COMPLEX ANALYTIC PROOFS OF BERNSTEIN'S THEOREM

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1. Introduction

In this paper we present three complex analytic approaches to prove Bernstein's theorem.

Theorem (Bernstein)**.** *If S is a minimal graph of some function $f : \mathbb{R}^2 \to \mathbb{R}$ defined over all of $\mathbb{R}^2$, then S is a plane.*

The most important hypothesis of this theorem is that the graph be defined over all of $\mathbb{R}^2$ - there are numerous examples of minimal surfaces in $\mathbb{R}^3$ which are easily described, but which are not defined for all x, y coordinates - for example, the upper half of a catenoid $r = \cosh(z)$ omits the circle $r^2 = x^2 + y^2 < 1$, and the ruled helicoid $z = y \tan x$ fails to include those (x, y) for which $\tan x = \pm\infty$ and $y \neq 0$.

Thus, Bernstein's theorem makes a global statement about the behavior of minimal graphs - and thus we need methods similarly capable. In the first proof, from the function f whose graph satisfies the hypotheses, we construct a complex holomorphic (more specifically - entire) function which avoids the upper half plane, and with the Casorati-Weierstrass conclude that it is a constant. Some additional manipulation is done to show that f has constant first derivatives of x_1, x_2 - so it describes a plane.

In the second proof, we first show that the entire minimal graph S is conformally equivalent to the complex plane C. Instead of directly constructing an entire holomorphic function like the first proof, we invoke the uniformization theorem in the Riemann surface theory to alleviate our works. Next, we observe that the Gauss map on S is conformal and invoke Picard's theorem to finish the proof.

The third proof is of a more functional analytic nature. In this proof we use some results of differential forms and introduce the notion of calibrations to establish a growth bound on the surface area of S. It turns out that this growth bound, which will be shown to be quadratic, allows us to very easily prove that S has a property called parabolicity. This property concerns the behavior of a certain class of functions on S, and we will exploit this property by defining such a function describing the unit normal on S. The parabolicity of S will then imply that the unit normal is in fact constant, meaning that S must be a plane.

2. Preparations for the first two proofs

Definition 1. *Local coordinates ξ_1, ξ_2 on a Riemannian surface (Σ, g) is said to be **isothermal** if $g_{ij} = \lambda^2 \delta_{ij}$ for some function $\lambda > 0$.*

It is a fact that local isothermal coordinates always exists on a Riemannian surface. We will not give a proof for this fact, but a justification can be found in the remark after Proposition 10. Intuitively speaking, isothermal coordinates are parametrizations that are suitable for doing complex analysis. See Remark 9 for a further explanation.

Using isothermal coordinates, we propose an alternative description for minimal surfaces. See [6].

Suppose a surface $\Sigma \subset \mathbb{R}^n$ is parametrized by $x(\xi)\colon D \to \mathbb{R}^3$ where D is a domain in $\mathbb{R}^2$ and (ξ_1, ξ_2) are isothermal coordinates. Let $\zeta = \xi_1 + i\xi_2$ and

$$\phi_k = \frac{\partial x_k}{\partial \xi_1} - i\frac{\partial x_k}{\partial \xi_2},\ k = 1, \cdots n.$$

Then we have

$$\sum_{k=1}^n \phi_k^2 = g_{11} - g_{22} - 2ig_{12} = \lambda - \lambda - 0 = 0; \tag{1}$$

$$\sum_{k=1}^n |\phi_k|^2 = g_{11} + g_{22} = 2\lambda^2 > 0. \tag{2}$$

Proposition 2. *Let $x(\xi)$ be as above. The following are equivalent:*
(I) Σ is minimal;
(II) x_k is harmonic in (ξ_1, ξ_2) for all k;
(III) ϕ_k is holomorphic in ζ for all k.

Proof. (I)$\Leftrightarrow$(II): We compute

$$\langle \Delta x, \partial_1 x\rangle = \langle \partial_{11} x, \partial_1 x\rangle + \langle \partial_{22} x, \partial_1 x\rangle$$
$$= \tfrac{1}{2}\partial_1 g_{11} + \partial_2 g_{12} - \langle \partial_2 x, \partial_{12} x\rangle = \tfrac{1}{2}\partial_1(g_{11} - g_{22}) + \partial_2 g_{12} = 0.$$

Similarly $\langle \Delta x, \partial_2 x\rangle = 0$. Therefore Δx is normal to Σ. Also note that $\partial_1 x, \partial_2 x$ are orthogonal with length λ, so $\Delta x = \partial_{11} x + \partial_{22} x = \lambda^2 H$, where H is the mean curvature vector of S.
(II)$\Leftrightarrow$(III): ϕ_k is holomorphic if and only if it the Cauchy-Riemann equation is satisfied:

$$\partial_{11} x_k = -\partial_{22} x_k,\ \partial_{12} x_k = \partial_{21} x_k.$$

The second equation is automatic, while the first is exactly the condition that x_k is harmonic. □

Therefore, in the minimal case we can recover x_k (up to constant) from ϕ_k via the complex integration

$$x_k = Re \int \phi_k d\zeta. \tag{3}$$

3. First approach

I follow the approach of Osserman [6], rearranged for clarity and with some technicalities omitted. Unfortunately, the proof requires much computation that is also omitted.

The main sketch of the proof is as follows: a particular mapping ξ from the x_1, x_2 plane to ξ_1, ξ_2 is constructed, such that this map is invertible; that this inverse is defined over the entire ξ_1, ξ_2 plane (as opposed to part of it); and with ξ_1, ξ_2 being isothermal coordinates of the surface. From this mapping ξ^{-1}, we construct a holomorphic/analytic entire function, which has negative imaginary part everywhere. It must therefore be a (complex) constant, and after some manipulation we obtain another set of isothermal coordinate u_1, u_2. The coordinate functions x_1, x_2 are shown to have constant derivatives with respect to u_1, u_2, and thus they describe a plane.

Let S, f be as in the statement. Let (x_1, x_2) be coordinates in the domain of f. Abbreviating the first derivatives by $p = f_{x_1}, q = f_{x_2}$, recall that the minimal surface equation for f is

$$(1+q^2)f_{x_1x_1} + (1+p^2)f_{x_2x_2} = 2pqf_{x_1x_2}.$$

Lemma 3. *The above equation is equivalent to both*

$$\frac{\partial}{\partial x_1}\left(\frac{pq}{W}\right) = \frac{\partial}{\partial x_2}\left(\frac{1+p^2}{W}\right),$$
$$\frac{\partial}{\partial x_1}\left(\frac{1+q^2}{W}\right) = \frac{\partial}{\partial x_2}\left(\frac{pq}{W}\right),$$

where $W = \sqrt{1+p^2+q^2}$.

Proof. Computation left to the reader. □

Inspecting the first equation, we can find a function $F(x_1, x_2)$ for which the first equation equals $\frac{\partial^2 F}{\partial x_1 \partial x_2}$ (valid over a bounded region such as a circle in the x-plane with radius R). Integrating, we obtain

$$\frac{\partial F}{\partial x_1} = \frac{1+p^2}{W}, \frac{\partial F}{\partial x_2} = \frac{pq}{W}.$$

Similarly, over the same region we can find G for which

$$\frac{\partial G}{\partial x_1} = \frac{pq}{W}, \frac{\partial G}{\partial x_2} = \frac{1+q^2}{W}.$$

Lemma 4. *The 2x2 matrix M with these entries in the above order is positive definite.*

Proof. A 2x2 matrix is positive definite if and only if both its trace and determinant are positive. The diagonal entries are clearly positive, and thus so is the trace. The determinant of this matrix is 1. □

Before ξ is defined, consider an intermediate mapping v, defined by $x = (x_1, x_2) \to v(x) = (v_1, v_2) \in \mathbb{R}^2$ by $v_1 = F, v_2 = G$.

Lemma 5. *For any $x \neq y \in \mathbb{R}^2$, we have*

$$(v(y) - v(x)) \cdot (y - x) > 0.$$

In the above statement, we abbreviate $v((y_1, y_2))$ as $v(y)$, letting $y = (y_1, y_2)$, and so on.

Proof. Letting $f(t) = (y-x) \cdot v(ty + (1-t)x)$ (the dot product of two vectors in $\mathbb{R}^2$), we can delicately compute using the Chain rule that

$$\begin{aligned}
f'(t) &= (y-x) \cdot \left(\frac{\partial v}{\partial x_1}(y_1 - x_1) + \frac{\partial v}{\partial x_2}(y_2 - x_2) \right) \\
&= (y_1 - x_1) \left(\frac{\partial v_1}{\partial x_1}(y_1 - x_1) + \frac{\partial v_1}{\partial x_2}(y_2 - x_2) \right) \\
&+ (y_2 - x_2) \left(\frac{\partial v_2}{\partial x_1}(y_1 - x_1) + \frac{\partial v_2}{\partial x_2}(y_2 - x_2) \right) \\
&= (y_1 - x_1) \left(\frac{\partial F}{\partial x_1}(y_1 - x_1) + \frac{\partial F}{\partial x_2}(y_2 - x_2) \right) \\
&+ (y_2 - x_2) \left(\frac{\partial G}{\partial x_1}(y_1 - x_1) + \frac{\partial G}{\partial x_2}(y_2 - x_2) \right) \\
&= (y-x) \cdot M(y-x) \\
&> 0,
\end{aligned}$$

because M is positive definite and $y \neq x$. Evaluating $f(1) > f(0)$, we obtain

$$v(y) \cdot (y-x) > v(x) \cdot (y-x)$$

and the lemma is proven by rearranging. $\square$

Now, construct $\xi_i(x_1, x_2) = x_i + u_i(x_1, x_2)$. Then for $y \neq x$ again, we have that

$$\begin{aligned}
|\xi(y) - \xi(x)||y-x| &\geq (\xi(y) - \xi(x)) \cdot (y-x) \\
&= (y - x + u(y) - u(x)) \cdot (y-x) \\
&= (y-x) \cdot (y-x) + (u(y) - u(x)) \cdot (y-x) \\
&> (y-x) \cdot (y-x) \\
&= |y-x|^2 \\
\implies |\xi(y) - \xi(x)| &> |y-x|,
\end{aligned}$$

where the first inequality is the Cauchy Schwarz inequality. This inequality also implies that the map ξ is one-to one. At this point, we may abandon the v mapping.

In the next step, it will be shown that ξ takes values on the entire ξ_1, ξ_2 plane, and not just part of it. I claim that if the original bounded domain Δ of x contains a large disc $N_R(0)$ around 0, then $\xi(\Delta)$ also has a large disc $N_R(\xi(0))$ around $\xi(0)$. Indeed, because ξ is continuously differentiable, the interior of δ must map to the interior of $\xi(\Delta)$, so that the boundary of the latter must have preimage in the boundary of the former. Thus, if $\xi_0 \in \partial\xi(\Delta)$, then $\xi_0 = \xi(x_0)$ for $x_0 \in \partial\Delta$, ie. $|x_0 - x| = R \implies |\xi(x_0) - \xi(x)| \geq R$.

This fact is important because we can then expand our circle of consideration (in the $x_1 - x_2$ plane) to arbitrarily large size, and the above tells us that the ξ_1, ξ_2 grow even bigger - so that when the entire x_1, x_2 plane is considered, then so is the entire ξ_1, ξ_2 plane.

Now, we must invert the $(x_1, x_2) \to (\xi_1, \xi_2)$ mapping, to describe the coordinate functions x_1, x_2 as functions of the parameters ξ_1, ξ_2. We can compute that the Jacobian of this forward transformation is $J := 2 + \frac{2+p^2+q^2}{W} > 0$ so that inversion is indeed possible. We can then check that ξ_1, ξ_2 are isothermal coordinates by using

$$g_{ij} = \frac{\partial x}{\partial \xi_i} \cdot \frac{\partial x}{\partial \xi_j} = \frac{\partial x_1}{\partial \xi_i}\frac{\partial x_1}{\partial \xi_j} + \frac{\partial x_2}{\partial \xi_i}\frac{\partial x_2}{\partial \xi_j} + \frac{\partial x_3}{\partial \xi_i}\frac{\partial x_3}{\partial \xi_j}.$$

$\frac{\partial x_i}{\partial \xi_j}$ for $i = 1, 2$ can be read off the inverse of the Jacobian matrix. For $i = 3$, use the chain rule to find $\frac{\partial x_3}{\partial \xi_j} = \frac{\partial x_3}{\partial x_1}\frac{\partial x_1}{\partial \xi_j} + \frac{\partial x_3}{\partial x_2}\frac{\partial x_2}{\partial \xi_j} = p\frac{\partial x_1}{\partial \xi_j} + q\frac{\partial x_2}{\partial \xi_j}$. Doing all this computation, we will find $g_{11} = g_{22} = \frac{W}{J}$, and $g_{12} = g_{21} = 0$ - so that ξ_1, ξ_2 indeed describe isothermal coordinates.

The next few steps of the proof can be found in Osserman as theorem 5.1, but is here split into many steps.

We know ξ_1, ξ_2 are isothermal parameters. From our preliminary discussion, constructing $\zeta = \xi_1 + i\xi_2 \in \mathbb{C}$ and $\phi_k = \frac{\partial x_k}{\partial \xi_1} - i\frac{\partial x_k}{\partial \xi_2}$, both ϕ_1 and ϕ_2 are holomorphic functions of ζ. Because ξ_1, ξ_2 were previously shown to both cover the entire ξ plane, ζ traverses the entire complex plane - so that ϕ_1, ϕ_2 are actually both entire.

Theorem 6. $\frac{\phi_2}{\phi_1}$ *is also entire, and more specifically, is a (complex) constant function as well.*

This is the first truly 'global' argument made in this proof, and is the reason for bringing in methodology from complex analysis.

Proof. First, we need to show that ϕ_1 avoids zero so that $\frac{\phi_2}{\phi_1}$ has non-zero denominator, and so will be entire as well.

Note the identity $Im(\frac{\phi_2}{\phi_1}) = \frac{1}{|\phi_1|^2} Im(\bar{\phi_1}\phi_2)$. To evaluate the latter term, plug in the definitions of ϕ_k and take the imaginary part, to obtain $Im(\bar{\phi_1}\phi_2) = \frac{\partial x_1}{\partial \xi_2}\frac{\partial x_2}{\partial \xi_1} - \frac{\partial x_1}{\partial \xi_1}\frac{\partial x_2}{\partial \xi_2} = -\frac{\partial(x_1, x_2)}{\partial(\xi_1, \xi_2)} < 0$.Therefore, ϕ_1 and ϕ_2 both avoid zero (otherwise the above quantity would equal zero), so the function $\frac{\phi_2}{\phi_1}$ is entire.

We have shown the entire function $\frac{\phi_2}{\phi_1}$ avoids the upper half plane - so by the Casorati-Weierstrass theorem, it must be a constant function $a - bi$ for some $b > 0$. □

Theorem 7. *There exist another set of isothermal coordinates* u_1, u_2 *which are simply a linear transformation of* x_1, x_2*:* $u_1 = x_1, u_2 = \frac{1}{b}(x_2 - au_1)$.

Proof. We have that $\phi_2 = (a - bi)\phi_1$. Plugging in ϕ_1, ϕ_2's definitions and equating real and imaginary parts, we obtain

$$\frac{\partial x_2}{\partial \xi_1} = a\frac{\partial x_1}{\partial \xi_1} - b\frac{\partial x_1}{\partial \xi_2}$$

$$\frac{\partial x_2}{\partial \xi_2} = b\frac{\partial x_1}{\partial \xi_1} + a\frac{\partial x_1}{\partial \xi_2}.$$

Letting $u_1 = x_1, u_2 = \frac{1}{b}(x_2 - au_1) \iff x_2 = au_1 + bu_2$ be the linear transformation, we have

$$\frac{\partial u_1}{\partial \xi_1} = \frac{\partial u_2}{\partial \xi_2}, \frac{\partial u_2}{\partial \xi_1} = -\frac{\partial u_1}{\partial \xi_2}.$$

These are precisely the Cauchy-Riemann equations for the function $u = u_1 + iu_2$ as a function of $\zeta = \xi_1 + i\xi_2$. Thus, u is a holomorphic function of ζ.

u's derivative can never be zero somewhere - for this would imply $\frac{\partial u_1}{\partial \xi_1}, \frac{\partial u_2}{\partial \xi_1}, \frac{\partial u_1}{\partial \xi_2}, \frac{\partial u_2}{\partial \xi_2}$ are all 0 there. From the linear relationship between u and x, this would imply $\phi_2, \phi_1 = 0$ there, which was shown to never occur. Thus, again using facts from complex analysis, u is a conformal map from $\mathbb{C}$ to $\mathbb{C}$.

Since ξ_1, ξ_2 are isothermal coordinates taking the x plane to the ξ, and u a conformal map from ξ to the u plane, we conclude by taking the composition $u \circ \xi : x \to u$ that u_1, u_2 are isothermal coordinates as well - since conformal maps preserve isothermalness.

□

We come to the final step of the proof. u_1, u_2 are isothermal parameters of x_1, x_2. Just as ϕ_i was defined as $\frac{\partial x_k}{\partial \xi_1} - i\frac{\partial x_k}{\partial \xi_2}$, define $\psi_k = \frac{\partial x_k}{\partial u_1} - i\frac{\partial x_k}{\partial u_2}$. By the linear relation between x and u, ψ_1, ψ_2 are constants. But from our preliminary discussion, $\psi_1^2 + \psi_2^2 + \psi_3^2 = 0$, so $\psi_3 = \frac{\partial x_3}{\partial u_1} - i\frac{\partial x_3}{\partial u_2}$ is a (complex) constant. Thus, x_3 has constant derivatives with respect to u_1, u_2, and again by linearity, with respect to x_1, x_2 as well - the surface is a plane.

4. Second approach

A map between two Riemannian surfaces is said to be *conformal* if it is a diffeomorphism that preserves angles. Clearly conformality is an equivalence relation between Riemannian surfaces.

Given an abstract surface Σ, two Riemannian metrics g^1, g^2 on it are said to be *conformally equivalent* if the identity map $(M, g^1) \to (M, g^2)$ is a conformal equivalence.

Definition 8. *A* ***conformal structure*** *on a surface is a conformally equivalent class of Riemannian metrics on this surface. A* ***Riemann surface*** *is an oriented surface equipped with a conformal structure.*

Remark 9. Equivalently, a Riemann surface is a one dimensional complex manifold. The equivalence comes from the observation that conformal maps between two domains in $\mathbb{C}$ are exactly biholomorphic or anti-biholomorphic maps, and that the orientation rules out the anti-biholomorphic ones.

Here is a powerful classification theorem for simply connected Riemann surfaces.

Proposition 10 (Uniformization Theorem)**.** *Any simply connected Riemann surface is conformally equivalent to exactly one of the following:*

- *The Riemann sphere* $\mathbb{CP}^1 = \mathbb{S}^2$.
- *The complex plane* $\mathbb{C}$.
- *The unit disk* $\mathbb{D}$ *in* $\mathbb{C}$.

The proof is complicated. We refer interested readers to any standard textbook in Riemann surfaces, e.g. Donaldson [3], Forster [5]. As a corollary of this we obtain the local existence of isothermal coordinates (although this is a much easier statement).

As an application for isothermal coordinates, we first prove the following lemma.

Lemma 11. *Suppose* $n = 3$*. The Gauss map* $N\colon \Sigma \to \mathbb{S}^2$ *on a minimal surface* Σ *is conformal.*

Proof. It suffices to check locally. Let (ξ_1, ξ_2) be isothermal coordinates near some point $p \in \Sigma$. Let A denotes the second fundamental form of $\Sigma \hookrightarrow \mathbb{R}^3$. Then $A_{ij} = \langle \partial_i N, \partial_j x\rangle$.

Note $\partial_i N \in T_{N(p)}\mathbb{S}^2 = T_p\Sigma$ and that $\partial_1 x, \partial_2 x \in T_p\Sigma$ are orthogonal with length λ, we have

$$\partial_i N = \frac{1}{\lambda^2}(A_{i1}\partial_1 x + A_{i2}\partial_2 x).$$

Therefore

$$|\partial_i N|^2 = \frac{1}{\lambda^2}(A_{i1}^2 + A_{i2}^2);$$

$$\langle \partial_1 N, \partial_2 N\rangle = \frac{1}{\lambda^2}(A_{11}A_{21} + A_{12}A_{22}).$$

Symmetry of A implies $A_{12} = A_{21}$, minimality of Σ implies $A_{11} + A_{22} = 0$. Therefore the equations above yields $|\partial_1 N|^2 = |\partial_2 N|^2$ and $\langle \partial_1 N, \partial_2 N\rangle = 0$, which implies that $N \circ x$ is conformal. Since conformality is an equivalence relation we conclude that N is conformal. □

Now let's get back to Theorem 1. The surface S inherited a Riemannian structure from $\mathbb{R}^3$ which descends to a conformal structure. Therefore S is naturally a simply connected Riemann surface. The key for this approach is the following lemma. Our proof presented below follows Osserman [6].

Lemma 12. *S is conformally equivalent to $\mathbb{C}$.*

Proof of Bernstein's Theorem. By Lemma 12, there is a conformal map $\phi\colon \mathbb{C} \to S$. By Lemma 11, the Gauss map $N\colon S \to \mathbb{S}^2$ is conformal. Therefore the composition $N \circ \phi \colon \mathbb{C} \to \mathbb{S}^2 = \mathbb{CP}^1$ is conformal, thus holomorphic or anti-holomorphic as a map between Riemann surfaces.

By Picard's theorem, either $N \circ \phi$ is constant, or its image omits at most two points in $\mathbb{S}^2$. Since S is a graph over $\mathbb{R}^2$, we see the image is contained in a hemisphere. Therefore N must be constant and Bernstein's theorem follows. $\square$

It remains to prove the Lemma 12. Since S is noncompact, it is not conformally equivalent to $\mathbb{S}^2$. By uniformization theorem, it suffices to rule out the possiblity that S is hyperbolic. Assume for contrary that we have a conformal equivalence $x(\xi) \colon \mathbb{D} \to S$. Let ϕ_k be defined as in Section 2. The following result might be a little surprising.

Lemma 13. *When $n = 3$, all holomorphic solution to* (1) *in a domain $U \subset \mathbb{C}$ are given by*

$$\phi_1 = \tfrac{1}{2} f(1-g^2),\ \phi_2 = \tfrac{i}{2} f(1+g^2),\ \phi_3 = fg \tag{4}$$

or

$$\phi_1 = if,\ \phi_2 = f,\ \phi_3 = 0, \tag{5}$$

where f is holomorphic in U, g is meromorphic in U, such that f has a zero of order at least $2m$ wherever g has a pole of order m.

Proof. It is straightforward to check (4)(5) are solutions.

Conversely, let (ϕ_1, ϕ_2, ϕ_3) be any solution. If $\phi_1 - i\phi_2 = 0$, then $\phi_3 = 0$ and we obtain (5) by letting $f = \phi_2$.

Now we shall assume $f := \phi_1 - i\phi_2$ is not constantly zero. Let $g := \phi_3/f$. Then we have

$$f(\phi_1 + i\phi_2) = \phi_1^2 + \phi_2^2 = -\phi_3^2 = -f^2 g^2,$$

from which we obtain $\phi_1 + i\phi_2 = -fg^2$, and (4) follows. $\square$

Therefore, the map $x(\xi)$ is given by (3) where ϕ_k are given by (4) or (5). This is called the **Weierstrass representation** for S. Note this formula applies to any simply connected minimal surfaces, where the domain of parametrization can be choosen to be either $\mathbb{D}$ or $\mathbb{C}$.

Lemma 14. *Let $f \colon \mathbb{D} \to \mathbb{C}^* = \mathbb{C} \backslash \{0\}$ be a holomorphic function. Then there exist a divergent path γ in $\mathbb{D}$ such that*

$$\int_\gamma |f(z)||dz| < \infty.$$

Proof. Let g be a primitive of f with $g(0) = 0$. Then g is a biholomorphic map from $\mathbb{D}$ onto some bounded domain in $U \subset \mathbb{C}$ containing 0. Choose $w_0 \in \mathbb{C} \backslash U$ with $|w_0|$ minimal and let $C \colon [0,1) \to \mathbb{C}$ denotes the path $t \mapsto tw_0$ in U. Then $\gamma = g^{-1}(C)$ is a divergent path in $\mathbb{D}$ satisfying

$$\int_\gamma |f(z)||dz| = \int_C |f \circ g^{-1}(w)| \left|\frac{dz}{dw}\right| |dw| = \int_C |dw| = |w_0| < \infty. \qquad \square$$

Back to the proof of Lemma 12. Let $x(\xi)$ be as above. If ϕ_k are given by (5), then x_3 is constant, which implies S is a plane which is conformally equivalent to $\mathbb{C}$, so the lemma holds. Below we assume ϕ_k are given by (4). Then we have

$$N = \frac{1}{\lambda^2}\left(\partial_1 x \times \partial_2 x\right) = \frac{1}{\lambda^2}(Im(\phi_2\bar{\phi}_3), Im(\phi_3\bar{\phi}_1), Im(\phi_1\bar{\phi}_2))$$
$$= \left(\frac{2Re(g)}{|g|^2+1}, \frac{2Im(g)}{|g|^2+1}, \frac{|g|^2-1}{|g|^2+1}\right). \tag{6}$$

By reversing the orientation (switching ξ_1, ξ_2) if necessary, we may assume N points downward, then (6) implies $|g| < 1$.

Next, by (2) we have

$$\lambda = (\frac{1}{8}|f|^2|1+g^2|^2 + \frac{1}{8}|f|^2|1-g^2|^2 + \frac{1}{2}|f|^2|g|^2)^{1/2} = \frac{|f|}{2}(1+|g|^2) \le |f|. \tag{7}$$

In particular since $\lambda > 0$, $f\colon \mathbb{D} \to \mathbb{C}$ does not vanishes. Now let γ be choicen as in Lemma 14. Then $x \circ \gamma$ is a divergent path in S, hence has infinite length. However,

$$length(x \circ \gamma) = \int_{x\circ\gamma} ds = \int_\gamma \lambda |d\zeta| \le \int_\gamma |f(\zeta)||d\zeta| < \infty,$$

which is a contradiction.

5. Third approach

In this approach, the core idea is to show that every minimal graph S "grows quadratically in area" relative to open balls in the ambient space. To reach this result we first review and generalize some results from 18.02 and then introduce the notion of calibrations. Once we've used calibrations to establish the quadratic growth of the surface area of S, we use a log cutoff trick in the proof of Proposition 23 to show that S has a certain property called parabolicity. Once this property is established, Bernstein's theorem follows almost immediately when the parabolicity of S is used to show that a certain function on S is in fact constant.

To set notational conventions straight, let ∂M denote the *boundary* of a surface M, meaning let ∂M denote the set of points in M which are not regular points, as defined in class.

One of the first results we're going to use for this approach is Stokes's theorem, but for ease of notation it's going to be written slightly differently than in the usual 18.02 way. In order to make sense of this new form, we introduce the following definitions.

Lemma 15 (Stokes's Theorem)**.** *Given an n-dimensional surface $\Sigma \subset \mathbb{R}^3$ and a smooth $(n-1)$-form ω on $\mathbb{R}^3$, we have the equality*

$$\int_{\partial\Sigma} \omega = \int_{\Sigma} d\omega \tag{$\dagger$}$$

Proof. We won't give a proof of this result since most proofs involve a fair amount of theory that won't be used in this paper, but the interested reader should consult Munkres's textbook on manifolds or (MIT Professor Victor) Guillemin's book on differential forms. ☐

For $n = 2$ one would compute the left-hand side by parametrizing the (1-dimensional) boundary $\partial\Sigma$ of the surface Σ by a variable, say t, and then computing the line integral

$$\int_{\partial\Sigma} f(x,y)dx + g(x,y)dy = \oint \left(f(x(t),y(t))\frac{dx}{dt} + g(x(t),y(t))\frac{dy}{dt} \right) dt,$$

where we have written $\omega = f(x,y)dx + g(x,y)dy$ without loss of generality. The right-hand side of ($\dagger$) is computed by first noting that $d\omega = (\frac{dg}{dx} - \frac{df}{dy})dx \wedge dy$, following from the definition of d found in any source on differential forms, and then computing the double integral

$$\int_{\Sigma} d\omega = \iint_{\Sigma} (\frac{dg}{dx} - \frac{df}{dy})dxdy$$

using the parametrization of Σ. This theorem, which takes the form of Green's theorem in $\mathbb{R}^3$, allows one to choose whichever side is easier to compute and also yields the following result as a consequence.

Corollary 16. *Given two surfaces $\Sigma, \Sigma' \subset \mathbb{R}^3$ having identical boundary (i.e., $\partial\Sigma = \partial\Sigma'$) and a smooth 1-form ω, we have that $\int_{\Sigma} d\omega = \int_{\Sigma'} d\omega$.*

Proof. This is a generalization of the technique used in 18.02 to show that line integrals over conservative vector fields depend only on their endpoints. Since $\partial\Sigma = \partial\Sigma'$ by assumption, we can actually form the surface $\Sigma \cup \Sigma'$ in a meaningful way. That is to say, this union does indeed give us a regular surface, as opposed to, say, the disjoint union of Σ and some arbitrary point not on Σ (which doesn't even have a well-defined dimension). As a result, we can apply Stokes's theorem to this surface.

Letting $\Omega = \Sigma \cup \Sigma'$, we see that $\partial\Omega = \emptyset$, which means that the left-hand side of ($\dagger$) vanishes, where we take integrals over $\partial\Omega$ and Ω. The right-hand side, however, can be rewritten as

$$\int_{\Omega} d\omega = \int_{\Sigma} d\omega - \int_{\Sigma'} d\omega,$$

where the minus sign appears because we have to keep track of orientation (ω is sensitive to orientation-reversal). Since this quantity must equal the left-hand side of (†), which we already showed to be equal to 0, we have that $\int_\Sigma d\omega = \int_{\Sigma'} d\omega$, as desired. □

From here on let $\Omega \subset \mathbb{R}^2$ and $u : \Omega \to \mathbb{R}$ be a C^2 function satisfying the minimal surface equation, and let $\text{Graph}_u = \{(x, y, u(x, y)) \in \mathbb{R}^3 \mid (x, y) \in \Omega\}$. The graphical nature of Graph_u allows us to make the following definition.

Definition 17. *Given vectors $X, Y \in \Omega \times \mathbb{R} \subset \mathbb{R}^3$, let* ***N*** *denote the unit normal to the plane spanned by X, Y so that $N = \frac{X \times Y}{|X \times Y|}$, and let* ***(X,Y,N)*** *denote the matrix with columns given by X, Y, N, respectively. The* ***calibration*** *ω is defined to be the 2-form $\omega(X, Y) = det(X, Y, N)$, where $det(X, Y, N)$ denotes the determinant of the matrix (X, Y, N) defined above.*

Lemma 18. *The calibration ω defined above satisfies $d\omega = 0$ and, supposing X, Y are orthonormal vectors, $|\omega(X, Y)| \leq 1$, with this inequality being saturated if and only if X, Y lie in the tangent space of $Graph_u$ at some point.* [2, Eqs 1.14-1.16]

Proof. [2] □

Proposition 20 below is one of the key steps in this proof of Bernstein's theorem, and the following lemma should be thought of as a comparison result like those covered in lecture. It allows us to use the surface areas of "nice" surfaces like spheres to establish bounds on the surface area of Graph_u by exploiting its area-minimizing property. Once we've done this, proposition 20 follows almost immediately. Functionally, the following lemma plays the same role in this proof as the ODE-PDE comparison principle played in showing convexity of CSF solutions in class. By bounding our object of interest by an object which is easier to study, we can constrain its behavior.

Lemma 19. *With u as above, if $\Sigma \subset \Omega \times \mathbb{R}$ is any other surface with $\partial\Sigma = \partial Graph_u$, then*

$$Area(Graph_u) \leq Area(\Sigma)$$

Proof. (*see* [2]) By assumption, Σ and Graph_u have the same boundary, and $d\omega = 0$ by the previous lemma, so our corollary to Stokes's theorem above gives us the equality

$$\int_{\text{Graph}_u} \omega = \int_\Sigma \omega$$

The rest of the previous lemma then gives us the inequality

$$\text{Area}(\text{Graph}_u) = \int_{\text{Graph}_u} \omega = \int_\Sigma \omega \leq \text{Area}(\Sigma)$$

□

In other words, the fact that u satisfies the minimal surface equation tells us, as we expect, that Graph_u minimizes area with respect to "nearby" surfaces $\Sigma \subset \Omega \times \mathbb{R}$. Here, "nearby" means having the same boundary. Now we use the previous results to prove the following.

Proposition 20. *With u as above, if $D_r \subset \Omega$, then $Area(B_r \cap \Omega) \leq \frac{Area(S^2)}{2} r^2 = 2\pi r^2$, where $D_r \subset \mathbb{R}^2$ and $B_r \subset \mathbb{R}^3$ denote the closed unit disk in 2-space and the closed unit ball in 3-space.* [2, Cor 1.2]

Proof. We derive an extremely crude bound on the area of $B_r \cap \mathrm{Graph}_u$ by noting that $\partial B_r \cap \mathrm{Graph}_u$ divides ∂B_r into two connected components, one of which must have area at most $\frac{\mathrm{Area}(S^2)}{2} r^2 = 2\pi r^2$. By the previous lemma, this gives an upper bound on $\mathrm{Area}(B_r \cap \mathrm{Graph}_u)$. $\square$

Before proving Proposition 22 below, we first recall the definition of a parabolic surface.

Definition 21. *We say that a surface Σ is* ***parabolic*** *if it does not admit any positive superharmonic function u (i.e., $u > 0$ and $\Delta_\Sigma u \leq 0$) such that u is not constant.*

Proposition 22. *If Σ is a complete surface so that for all $s > 0$ we have $B_s^\Sigma \leq Cs^2$, then Σ is parabolic.* [2, Prop 1.37]

Proof. Following our definition above, assume that there is a function u on Σ with $u > 0$ and $\Delta_\Sigma u \leq 0$. Set $w = \log u$ so that $|\nabla_\Sigma w|^2 \leq -\Delta_\Sigma w$. Let $r(x)$ denote the distance from x to p and $R > 0$ be some positive constant which will be manipulated later, and define the *cutoff function* η by

$$\eta = \begin{cases} 1 & r^2 \leq R \\ 2 - \frac{\log r^2}{\log R} & R < r^2 \leq R^2 \\ 0 & r^2 > R^2 \end{cases}$$

It is not too hard to convince oneself that $2ab \leq \frac{1}{2}a^2 + 2b^2$ by computing $(a+b)^2$ and liberally removing (nonnegative) terms from the left hand side. Combining this inequality with an application of Stokes's theorem gives

$$\begin{aligned} \int \eta^2 |\nabla_\Sigma w|^2 &\leq -\int \eta^2 \Delta_\Sigma w \\ &\leq 2 \int \eta |\nabla_\Sigma \eta| |\nabla_\Sigma w| \\ &\leq \frac{1}{2} \int \eta^2 |\nabla_\Sigma w|^2 + 2 \int |\nabla_\Sigma \eta|^2 \end{aligned}$$

From here we substitute the definition of η and using our area bound $\mathrm{Area}(B_s^\Sigma) \leq Cs^2$, we obtain

$$
\begin{aligned}
\int_{B^{\Sigma}_{\sqrt{R}}} |\nabla_{\Sigma} w|^2 &\le \int \eta^2 |\nabla_{\Sigma} w|^2 \\
&\le 4 \int |\nabla_{\Sigma} \eta|^2 \\
&\le \frac{16}{(\log R)^2} \sum_{\ell=\frac{1}{2}\log R}^{\log R} \int_{(B^{\Sigma}_{e^{\ell}} - B^{\Sigma}_{e^{\ell-1}})} r^{-2} \\
&\le \frac{16}{(\log R)^2} \sum_{\ell=\frac{1}{2}\log R}^{\log R} Ce^2 \\
&\le \frac{8Ce^2}{\log R}
\end{aligned}
$$

Now we let $R \to \infty$ to see that w must be constant, as the integral of $|\nabla_{\Sigma} w|^2$ over all of Σ is identically zero.

□

In particular, a surface satisfying the hypotheses of Bernstein's theorem is parabolic. Before we prove Bernstein's theorem we'll need the following equation. It will be stated without proof since it is mostly computational, and the interested reader should consult [2, pp. 47] for a more thorough treatment. If we consider $u = \langle N, (0,0,1) \rangle$ and consider the graph Graph_u of this function, the second variational formula gives

$$\Delta_{\Sigma} u = -|A|^2 u \le 0, \tag{8}$$

where A denotes the shape operator on Σ.

Proof of Bernstein's Theorem. At this point Bernstein's theorem follows straightforwardly. Equation (8) tells us that $u = \langle N, (0,0,1) \rangle > 0$ is superharmonic. Applying Proposition 22, we get that u must be constant. As u gives the inner product between the unit normal N on Σ and the constant vector $(0,0,1)$ pointing upward in the z-direction, this tells us Σ is planar. □

6. Ending Remarks

6.1. **Two classifications of Riemann surfaces.** All connected Riemann surfaces are classified into three types: **elliptic, parabolic, hyperbolic**, according to its universal cover being conformally equivalent to the Riemann sphere, the complex plane, or the hyperbolic disk [5]. In this language, Lemma 12 says that S is parabolic.

In the third approach above we used a different convention by functional analysts: a connected Riemann surface is said to be **elliptic** if it is compact, **hyperbolic** if it possesses a nonconstant negative subharmonic function, and **parabolic** otherwise.

In general these two definitions are not equivalent. One obvious reason is that in the first definition, the only elliptic Riemann surface is $\mathbb{S}^2$ [5], whereas any closed surface is elliptic according to the second definition. One less trival example is that every finitely punctured closed surface is hyperbolic in the first definition, but not in the second [4].

Proposition 23. *For a simply connected Riemann surfaces, the two definitions agree.*

Therefore, Lemma 12 and Lemma 22 are equivalent.

Proof. Let S be a simply connected Riemann surface. If S is compact, then it can only be $\mathbb{S}^2$, which is ellptic in both definitions. Below we assume S is noncompact.

Since (sub)harmonicity is a conformal invariant, it suffices to prove $\mathbb{D}$ possess a nonpositive nonconstant subharmonic function while $\mathbb{C}$ does not possess one. For $\mathbb{D}$, such a function is given by $f(x, y) = x^2 - 1$. For $\mathbb{C}$, the statement is just Liouville's Theorem for subharmonic functions. □

6.2. **A generalization.** The second approach presented above can be slightly improved to prove the following generalization of Bernstein's Theorem.

Proposition 24. *Let S be any complete oriented minimal surface in $\mathbb{R}^3$. If S is not a plane, then the image of the Gauss map is dense in $\mathbb{S}^2$.*

Proof. Suppose the Gauss map does not have dense image. Rotate the axis if necessary we may assume it omit a neighborhood of $(0, 0, 1) \in \mathbb{S}^2$.

Realize $\tilde{S}$, the universal cover of S, as an immersed surface in $\mathbb{R}^3$ with the same image as S. Then the Gauss map of $\tilde{S}$ also omits a neighborhood of $(0, 0, 1)$.

Now everything in the proof of Lemma 12 goes through for $\tilde{S}$, except that we conclude from (6) that $|g| \leq C$ for some constant $C > 0$ and (7) becomes $\lambda \leq M|f|$ for some constant $M > 0$. Completeness is used to guarantee that every divergent path has infinite length. The rest of the proof are identical with the proof of Bernstein's theorem. □

6.3. **Parametrize Minimal Surfaces by Riemann Surfaces.** In this section we want to say a little more about the alternative description for minimal surfaces introduced in Section 2, which has been used in both our first and second proofs.

Recall that we have a minimal surface $\Sigma \subset \mathbb{R}^n$ equipped with a global isothermal parametrization $x(\xi) \colon D \to \Sigma$, where D is a domain in $\mathbb{R}^2$. The functions $\phi_k = \partial_1 x_k - i\partial_2 x_k$ are holomorphic and satisfy (1)(2).

How about the converse? Suppose we have arbitrary holomorphic function $\phi_1, \cdots, \phi_n$ defined on a domain $D \subset \mathbb{C} = \mathbb{R}^2$ satisfying $\sum_{k=1}^n \phi_k^2 = 0$ and $\sum_{k=1}^n |\phi_k|^2 > 0$, when can we recover the original surface $\Sigma \subset \mathbb{R}^n$ (and the parametrization $x(\xi)$)? Clearly, if we can recover Σ, then up to constant, x_k has to satisfy (3). But when D is not simply connected, these integrals are not necessarily well-defined. Another problem is that, even if (3) are well-defined for all x_k, we have no guarantee that $x(\xi)$ is an embedding.

The following proposition addresses these two issues, while generalize the domain of parametrization to a more general setting of a Riemann surface. Note that the proposition essentially provides an alternative definition for minimal surfaces immersed in some $\mathbb{R}^n$, see [1].

Proposition 25. *Let S_0 be a Riemann surface, α_k be holomorphic 1-forms on S_0, $k = 1, \cdots, n$. Let ζ be a local complex coordinate on S_0 and write $\alpha_k = \phi_k d\zeta$. Assume that $x_k := \operatorname{Re} \int \alpha_k$ are well-defined and that*

$$\sum_{k=1}^{n} \phi_k^2 = 0; \tag{9}$$

$$\sum_{k=1}^{n} |\phi_k|^2 > 0. \tag{10}$$

Then the parametrization

$$x = (x_1, \cdots, x_n) \colon S_0 \to \mathbb{R}^n \tag{11}$$

gives an immersed minimal surface.

Conversely, for any minimal surface Σ immersed in $\mathbb{R}^n$, we can find such S_0 and α_k such that Σ is parametrized by S_0 via (11).

Proof. Let S_0, α_k, x_k be given. To check minimality it suffices to work locally. Write $\alpha_k = \phi_k d\zeta$ for a complex coordinate $\zeta = \xi_1 + i\xi_2$ on S_0. Then x_k are harmonic in ξ_1, ξ_2 and that $\phi_k = \partial_1 x_k - i\partial_2 x_k$.

Now

$$0 < \sum_{k=1}^{n} |\phi_k|^2 = g_{11} + g_{22}$$

shows that the parametrized surface Σ is regular;

$$0 = \sum_{k=1}^{n} \phi_k^2 = g_{11} - g_{22} - 2ig_{12}.$$

shows $g_{11} = g_{22}$, $g_{12} = 0$, hence ξ_1, ξ_2 are isothermal coordinates on Σ.

Finally, by computation in the proof of Proposition 2,

$$2\lambda^2 H = \Delta x = 0,$$

which implies $H = 0$, i.e. Σ is minimal.

Conversely, let $\Sigma \subset \mathbb{R}^n$ be an immersed minimal surface. Then the submanifold metric on Σ descends to a conformal structure which makes Σ a Riemann surface, denoted S_0. Let $x_1, \cdots, x_n$ denotes the coordinates in $\mathbb{R}^n$. For each $k = 1, \cdots, n$, let $\alpha_k = 2\partial x_k$ be a 1-form. Then $x_k = \operatorname{Re} \int \alpha_k$. Let $\zeta = \xi_1 + i\xi_2$ be a complex coordinate, we see that α_k are in fact holomorphic 1-forms on S_0 since

$$2\bar{\partial}\alpha_k = (\Delta x_k) d\zeta \wedge d\bar{\zeta} = 0.$$

Here the last step we again used Proposition 2 together with minimality of Σ.

Finally, write $\alpha_k = \phi_k d\zeta$, then $\phi_k = \partial_1 x_k - i\partial_2 x_k$. As before

$$\sum_{k=1}^{n} \phi_k^2 = g_{11} - g_{22} - 2ig_{12} = 0;$$

$$\sum_{k=1}^{n} |\phi_k|^2 = g_{11} + g_{22} > 0.$$

Thus the Riemann surface S_0, holomorphic 1-forms α_k, together with the immersion $x \colon S_0 \hookrightarrow \mathbb{R}^n$ satisfy the requirements. $\square$

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