ON THE CONJUGACY OF CARTAN SUBALGEBRAS

QIUYU REN

1. Introduction

To classify all semisimple Lie algebras (finite dimensional, over an algebraically closed field of characteristic 0), one first goes through the root space decomposition. Then he classifies all irreducible root systems up to isomorphism. Next he may explicitly construct simple Lie algebras correspond to each of these irreducible root systems. Now finally he claims that he has classified all simple Lie algebras up to isomorphisms, and therefore all semisimple Lie algebras up to isomorphism are finite direct sums of these simple ones.

There is a flaw in this argument. It may happen that two simple Lie algebras correspond to two different root systems are isomorphic. To rule out this possibility, one way is to show that when we do the root space decomposition, different choices of maximal toral subalgebra give rises to isomorphic root systems.

We show that two different maximal toral subalgebras H, H' of a semisimple Lie algebra L are conjugate under the automorphism group of L (in fact, the inner automorphism group of L). This would be sufficient for this final step of classification theorem for semisimple Lie algebras.

We will also prove a generalization of this for any (not necessarily semisimple) Lie algebra L. Namely that the Cartan subalgebras of L are conjugate to each other. Our proof will follow that of Bourbaki [1].

In this article the base field k is always assumed to be algebraically closed with characteristic zero.

2. Conjugacy of maximal toral subalgebras

Let L be a finite dimensional semisimple Lie algebra. For a maximal toral subalgebra $H \subset L$, we have the root space decomposition

$$L = H \oplus \oplus_{\alpha \in R} L_\alpha,$$

where $R \subset H^*$ is the set of roots.

For any nilpotent element in L we write $e(x) = \exp(\operatorname{ad} x)$. Then the inner automorphism group of L, denoted $\operatorname{Inn} L$, is the subgroup of $\operatorname{Aut} L$ generated by all $e(x)$ with $\operatorname{ad} x$ nilpotent. Let $E(H) < \operatorname{Inn} L$ be the subgroup generated by $e(x)$, $x \in L_\alpha$ for some root α. The main theorem we are going to prove is:

Theorem 1. *Let H, H' be two maximal toral subalgebras of the finite dimensional semisimple Lie algebra L. Then*
(i) $E = E(H)$ is independent of choice of H.
(ii) $H' = \sigma H$ for some $\sigma \in E$.

We first do some preparations for the proof. An element $h \in H$ is said to be regular if $\alpha(h) \neq 0$ for all $\alpha \in R$. Let H_{reg} be the set of regular elements of H. Regard H as an affine space $\mathbb{A}^n(k)$ with the Zariski topology. Then each $\alpha \in R$ is a polynomial function (of degree 1). Since R is finite, we see that $H_{\mathrm{reg}} = \mathbb{A}^n \backslash \cup_{\alpha \in R} \{\alpha = 0\}$ is a nonempty open set. A basic fact in algebraic geometry we'll use is that:

Proposition 2. *Let $F\colon \mathbb{A}^N \to \mathbb{A}^N$ be a polynomial map such its differential dF is nonsingular at some point $P \in \mathbb{A}^N$. Then for any nonempty open set U, $F(U)$ contains a nonempty open set.*

For a proof, see Bourbaki [1, Chapter VII, App. I]. We now prove the main theorem.

Proof of Theorem 1. Let $L = H \oplus \oplus_{i=1}^r L_{\alpha_i}$ be the root space decomposition with respect to H. Let $F\colon H \times \prod_{i=1}^r L_{\alpha_i} = L \to L$, $(h, x_1, \cdots, x_r) \mapsto e(x_1) \cdots e(x_r)(h)$. Let $L'_{\alpha'_i}$, $i = 1, \cdots, r'$, $F'\colon L \to L$ be similarly defined with respect to H'. Claim: $F(H_{\mathrm{reg}} \times \prod_{i=1}^r L_{\alpha_i})$ contains a dense open set in L.

We have remarked that H_{reg} is a nonempty open set in H. It follows that $H_{\mathrm{reg}} \times \prod_{i=1}^r L_{\alpha_i}$ is a nonempty open set in L. By Proposition 2, it suffices to show that $dF|_P$ is nonsingular for some $P \in L$. Take any $h_0 \in H_{\mathrm{reg}}$, $P = (h_0, 0, \cdots, 0)$, we compute that

$$dF|_P(h, 0, \cdots, 0) = \left.\frac{d}{dt}\right|_{t=0} F(h_0 + th, 0, \cdots, 0) = \left.\frac{d}{dt}\right|_{t=0} (h_0 + th) = h;$$

$$dF|_P(0, 0, \cdots, x_i, \cdots, 0) = \left.\frac{d}{dt}\right|_{t=0} F(h_0, 0, \cdots, tx_i, \cdots, 0)$$
$$= \left.\frac{d}{dt}\right|_{t=0} \exp(\operatorname{ad} tx_i)(h_0) = \operatorname{ad} x_i(h_0) = -\operatorname{ad} h_0(x_i) = -\alpha_i(h_0)x_i.$$

Note the regularity of h_0 means $\alpha_i(h_0) \neq 0$. Therefore we see that $dF|_P$ is surjective, thus nonsingular. The claim is proved.

Back to our proof. By the same argument $F'(H'_{\mathrm{reg}} \times \prod_{i=1}^{r'} L'_{\alpha'_i})$ contains a nonempty open set in L. Now basic algebraic geometry tells us $F(H_{\mathrm{reg}} \times \prod_{i=1}^r L_{\alpha_i}) \cap F'(H'_{\mathrm{reg}} \times \prod_{i=1}^{r'} L'_{\alpha'_i}) \neq \emptyset$. Choose a common element $\sigma(h) = \sigma'(h')$, where $\sigma \in E(H)$, $\sigma' \in E(H')$, $h \in H_{\mathrm{reg}}$, $h' \in H'_{\mathrm{reg}}$. Take centralizer, we have

$$\sigma C_L(h) = C_L(\sigma(h)) = C_L(\sigma'(h')) = \sigma' C_L(h'). \tag{1}$$

But the regularity of h, h' implies that $C_L(h) = H$, $C_L(h') = H'$. Therefore $\sigma H = \sigma' H'$. (Note that up to this point we have already resolved the problem raised in the introduction.)

It remains to prove (i). This is immediate: Note $\sigma \circ e(x) \circ \sigma^{-1} = e(\sigma(x))$, we have $\sigma E(H) \sigma^{-1} = E(\sigma H)$. But $\sigma \in E(H)$, so $E(H) = \sigma E(H) \sigma^{-1} = E(\sigma H)$. Similarly $E(H') = E(\sigma' H')$, and (i) follows since $\sigma H = \sigma' H'$. □

3. Conjugacy of CSAs

In this section let L be any finite dimensional Lie algebra. A **Cartan Subalgebra (CSA)** of L is a nilpotent subalgebra that is the normalizer of itself in L. In semisimple case the definition turns out to coincide with that of a maximal toral subalgebra. For a direct proof see Bourbaki [1, Chapter VII, §2]. For an indirect proof, first note a maximal toral subalgebra is a CSA, then apply the two conjugacy theorems (see below)! (In particular, we'll need the independence of $E(H)$ on H)

For this arbitrary L, Theorem 1 is still true with "maximal toral subalgebra" replaced by "CSA". There is no significant difference for the proof but we need to generalize some concept into this setting. Here is a brief summation:

- Generalization of root space decomposition.
  If $H \subset L$ is a nilpotent subalgebra, then we have the following decomposition (Proposition 4):
  $$L = L^0 \oplus \oplus_{\alpha \in R} L^\alpha (= L^0(H) \oplus \oplus_{\alpha \in R} L^\alpha(H)),$$
  where $R \subset H^* \backslash \{0\}$, $L^\alpha = \{x \in L \colon (\operatorname{ad} h - \alpha(h))^n(x) = 0$ for some $n \geq 0\}$. R is such that $L^\alpha \neq \{0\}$ for $\alpha \in R$. This is called the primary decomposition of L with respect to H. Clearly $H \subset L^0$ since H is nilpotent. Moreover, if H is a CSA, then $H = L^0$ (Lemma 5).
- Generalization of a regular element.
  Let the notations be as above. An element $h \in H$ is said to be regular if $\alpha(h) \neq 0$ for all $\alpha \in R$. Let H_{reg} be the set of regular elements in H. Then H_{reg} is a nonempty open subset of H (the proof is similar as in semisimple case). Moreover, it is clear that $H = L^0(h) (= \{x \in L \colon (\operatorname{ad} h)^n(x) = 0$ for some $n \geq 0\})$ for any $h \in H_{\mathrm{reg}}$.
- Generalization of the automorphism group $E(H)$.
  Let the notations be as above. Let $E(H) < \operatorname{Inn} L$ be the subgroups generated by $e(x)$, $x \in L^\alpha$ for some $\alpha \in R$ (see Lemma 6).

One can now readily recover the proof of Theorem 1 for this general setting (but note in (1), one should take $L^0(\cdot)$ instead of centralizer). There are a few claims to check. Let's fill in the gaps.

Lemma 3. *Let V be a finite dimensional vector space over k, T, T' be linear operators on V. Let $V^a = \{x \in V \colon (T - a)^n x = 0$ for some $n \geq 0\}$. Suppose that $(\operatorname{ad} T)^m(T') = 0$ for some $m \geq 0$, then V^a is stable under T'.*

Proof. Let $d = \dim V$. By linear algebra we can also write $V^a = \{x \in V : (T-a)^d x = 0\}$.
We use induction on m to prove the lemma. When $m = 0$, we have $T' = 0$, there is nothing to prove.
Suppose $m > 0$. Let $x \in V^a$ be arbitrary. We have

$$\begin{aligned}(T-a)^{2d}T'x =& ((T-a)^{2d}T' - T'(T-a)^{2d})x \\ =& \sum_{i=1}^{2d}((T-a)^{2d-i}[T,T'](T-a)^{i-1})x.\end{aligned}$$

But $(\operatorname{ad} T)^{m-1}[T,T'] = (\operatorname{ad} T)^m(T') = 0$, so by induction hypothesis V^a is stable under $[T,T']$. Clearly V^a is also stable under $T - a$. This means $[T,T'](T-a)^{i-1}x \in V^a$ for all i. Now $((T-a)^{2d-i}[T,T'](T-a)^{i-1})x = 0$ no matter $i - 1 \geq d$ or $2d - i \geq d$. Hence $(T-a)^{2d}T'x = 0$, so $T'x \in V^a$ by definition. □

Proposition 4. *If $H \subset L$ is nilpotent subalgebra, then we have the primary decomposition*

$$L = L^0 \oplus \oplus_{\alpha \in R} L^\alpha,$$

Proof. The proof is similar to the usual root space decomposition (which is an application of simultaneous diagonalization). One need to show that the sum is direct and is all of L. The first part is straightforward: Suppose $x_\beta = \sum_\alpha x_\alpha$ is a nontrivial identity, then x_β is vanished by both $(\operatorname{ad} h - \beta(h))^m$ and $\prod_\alpha (\operatorname{ad} h - \alpha(h))^m$ for some large m. Since k is infinite we may choose h such that $\beta(h)$ and all $\alpha(h)$ are distinct. Then Bézout's theorem yields $x_\beta = 0$, a contradiction.

For the second part, first note that for any $h \in H$, linear algebra tells us that with respect to $\operatorname{ad} h$, V can be decomposed into V^a for some a's. Moreover, for any $h' \in H$ we have $(\operatorname{ad} \operatorname{ad} h)^m(\operatorname{ad} h') = \operatorname{ad}((\operatorname{ad} h)^m(h')) = 0$ for large m since H is nilpotent. Now Lemma 3 and induction on $\dim L$ finish the proof. □

Lemma 5. *Suppose H is a CSA of L, then $L^0 = H$.*

Proof. Let $h \in H$ be arbitrary. For every $x \in L^0$, we can find n such that $(\operatorname{ad} h)^n(x) = 0 \in H$. Since L is finite dimensional this n can be choosen to be uniform in x. Consider the adjoint representation of H on L_0/H, the above argument shows that every $\operatorname{ad} h$ for $h \in H$ is nilpotent. Suppose $L^0 \neq H$, then Engel's theorem yields an element $x \in L_0 \backslash H$ such that $\operatorname{ad} h(x) \in H$ for all $h \in H$. In other words x belongs to the normalizer of H, this contradicts the assumption that H is a CSA. □

Lemma 6. *(i) $[L^\alpha, L^\beta] \subset L^{\alpha+\beta}$ for any α, β; (ii) Every $x \in L^\alpha$ where $\alpha \in R$ is ad-nilpotent. Therefore $E(H) < \operatorname{Inn} L$ is well-defined.*

Proof. (i) Use induction on ℓ and the Jacobi identity, it is easily seen that for any $x, y, h \in L$, $a, b \in k$ we have

$$(\operatorname{ad} h - a - b)^{\ell}[x, y] = \sum_{i=0}^{\ell} \binom{\ell}{i} [(\operatorname{ad} h - a)^i x, (\operatorname{ad} h - b)^{\ell - i} y].$$

Now let $x \in L^{\alpha}$, $y \in L^{\beta}$, $h \in H$. Find m, n such that $(\operatorname{ad} h - \alpha(h))^m x = (\operatorname{ad} h - \beta(h))^n y = 0$. Then the above identity applied to $a = \alpha(h)$, $b = \beta(h)$ and $\ell = m + n$ yields $(\operatorname{ad} h - (\alpha + \beta)(h))^{m+n}[x, y] = 0$. Since h is arbitrary we see $[x, y] \in L^{\alpha+\beta}$. Since x, y are arbitrary the statement follows.
(ii) Since L is finite dimensional, we see R is finite. Now, (i) says that $(\operatorname{ad} x)^n (L^{\beta}) \subset L^{n\alpha+\beta} = 0$ for all $\beta \in R$, for sufficiently large n. Therefore $\operatorname{ad} x$ is nilpotent. □

Theorem 7. *Let H, H' be two CSAs of the finite dimensional Lie algebra L. Then*
(i) $E = E(H)$ is independent of choice of H.
(ii) $H' = \sigma H$ for some $\sigma \in E$.

Proof. Exercise. □

4. A comment on the group E

Recall $E(H) < \operatorname{Inn} L$ is defined as the subgroup generated by $e(x)$, $x \in L^{\alpha}$ for some $\alpha \in R$, and in our conjugacy theorem we have incidentally showed that $E = E(H)$ is independent of the choice of H. Now, E seems to possess a large part of $\operatorname{Inn} L$. Can we actually prove that $E = \operatorname{Inn} L$?

A little thought tells us that this is not true in general.

Example 8. Let L be a Lie algebra spanned by x, y, with $[x, y] = x$. Then L itself is a CSA. It follows that $E = E(L)$ is the trivial group. However, $e(x) = 1 + \operatorname{ad} x \in \operatorname{Inn} L$ is nontrivial since $e(x)(y) = x + y$. Therefore $E \neq \operatorname{Inn} L$.

However, this is true when L is semisimple.

Theorem 9. *If L is semisimple, then $E = \operatorname{Inn} L$.*

For a proof, one can see for example Fulton, Harris [2, Proposition D.40]. Note that the proof actually shows that the automorphism group $\operatorname{Aut} L$ is the semidirect product of $\operatorname{Inn} L$ and $\Gamma(L)$, where $\Gamma(L)$ is the automorphism groups of the Dynkin diagram associated to the root system of L.

References

[1] N. Bourbaki, *Lie groups and Lie algebras. Chapters 7-9*, Springer-Verlag, 2004
[2] W. Fulton, J. Harris, *Representation Theory*, Springer-Verlag, New York, 1991