12. *(1) If $n = 2m-1$ is odd, then the ring of invariant polynomials, $S(\mathfrak{so}(n)^*)^{SO(n)}$, is freely generated by $p_1, \cdots, p_{m-1}$ as an algebra over $\mathbb{R}$.*

(2) If $n = 2m$ is even, then $S(\mathfrak{so}(n)^*)^{SO(n)}$ is freely generated by $p_1, \cdots, p_{m-1}, e$ as an algebra over $\mathbb{R}$. □

In the case $n = 2m$, the element $e \in S^n(\mathfrak{so}(n)^*)^{SO(n)}$ is called the **Euler class** for oriented real n-bundles. In the case $n = 2m - 1$, we define the **Euler class** $e = e^{(n)}$ for oriented real n-bundles to be the zero element in $S(\mathfrak{so}(n)^*)^{SO(n)}$.

Suppose $E \to B$ is an oriented real n-bundle. Equip it with a metric and let $P = \mathcal{F}(E) \to B$ be the associated special orthogonal frame bundle. The image of e under the Chern-Weil map (26), denoted $e(E)$, is called the **Euler class** of $E \to B$, which is a cohomology class in $H^n(B)$. It is zero if n is odd.

5.4 Reduction of Structure Group

5.4.1 Reduction of Structure Group in Equivariant Cohomology

Let $H \hookrightarrow G$ be an embedding of compact Lie groups. Any G-space M is also an H-space. We now define a reduction map

$$H_G^*(M) \to H_H^*(M) \tag{29}$$

naturally in M, in both topological and smooth setups.

In the topological world, let $EG \to BG$ be a universal G-bundle. Then $EG \to EG/H = BH$ gives a universal H-bundle. We define (29) to be the composition

$$H_G^*(M) = H^*(EG \times_G M) \to H^*(EG \times_H M) = H_H^*(M).$$

In the smooth world, let E be an acyclic regular G^* algebra, which is also an acyclic regular H^* algebra (cf. Exercise 3.16). We define (29) to be the composition

$$H_G^*(M) = H((E \otimes \Omega(M))_{bas_{G^*}}) \to H((E \otimes \Omega(M))_{bas_{H^*}}) = H_H^*(M).$$

Here the subscript in *bas* indicate which group we are taking basic element with respect to.

Exercise 5.13. (1) Show that under the identification $H_G^* = H_{G,dR}^*$, the identifications in Proposition 2.5 and Proposition 3.12 are the same.

(2) Check that for a G-manifold M, the two definitions above for (29) agree.
(Hint: One may want to use an infinite dimensional manifold model for $EG \to BG$)

In the smooth setup, we can be more explicit about the reduction map (29). Take the acyclic regular G^* algebra E to be the Weil algebra $W(\mathfrak{g})$. The restriction map $\mathfrak{g}^* \to \mathfrak{h}^*$ induces the unique H^* algebra morphism $W(\mathfrak{g}) \to W(\mathfrak{h})$. Therefore (29) can be naturally identified as the composition

$$H_G^*(M) = H((W(\mathfrak{g})\otimes\Omega(M))_{bas_{G^*}}) \to H((W(\mathfrak{g})\otimes\Omega(M))_{bas_{H^*}}) \to H((W(\mathfrak{h})\otimes\Omega(M))_{bas_{H^*}}) = H_H^*(M).$$

In terms of the Cartan model, the restriction $\mathfrak{g}^* \to \mathfrak{h}^*$ induces a map $\Omega_G(M) = (S(\mathfrak{g}^*)\otimes\Omega(M))^G \to (S(\mathfrak{h}^*) \otimes \Omega(M))^H = \Omega_H(M)$, which in turn induces the map (29).

5.4.2 Chern-Weil Map and Reduction of Structure Group

In this section we examine the compatibility of Chern-Weil map (25) with changes of structure group. Readers may consult [5, Section 4] for basic notions about extension and reduction of principal bundles.

Let $H \hookrightarrow G$ be as before. Let $P \to B$ be a principal G-bundle, and $Q \to B$ a principal H-bundle which is a reduction of $P \to B$, i.e. there is an H-equivariant inclusion $Q \hookrightarrow P$ compatible with the projection maps. It is natural to guess that the Chern-Weil maps for these two bundles are compatible.

Proposition 5.14. *The diagram*
$$\begin{tikzcd} S(\mathfrak{g}^*)^G \arrow[r, phantom, "="] \arrow[d] & H^*(BG) \arrow[d] \arrow[dr] & \\ & & H^*(B) \\ S(\mathfrak{h}^*)^H \arrow[r, phantom, "="] & H^*(BH) \arrow[ur] & \end{tikzcd}$$
is commutative.

The commutativity of the square on the left is justified in the previous section. Since the topological Chern-Weil map is the pullback by classifying map, it remains to prove the following proposition.

Proposition 5.15. *Let $B \to BH$ be the classifying map for $Q \to B$, then the composition map $B \to BH \to BG$ is the classifying map for $P \to B$.*

Proof. Let f denote the composition map $B \to BH \to BG$. The universal property of pullback yields a dotted map in the following commutative diagram:

$$\begin{tikzcd}
Q \arrow[dr, dashed] \arrow[drrr] \arrow[ddr] & & & \\
& f^*BG \arrow[rr] \arrow[d] & & E \arrow[d] \arrow[dl] \\
& B \arrow[r] & BH \arrow[r] & BG,
\end{tikzcd}$$

which implies f^*BG is an extension of Q, thus is isomorphic to P as principal G-bundles. □

5.4.3 Chern Classes and Pontrjagin Classes

In this section we examine the restriction map $r = r_{U(n),O(n)} \colon S(\mathfrak{u}(n)^*)^{U(n)} \to S(\mathfrak{o}(n)^*)^{O(n)}$.

The Chern classes c_k are defined by the polynomial equation (27) in $X \in \mathfrak{u}(\mathfrak{n})$. Restricting this to a polynomial equation in $X \in \mathfrak{o}(\mathfrak{n})$ and use (28), we obtain

$$\sum r(c_k)(X)t^k = \det\left(I - \frac{t}{2\pi i}X\right) = \sum p_k(X)(-it)^{2k}.$$

Compare coefficient, we see that

$$r(c_{2k}) = (-1)^k p_k,\ r(c_{2k+1}) = 0,\ k = 0, 1, \cdots. \tag{30}$$

These relations completely determine r. Now we turn to geometric interpretation in characteristic classes of vector bundles. Let $E \to B$ be a real n-bundle equipped with a metric. Then $E \otimes \mathbb{C} \to B$ is a complex n-bundle equipped with an induced Hermitian metric. Moreover, the orthogonal frame bundle of E is a reduction of the unitary frame bundle of $E \otimes \mathbb{C}$ to $O(n)$. Therefore Proposition 5.14 and (30) implies that

$$c_{2k}(E \otimes \mathbb{C}) = (-1)^k p_k(E),\ c_{2k+1}(E \otimes \mathbb{C}) = 0,\ k = 0, 1, \cdots.$$

5.4.4 The Euler Class and the Top Chern Class

Regarding $\mathbb{C}^n$ as $\mathbb{R}^{2n}$ by breaking each complex coordinate $z_k = x_k + iy_k$ into two adjacent real coordinates x_k, y_k, we obtain an inclusion $j \colon \mathbb{C}^{n\times n} \hookrightarrow \mathbb{R}^{2n\times 2n}$ by replacing each entry $z = x + iy$ by a matrix $\left(\begin{smallmatrix} x & -y \\ y & x \end{smallmatrix}\right)$.

Exercise 5.16. For any $A \in \mathbb{C}^{n\times n}$, we have $\det(j(A)) = |\det(A)|^2$.

The map j restricts to a map $U(n) \hookrightarrow SO(2n)$. The induced Lie algebra map is also given by a restriction of j. In this section we examine the induced restriction map $r = r_{SO(2n),U(n)} \colon S(\mathfrak{so}(2n)^*)^{SO(2n)} \to S(\mathfrak{u}(n)^*)^{U(n)}$.

Take $X = j(A)$ in (28) for $A \in \mathfrak{u}(n)$, we obtain

$$\begin{aligned}\sum r(p_k)(A)t^{2k} &= \det\left(I_{2n} - \frac{t}{2\pi}j(A)\right) = \det\left(j\left(I_n - \frac{t}{2\pi}A\right)\right) \\ &= \left|\det\left(I_n - \frac{t}{2\pi}(A)\right)\right|^2 = \left|\sum c_k(A)(it)^k\right|^2.\end{aligned}$$

Compare coefficient and we obtain that

$$r(p_k) = \sum_{\ell}(-1)^{k-\ell}c_\ell c_{2k-\ell}. \tag{31}$$

We still need to determine $r(e)$. We compute that $r(Pf)(A) = Pf(j(A)) = i^n \det(A)$, therefore

$$r(e) = \frac{1}{(2\pi)^n}r(Pf) = \frac{1}{(2\pi)^n}Pf \circ j = \left(\frac{i}{2\pi}\right)^n \det = c_n. \tag{32}$$

Exercise 5.17. Check that $Pf(j(A)) = i^n \det(A)$ for any $A \in \mathfrak{u}(n)$.

Geometrically, let $E \to B$ be a complex n-bundle, which is automatically an oriented real $2n$-bundle by breaking each complex coordinate into two adjacent real ones in the usual way. Then (31)(32) and similar argument as in the previous section imply

$$p_k(E) = \sum_{\ell}(-1)^{k-\ell}c_\ell(E)c_{2k-\ell}(E)$$

and

$$e(E) = c_n(E).$$

5.4.5 Whitney Sum Formulae

Choose $p, q \geq 0$ with $p+q = n$. There is an embedding $U(p)\times U(q) \hookrightarrow U(n)$ defined in the usual way. We examine the restriction map $r = r_{U(n),U(p)\times U(q)} \colon S(\mathfrak{u}(n)^*)^{U(n)} \to S((\mathfrak{u}(p) \oplus \mathfrak{u}(q))^*)^{U(p)\times U(q)} = S(\mathfrak{u}(p)^*)^{U(p)} \otimes S(\mathfrak{u}(q)^*)^{U(q)}$.

Restrict the value of indeterminant $X \in \mathfrak{u}(n)$ in (27) to block matrices $\left(\begin{smallmatrix} Y & 0 \\ 0 & Z \end{smallmatrix}\right)$, $Y \in \mathfrak{u}(p), Z \in \mathfrak{u}(q)$. We obtain

$$\sum r(c_k^{(n)})(X)t^k = \det\left(I_p - \frac{1}{2\pi i}Y\right)\det\left(I_q - \frac{1}{2\pi i}Z\right) = \left(\sum c_i^{(p)}t^i\right)\left(\sum c_j^{(q)}t^j\right).$$

Take $t = 1$, we can rewrite this in terms of total Chern classes as

$$r(c^{(n)}) = c^{(p)}c^{(q)}.$$

Compare each degree we obtain more concrete formula

$$r(c_k^{(n)}) = \sum_{i+j=k} c_i^{(p)}c_j^{(q)}.$$

Any of these three equivalent formulae is known as the **Whitney sum formula** for Chern classes.

Geometrically, if a complex n-bundle $E \to B$ splits to a direct sum of a p-bundle and a q-bundle, say $E = E_1 \oplus E_2$, then the formulae above translates to the Whitney sum formula for Chern classes of complex vector bundles:

$$c(E_1 \oplus E_2) = c(E_1)c(E_2),$$

and two equivalent formulations.

Similarly, by examining the reduction map $S(\mathfrak{o}(n)^*)^{O(n)} \to S(\mathfrak{o}(p)^*)^{O(p)} \otimes S(\mathfrak{o}(q)^*)^{O(q)}$ we obtain the Whitney sum formula for Pontrjagin classes of real vector bundles:

$$p(E_1 \oplus E_2) = p(E_1)p(E_2).$$

By examining $S(\mathfrak{so}(n)^*)^{SO(n)} \to S(\mathfrak{so}(p)^*)^{SO(p)} \otimes S(\mathfrak{so}(q)^*)^{SO(q)}$ we obtain the Whitney sum formula for Euler classes of oriented real vector bundles

$$e(E_1 \oplus E_2) = e(E_1)e(E_2).$$

Alternatively, these two formulae follow from the Whitney sum formula for Chern classes and the relation between Chern classes and Pontrjagin/Euler classes described in Section 5.4.3, 5.4.4.

5.5 The Splitting Principle

As an application of equivariant cohomology theory we have developed, we prove the following theorem, whose topological version is known as the splitting principle. This section follows [7, Section 6.8, 8.6].

Theorem 5.18. *For any (smooth) complex vector bundle $E \to B$, there exists a manifold $Fl(E)$ equipped with a map $f\colon Fl(E) \to B$, such that*

(1) The pullback map $f^*\colon H^*(B) \to H^*(Fl(E))$ is injective.

(2) The pullback bundle f^*E splits into direct sum of line bundles.

Such manifold $Fl(E)$ is called a **splitting manifold** for $E \to B$.

The splitting principle implies that if we want to check some algebraic relations between Chern classes of a complex vector bundle, by replacing it with the pullback bundle over its split manifold if necessary, we may as well assume that this bundle splits into line bundles.

Proof. Let n be the dimension of $E \to B$. Let $T \subset U(n)$ be the subgroup of diagonal unitary matrices. Then $S(\mathfrak{t}^*)^T = S(\mathfrak{t}^*) = \mathbb{R}[t_1, \cdots, t_n]$ is the polynomial ring in n variables $t_1, \cdots, t_n$, each of degree 2. Let N be the normalizer of T in $U(n)$. Then $\Sigma_n = N/T$ can be identified with the permutation group of $\{t_1, \cdots, t_n\}$. Since the restriction map $S(\mathfrak{u}(n)^*)^{U(n)} \to S(\mathfrak{t}^*)$ maps each Chern class c_k to some nonzero constant times the k-th symmetric polynomial function in $t_1, \cdots, t_n$, we see that the restriction map gives an isomorphism

$$S(\mathfrak{u}(n))^{U(n)} \xrightarrow{\cong} S(\mathfrak{t}^*)^{\Sigma_n} = S(\mathfrak{n}^*)^N. \tag{33}$$

Let $\mathcal{F}(E)$ be the frame bundle of $E \to B$. Let $Fl(E) = \mathcal{F}(E)/T$ and let $f\colon Fl(E) \to B$ be the projection map. We check that $(Fl(E), f)$ satisfies our requirements.

We first check (2). A point $p \in Fl(E)$ is an equivalence class $p = [(e_1, \cdots, e_n)]$ of unitary frame in the fiber E_x of $E \to B$, where $x = f(p) \in B$, which is the same as an ordered direct sum decomposition $E_x = L_1 \oplus \cdots \oplus L_n$ into line bundles. This induces a decomposition of the fiber in the pullback bundle f^*E above p into ordered direct sum of line bundles. The decompositions at all $p \in Fl(E)$ patches to a splitting of f^*E.

It remains to check (1). The map $f^*\colon H^*(B) \to H^*(Fl(E))$ is naturally identified with the reduction map

$$H^*_{U(n)}(\mathcal{F}(E)) \to H^*_N(\mathcal{F}(E)) \to H^*_T(\mathcal{F}(E)). \tag{34}$$

In terms of Cartan model for the $U(n)^*$ algebra $A = \Omega(\mathcal{F}(E))$, this composition map is induced by the composition

$$C_{U(n)}(A) \to C_N(A) \to C_T(A) \tag{35}$$

The second map in (35) is the inclusion $C_T(A)^{\Sigma_n} \hookrightarrow C_T(A)$. By Lemma 4.13, it induces isomorphism $H_N(A) \xrightarrow{\cong} H_T(A)^{\Sigma_n}$.

The first map in (35) induces a map in each terms E_r in the corresponding spectral sequences constructed in Section 4.5. Since $U(n)$, and therefore N, acts on $H(A)$ trivially by Lemma 4.14, the induced map in E_1 term is given by (cf. (16))

$$S(\mathfrak{u}(n)^*)^{U(n)} \otimes A \to S(\mathfrak{n}^*)^N \otimes A,$$

which is an isomorphism by (33). By Corollary 1.19, the induced map in cohomology is also an isomorphism.

In conclusion, (34) factors as

$$H^*_{U(n)}(\mathcal{F}(E)) \xrightarrow{\cong} H^*_N(\mathcal{F}(E)) \xrightarrow{\cong} H^*_T(\mathcal{F}(E))^{\Sigma_n} \hookrightarrow H^*_T(\mathcal{F}(E)),$$

which is an injection, as desired. □

6 Localization for Torus Actions

Thoughout this section, let T denote a torus, M denotes a T-manifold, and $F := M^T$ denotes the fixed point set of M. By a usual argument [15, Theorem 25.1][7, Proposition 10.9.1], each connected component of F is a closed embedded submanifold of M of even codimension. In this section we will describe, in two ways, that some useful information of M is actually captured by the behavior of M at or near F.

This section follows and generalizes [15, Chapter IV, V], which deals with the case $T = S^1$. Many results in this section can be generalized into ones for general compact Lie group G. We will not mention such generalizations. Interested readers may consult [7, Section 10, 11] for further information.

6.1 Localized Equivariant Cohomology

The construction of localized T-equivariant cohomology is algebraic. For basic notions and properties of localization of a ring, we refer readers to any introductory book in commutative algebra, e.g. [1].

Recall that the space of T-equivariant differential forms on M is given by the ring $\Omega_T(M) = (S(\mathfrak{t}^*) \otimes \Omega(M))^T = S(\mathfrak{t}^*) \otimes \Omega(M)^T$, which is an $S(\mathfrak{t}^*)$-superalgebra. Its cohomology, $H_T^*(M)$, is also an $S(\mathfrak{t}^*)$-superalgebra.

Let $F(\mathfrak{t}^*)$ denote the fraction field of $S(\mathfrak{t}^*)$, which inherit a $\mathbb{Z}$-grading in the usual way. Localize $\Omega_T(M)$ at the multiplicative subset $S = S(\mathfrak{t}^*)\backslash\{0\}$ of $S(\mathfrak{t}^*)$, we obtain the $F(\mathfrak{t}^*)$-superalgebra $\Omega_{T,loc}(M) := S^{-1}\Omega_T(M) = F(\mathfrak{t}^*) \otimes \Omega(M)^T$. Since localization commutes with cohomology, its cohomology is the $F(\mathfrak{t}^*)$-superalgebra $H^*_{T,loc}(M) := S^{-1}H_T^*(M)$, which is called the **localized T-equivariant cohomology** of M.

6.2 Equivariant Mayer-Vietoris Sequence

Recall that in ordinary de Rham cohomology theory, if $\{U, V\}$ is an open cover of M, then we have a short exact sequence of cochain complexes

$$0 \to \Omega(M) \to \Omega(U) \oplus \Omega(V) \to \Omega(U \cap V) \to 0.$$

If U, V are both G-invariant, by averaging over T, one sees that

$$0 \to \Omega_T(M) \to \Omega_T(U) \oplus \Omega_T(V) \to \Omega_T(U \cap V) \to 0,$$

is also short exact. We obtain

Theorem 6.1 (Mayer-Vietoris Sequence for Equivariant Cohomology)**.** *There is a long exact sequence*

$$\cdots \to H_T^q(M) \to H_T^q(U) \oplus H_T^q(V) \to H_T^q(U \cap V) \to H_T^{q+1}(M) \to \cdots . \qquad \square$$

Alternatively, one can resort to topological argument to prove Theorem 6.1, which might be more straightforward: the homotopy quotients U_T, V_T is an open cover of M_T. Therefore the usual Mayer-Vietoris sequence for $M_T = U_T \cup V_T$ implies the equivariant version above. Note that the fact T is abelian is not necessary.

Since localization preserves exactness, we obtain

Corollary 6.2 (Mayer-Vietoris Sequence for Localized Equivariant Cohomology)**.** *There is a long exact sequence*

$$\cdots \to H^q_{T,loc}(M) \to H^q_{T,loc}(U) \oplus H^q_{T,loc}(V) \to H^q_{T,loc}(U \cap V) \to H^{q+1}_{T,loc}(M) \to \cdots . \qquad \square$$

6.3 Borel Localization Theorem

Lemma 6.3. *Let $T = S^1 \oplus T'$ be a decomposition of T into subtori. If $M^{S^1} = \emptyset$, then $H^*_{T,loc}(M) = 0$.*

Proof. Let $X_1, \cdots, X_n$ be a basis for $\mathfrak{t}$ with $X_1 \in \mathfrak{s}^1$, and $u^1, \cdots, u^n$ be the dual basis, as elements in $S^2(\mathfrak{t}^*)$.

By assumption, the associated vector field $\underline{X_1}$ is nonvanishing. By using a partition of unity and averaging we can find a T-invariant element $\theta \in \Omega^1(M)$ such that $\iota_1\theta = 1$. Then, $t \cdot (\iota_i\theta) = \iota_i(t \cdot \theta) = \iota_i\theta$ for all $i = 1, \cdots, n$ and $t \in T$. Thus each ι_i is an invariant element in $\Omega^0(M) = \mathbb{R}$. Therefore $\ell_i = \iota_i\theta$ are constants with $\ell_1 = 1$.

Let

$$\alpha = \frac{\theta}{\ell_i u^i} \in \Omega^{-1}_{T,loc}(M).$$

Then we find that

$$d_T\alpha = \frac{d\theta - u^i \otimes \iota_i\theta}{\ell_i u^i} = \frac{d\theta}{\ell_i u^i} - 1$$

is an invertible element in $\Omega^0_{T,loc}(M)$ with inverse

$$\lambda = -\left(1 + \frac{d\theta}{\ell_i u^i} + \frac{(d\theta)^2}{(\ell_i u^i)^2} + \cdots + \frac{(d\theta)^m}{(\ell_i u^i)^m}\right),$$

where m is any integer with $2m \geq \dim(M) - 1$. Since $\iota_i d\theta = \mathcal{L}_i\theta - d\iota_i\theta = 0$ for all i, we see that $d_T\lambda = 0$. It follows that the element $\beta = \lambda\alpha \in \Omega^{-1}_{T,loc}(M)$ satisfies $d_T\beta = 1$.

Now we see any cocycle $c \in \Omega_{T,loc}(M)$ is also a coboundary since $c = d_T(\beta c)$. It follows that $H^*_{T,loc}(M) = 0$. □

Theorem 6.4 (Borel Localization Theorem)**.** *The inclusion $F \hookrightarrow M$ induces isomorphism in localized T-equivariant cohomology.*

Proof. Use induction on $\dim(T)$. When $\dim(T) = 0$ the statement is trivial since $F = M$.

Suppose $\dim(T) > 0$. Write T as a direct sum of tori $S^1 \oplus T'$. Since S^1 and T' commute, T' acts on each component of M^{S^1} respectively. Since $(\Omega_T(M^{S^1}), d_T) = S((\mathfrak{s}^1)^*) \otimes (\Omega_{T'}(M^{S^1}), d_{T'})$ and $(\Omega_T(F), d_T) = S((\mathfrak{s}^1)^*) \otimes (\Omega_{T'}(F), d_{T'})$ as cochain complexes, by induction hypothesis we see that the inclusion $F \hookrightarrow M^{S^1}$ induces an isomorphism

$$H^*_{T,loc}(M^{S^1}) = F((\mathfrak{s}^1)^*) \otimes H^*_{T',loc}(M^{S^1}) \xrightarrow{\cong} F((\mathfrak{s}^1)^*) \otimes H^*_{T',loc}(F) = H^*_{T,loc}(F).$$

Therefore, it remains to show that $M^{S^1} \hookrightarrow M$ induces isomorphism in localized T-equivariant cohomology.

Let U be a T-equivariant tubular neighbhorhood of M^{S^1} [10, Theorem 4.4], i.e. a neighborhood of M^{S^1} that is T-equivariantly diffeomorphic to the normal bundle of M^{S^1} in M. Let $V = M\backslash M^{S^1}$. Then both V and $U \cap V$ are T-manifolds without any point fixed by S^1. The Mayer-Vietoris sequence for localized T-equivariant cohomology applied to $\{U, V\}$ and Lemma 6.3 give isomorphism $H^*_{T,loc}(M) \xrightarrow{\cong} H^*_{T,loc}(U)$. Finally, by linearly shrinking onto the base, we see that the inclusion of M^{S^1} into its normal bundle is a T-equivariant deformation retract. By Proposition 2.4, we conclude that $M^{S^1} \hookrightarrow U$ induces isomorphism in T-equivariant cohomology, thus also in localized T-equivariant cohomology. The statement follows. □

The following corollaries are immediate. Note that Lemma 6.3 is a special case of Corollary 6.6.

Corollary 6.5. *The inclusion $F \hookrightarrow M$ induces $H^*_{T,loc}(M) \xrightarrow{\cong} F(\mathfrak{t}^*) \otimes H^*(F)$.* □

Corollary 6.6. *If $F = \emptyset$, then $H^*_{T,loc}(M) = 0$.* □

6.4 Equivariant Integration

For any oriented T-manifold X possibly with boundary, the integral operator $\int_X$ on $\Omega(X)$ induces an integral operator, still denoted $\int_X$, on the space of localized T-equivariant forms on X, $\Omega_{T,loc}(X)$. Similar to the usual integral, we have

Proposition 6.7 (Equivariant Stokes' Formula)**.** *Let $i\colon \partial X \to X$ denote the inclusion map and $\theta \in \Omega_{T,loc}(X)$ be arbitrary, then*

$$\int_X d_T\theta = \int_{\partial X} i^*\theta.$$

Proof. Use a basis expression as in Proposition 4.12, write $d_T = 1 \otimes d - u^i \otimes \iota_i$. By the usual Stokes' formula we deduce that

$$\int_X (1 \otimes d)\theta = \int_{\partial X} i^*\theta.$$

By degree reason we deduce that

$$\int_X (u^i \otimes \iota_i)\theta = 0.$$

The statement follows. ☐

Borel localization theorem says that the localized T-equivariant cohomology of M concentrates at F. Therefore, any closed localized T-equivariant differential form supported away from F is exact. In particular its integral is zero by equivariant Stokes' formula. Therefore, one should expect a formula expressing the integral of a closed T-equivariant form on M in terms of its value near F. This is the topic for Section 6.6.

We end this section with one more equivariant notion. Let $\pi\colon E \to B$ be a T-equivariant fiber bundle, i.e. a fiber bundle whose projection map is T-equivariant. Suppose the base B is oriented and the fiber F is compact oriented of dimension m. By convention, the orientation on E is taken to be the "base first" orientation, i.e. TE has the product orientation $TB \oplus TF$.

Then, the usual fiber integration operator $\pi_*\colon \Omega^q(E) \to \Omega^{q-m}(B)$ induces a fiber integration operator $\pi_*\colon \Omega^q_{T,loc}(E) \to \Omega^{q-m}_{T,loc}(B)$. The usual formula

$$\int_E \pi^*\alpha \wedge \beta = \int_B \alpha \wedge \pi_*\beta \tag{36}$$

for $\alpha \in \Omega(B)$, $\beta \in \Omega(E)$ directly implies the validity of the same formula for $\alpha \in \Omega_{T,loc}(B)$, $\beta \in \Omega_{T,loc}(E)$.

6.5 Equivariant Euler Classes

In this section, we introduce the notion of equivariant Euler class for a T-equivariant oriented vector bundle and state without proof one of its important property. For a proof we refer readers to [6] or [7, Section 10]. The corresponding development for usual Euler classes is well-treated in [2].

Topologically, a principal G-bundle $P \to B$ is called a **T-equivariant principal G-bundle** if $P \to B$ is T-equivariant such that the actions of G, T on P commute. The homotopy quotient of such $P \to B$ by T is a principal G-bundle $P_T \to B_T$. By definition, a **T-equivariant characteristic class** for $P \to B$ is the image of a fixed element $c \in H^*(BG) = S(\mathfrak{g}^*)^G$ in $H^*(B_T) = H^*_T(B)$.

By the same development as in Section 5, we can define the equivariant Chern classes, equivariant Pontrjagin classes, and equivariant Euler classes for various equivariant vector bundles.

In the differentiable setting, the T-equivariant Euler class e^T for a T-equivariant oriented real n-bundle $E \to B$ can be represented by a T-equivariant n-form on B, also denoted e^T. Equip E with a T-invariant metric and let $\pi\colon S(E) \to B$ denote the unit sphere bundle of $E \to B$. Then $\pi^* e^T = -d_T\sigma$ for some $\sigma \in \Omega_T^{n-1}(B)$ that is a global volume form, i.e. a form σ such that the fiber integral $\pi_*\sigma$ is the constant 1 function on B.

6.6 Localization Formula

Theorem 6.8 (Localization Formula)**.** *Suppose M is oriented. For a closed form $\mu \in \Omega_{T,loc}(M)$, we have*

$$\int_M \mu = \sum_X \int_X \frac{i_X^*\mu}{e_X}. \tag{37}$$

Here X runs over the connected components of F, i_X is the inclusion of X into M, and e_X is the equivariant Euler class of NX, the normal bundle for $X \hookrightarrow M$.

We first make a few clarifications. If one is willing to work up to a sign, he/she may skip much discussions about orientation.

- In the integrand on the right hand side of (37), $1/e_X \in \Omega_{T,loc}(X)$ denotes the inverse of e_X, and $i_X^*\mu/e_X$ denotes the product of $i_X^*\mu$ and $1/e_X$, which is irrelevant of the order of multiplication since $1/e_X$ has even degree. See [7, Section 10.8] for a justification that e_X is invertible.
- By convention, the orientation of NX is taken such that TM has the "fiber first" orientation: the product orientation $NX \oplus TX$. Equivalently, TX has the "base first" orientation, since NX is even-dimensional.
- Any submanifold X is always orientable. Its orientation is irrelevent since reversing its orientation reverses the sign of both the integral and the equivariant Euler class e_X.

Proof. Suppose X is a full dimensional component, say with the same orientation as M. Then we have $e_X = 1$, and $M\backslash X$ is still a T-manifold. Therefore we may substract X from M without changing the truthness of the statement. Below we shall assume F has no full dimensional component.

Assume without loss of generality that μ is homogeneous. Note both side of (37) is zero if $|\mu|$ has different parity from $\dim(M)$. Below we assume $|\mu|$ and $\dim(M)$ have the same parity.

Let $NF \to F$ denote the normal bundle of F in M. Let $S(NF) \to F$ denote its unit sphere bundle (with respect to a T-invariant metric). Then $S(NF)$ is a T-manifold in the usual way, without fixed point. Equip $S(NF)$ with the usual ("base first") orientation as a fiber bundle.

We begin the proof with a spherical blow-up of M at the (union of) submanifold(s) F. By definition, this is an oriented T-manifold $\tilde{M}$ with boundary obtained by replacing F with $S(NF)$, equipped with a projection map $p\colon \tilde{M} \to M$. Then, p restricted to $Int(\tilde{M})$ is an orientation preserving T-diffeomorphism onto $M\backslash F$, and p restricted to $\partial\tilde{M}$ is the bundle projection $S(NF) \to F$. The induced orientation on $\partial\tilde{M}$ is determined by "outward normal first". This outward normal direction is in the fiber of $S(NF) \to F$, which moreover differs from the usual fiber outward normal by a sign. Consequently, we see that $p|_{\partial\tilde{M}}$ change the orientation by a sign of $(-1)^{\dim(M)+1}$ (again, note each component of F has even codimension).

By dimensional reason and change of variable formula we see that

$$\int_M \mu = \int_{M\setminus F} \mu = \int_{Int(\tilde{M})} p^*\mu = \int_{\tilde{M}} p^*\mu.$$

Let $i\colon S(NF) = \partial\tilde{M} \hookrightarrow \tilde{M}$ denote the inclusion. Since $i^*p^*\mu$ is a closed form on $S(NF)$, a T-manifold without fixed point, we know a priori by Theorem 6.4 that $i^*p^*\mu = d_T\alpha$ for some $\alpha \in \Omega_{T,loc}(S(NF))$. For a component X of F, let α_X denote the restriction of α to $S(NX)$. By equivariant Stokes' formula,

$$\int_{\tilde{M}} p^*\mu = (-1)^{\dim(M)+1}\int_{S(NF)} \alpha = (-1)^{\dim(M)+1}\sum_X \int_{S(NX)} \alpha_X.$$

It remains to show that

$$(-1)^{\dim(M)+1}\int_{S(NX)} \alpha_X = \int_X \frac{i_X^*\mu}{e_X}.$$

Let $\pi\colon S(NX) \to X$ denote the projection map and π_* denotes the fiber integration. Choose a global volume form $\sigma_X \in \Omega_T(S(NX))$ with $d_T\sigma_X = -\pi^*e_X$. Since μ is closed, we have

$$d_T\left(-\sigma_X \wedge \frac{\pi^* i_X^*\mu}{\pi^* e_X}\right) = -d_T\sigma_X \wedge \frac{\pi^* i_X^*\mu}{-d_T\sigma_X} = \pi^* i_X^*\mu,$$

which is the same as $i^*p^*\mu$ restricted to $S(NX)$. Therefore we may take

$$\alpha_X = -\sigma_X \wedge \frac{\pi^* i_X^*\mu}{\pi^* e_X} = (-1)^{\dim(M)+1}\pi^*\left(\frac{i_X^*\mu}{e_X}\right) \wedge \sigma_X.$$

Now (36) implies

$$(-1)^{\dim(M)+1}\int_{S(NX)} \alpha_X = \int_X \frac{i_X^*\mu}{e_X} \wedge \pi_*\sigma_X = \int_X \frac{i_X^*\mu}{e_X},$$

as desired. □

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