Problem 1. Evaluate the following limits:

(a) \( \lim_{x \to 0} \frac{e^x - e^{-x}}{\sin(x)} \)

**Solution:** Note that we have an indeterminate form 0/0, so by L’Hospital,

\[
\lim_{x \to 0} \frac{e^x - e^{-x}}{\sin(x)} = \lim_{x \to 0} \frac{e^x + e^{-x}}{\cos(x)} = 2
\]

(b) \( \lim_{x \to 0} \frac{e^{e^x} - e}{e^x - 1} \)

**Solution:** Note that we have an indeterminate form 0/0, so by L’Hospital,

\[
\lim_{x \to 0} \frac{e^{e^x} - e}{e^x - 1} = \lim_{x \to 0} \frac{e^{e^x} e^x}{e^x} = \lim_{x \to 0} e^{e^x} = e
\]

(c) \( \lim_{x \to 0} \frac{e^{2x} - 2}{e^x - 1} \)

**Solution:** Note that the denominator goes to 0, but the numerator goes to -1. Since \( e^x - 1 \) is positive as \( x \to 0^+ \) and negative as \( x \to 0^- \), we conclude that

\[
\lim_{x \to 0} \frac{e^{2x} - 2}{e^x - 1} = DNE
\]

Problem 2. Use Mean Value Theorem to prove the following inequalities:

(a) \( e^{2x} \geq 2x + 1 \) for all \( x \geq 0 \)

**Solution:** First, for \( x = 0 \), this inequality holds. Now, for \( x > 0 \), since \( f(x) = e^{2x} \) is difffable and cts on \([0, x] \), by MVT, there exists \( c \in (0, x) \) such that

\[
f'(c) = 2e^{2c} = \frac{f(x) - f(0)}{x - 0}
\]

Since \( c > 0 \), \( f'(c) = 2e^{2c} \geq 2e^0 = 2 \). So,
\[ f(x) - f(0) \geq 2x \Rightarrow e^{2x} - 1 \geq 2x \Rightarrow e^{2x} \geq 2x + 1, \text{ as desired.} \]

(b) \( \sin(x) \leq x \) for all \( x \geq 0 \)

**Solution:** First, for \( x = 0 \), this inequality holds. Now, for \( x > 0 \), since \( f(x) = \sin(x) \) is differentiable and continuous on \([0, x]\), by MVT, there exists \( c \in (0, x) \) such that

\[
f'(c) = \cos(c) = \frac{f(x) - f(0)}{x - 0}
\]

Note that \( f'(c) = \cos(c) \leq 1 \), so

\[
f(x) - f(0) \leq x \Rightarrow \sin(x) \leq x, \text{ as desired.}
\]

(c) \( \ln(x) \leq x - 1 \) for all \( x > 0 \) (Hint: consider \( x > 1 \) and \( 0 < x < 1 \) separately)

**Solution:** First, for \( x = 1 \), this inequality holds. Now, for \( x > 1 \), since \( f(x) = \ln(x) \) is differentiable and continuous on \([1, x]\), by MVT, there exists \( c \in (1, x) \) such that

\[
f'(c) = \frac{1}{c} = \frac{f(x) - f(1)}{x - 1}
\]

Since \( c > 1 \), \( f'(c) = 1/c \leq 1 \). So,

\[
f(x) - f(1) \leq x - 1 \Rightarrow \ln(x) \leq x - 1
\]

Now for \( 0 < x < 1 \), since \( \ln(x) \) is differentiable and continuous on \([x, 1]\), by MVT, there exists \( c \in (x, 1) \) such that

\[
f'(c) = \frac{1}{c} = \frac{f(1) - f(x)}{1 - x}
\]

Since \( c < 1 \), \( f'(c) = 1/c \geq 1 \). So,

\[
f(1) - f(x) \geq 1 - x \Rightarrow -\ln(x) \geq 1 - x \Rightarrow \ln(x) \leq x - 1, \text{ as desired.}
\]