Problem 1. Find the limit of (it might not exist):

(a) \( \lim_{x \to 1} \frac{x^2 + x - 2}{x^3 - 1} \)

Solution: \( \lim_{x \to 1} \frac{x^2 + x - 2}{x^3 - 1} = \lim_{x \to 1} \frac{(x - 1)(x + 2)}{(x - 1)(x^2 + x + 1)} = \lim_{x \to 1} \frac{x + 2}{x^2 + x + 1} = \frac{3}{3} = 1 \)

(b) \( \lim_{x \to 0} \frac{e^{2x} + 2e^x - 3}{e^x - 1} \)

Solution: Let \( u = e^x \), then \( u \to 1 \) as \( x \to 0 \), so we write:

\( \lim_{x \to 0} \frac{e^{2x} + 2e^x - 3}{e^x - 1} = \lim_{u \to 1} \frac{u^2 + 2u - 3}{u - 1} = \lim_{u \to 1} \frac{(u - 1)(u + 3)}{u - 1} = \lim_{u \to 1} u + 3 = 4 \)

(c) \( \lim_{x \to \infty} \frac{|x - 1| + x^2 - 1}{3x^2 + 2} \)

Solution: As \( x \to \infty \), we see that \( x - 1 \) and \( 3x^2 + 2 \) are all positive, so we can drop the absolute values:

\( \lim_{x \to \infty} \frac{|x - 1| + x^2 - 1}{3x^2 + 2} = \lim_{x \to \infty} \frac{x - 1 + x^2 - 1}{3x^2 + 2} = \lim_{x \to \infty} \frac{x^2 + x - 2}{3x^2} = \lim_{x \to \infty} \frac{1 + \frac{1}{x} - \frac{2}{x^2}}{3} = \frac{1}{3} \)

(d) \( \lim_{x \to 1} \frac{|x - 1|}{x^2 - 1} \)

Solution: Note that \( x - 1 \) is positive as \( x \) approaches 1 on the positive side, but it is negative as \( x \) approaches 1 on the negative side. So, we do the limits separately:

\( \lim_{x \to 1^+} \frac{|x - 1|}{x^2 - 1} = \lim_{x \to 1^+} \frac{x - 1}{x^2 - 1} = \lim_{x \to 1^+} \frac{1}{x + 1} = \frac{1}{2} \)

\( \lim_{x \to 1^-} \frac{|x - 1|}{x^2 - 1} = \lim_{x \to 1^-} \frac{-(x - 1)}{x^2 - 1} = \lim_{x \to 1^-} \frac{-1}{x + 1} = -\frac{1}{2} \)

Therefore, this limit does not exist!

(e) \( \lim_{x \to \infty} \frac{2 + \sin(x)}{1 + \sin(x)} \)
Solution: Note that this limit does exist since the numerator is oscillating between \([1, 3]\) and the denominator is oscillating between \([0, 2]\).

\[
(f) \lim_{x \to \infty} \frac{2 + \sin(x)}{x(5 + \sin(x))}
\]

Solution: As before, we see that \(5 + \sin(x) \in [4, 6]\) and \(2 + \sin(x) \in [1, 3]\), so by the squeeze theorem,

\[
\frac{1}{6x} \leq \frac{2 + \sin(x)}{x(5 + \sin(x))} \leq \frac{3}{4x}
\]

\[
\Rightarrow \lim_{x \to \infty} \frac{1}{6x} \leq \lim_{x \to \infty} \frac{2 + \sin(x)}{x(5 + \sin(x))} \leq \lim_{x \to \infty} \frac{3}{4x}
\]

\[
\Rightarrow \lim_{x \to \infty} \frac{2 + \sin(x)}{x(5 + \sin(x))} = 0
\]

\[
(g) \lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x}
\]

Solution 1: \(\lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x} = \lim_{x \to 0} \frac{(\sqrt{x + 1} - 1)(\sqrt{x + 1} + 1)}{x(\sqrt{x + 1} + 1)} = \lim_{x \to 0} \frac{x + 1 - 1}{x(\sqrt{x + 1} + 1)} = \lim_{x \to 0} \frac{1}{\sqrt{x + 1} + 1} = \frac{1}{2}
\]

Solution 2: Let \(u = \sqrt{x + 1}\) and as \(x \to 0\), \(u \to 1\). Also, \(u^2 - 1 = x\). So, this is:

\[
\lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x} = \lim_{u \to 1} \frac{u - 1}{u^2 - 1} = \lim_{u \to 1} \frac{1}{u + 1} = \frac{1}{2}
\]

\[
(h) \lim_{x \to 0} \frac{(1 + x)^{1/3} - 1}{x}
\]

Solution: Let \(u = (1 + x)^{1/3}\), then as \(x \to 0\), \(u \to 1\). Furthermore, we see that \(u^3 - 1 = x\). So, this is equal to:

\[
\lim_{x \to 0} \frac{(1 + x)^{1/3} - 1}{x} = \lim_{u \to 1} \frac{u - 1}{u^3 - 1} = \lim_{u \to 1} \frac{1}{u^2 + u + 1} = \frac{1}{3}
\]

\[
(i) \lim_{x \to 1} \sin^{-1} \left[ \frac{x - 1}{x^2 - 1} \right]
\]

Solution: By continuity of \(\sin^{-1}\), we see that

\[
\lim_{x \to 1} \sin^{-1} \left[ \frac{x - 1}{x^2 - 1} \right] = \sin^{-1} \left[ \lim_{x \to 1} \frac{x - 1}{x^2 - 1} \right] = \sin^{-1} \left[ \frac{1}{2} \right] = \pi/6
\]
Problem 2. Prove the following limit values:

**Lemma 1:** \( \lim_{x \to a} x = a \) for all \( a \in (-\infty, \infty) \)

*Proof.* By the definition, we want to show that for any \( \epsilon > 0 \) (condition 1), we can find \( \delta > 0 \) (condition 2) such that

\[
|x - a| < \delta \Rightarrow |x - a| < \epsilon \quad \text{(condition 3)}
\]

First, we let \( \epsilon > 0 \) be any positive number (this is condition 1). Next, we simply pick \( \delta = \epsilon \) (this is condition 2).

Note that if \( |x - a| < \delta = \epsilon \), then \( |x - a| < \epsilon \), as desired (checks condition 3 holds).

This is true for all \( \epsilon > 0 \), so our statement (and its 3 conditions) is true and we are done.

\( \square \)

(a) \( \lim_{x \to 1} 4x^2 = 4 \)

**Solution:** Using Lemma 1 and limit laws, we see

\[
\lim_{x \to 1} 4x^2 = 4 \lim_{x \to 1} x^2 = 4(\lim_{x \to 1} x)^2 = 4(1)^2 = 4
\]

(b) \( \lim_{x \to 2} x^3 - 3x^2 + 1 = -3 \)

**Solution:** Using Lemma 1 and limit laws, we see

\[
\lim_{x \to 2} x^3 - 3x^2 + 1 = \lim_{x \to 2} x^3 - 3 \lim_{x \to 2} x^2 + \lim_{x \to 2} 1 = (\lim_{x \to 2} x)^3 - 3(\lim_{x \to 2} x)^2 + \lim_{x \to 2} 1 = -3
\]

(c) \( \lim_{x \to 1} \sqrt{x} + \frac{1}{x} = 2 \)

**Solution:** Using Lemma 1 and limit laws, we see

\[
\lim_{x \to 1} \sqrt{x} + \frac{1}{x} = \sqrt{\lim_{x \to 1} x} + \frac{1}{\lim_{x \to 1} 1/x} = 1 + 1 = 2
\]

(d) \( \lim_{x \to 0} \frac{1}{x^2} = \infty \)

**Solution:** By definition, this means that for any \( N > 0 \), we can find \( \delta > 0 \) such that
\[ |x - 0| < \delta \Rightarrow \frac{1}{x^2} > N \]

First, let \( N > 0 \) be any positive number. Note that we can solve for \( \delta \) by:
\[ \frac{1}{x^2} > N \Rightarrow x^2 < N \Rightarrow |x| < \sqrt{N} \]

Note that \( |x| < \delta \) and we want \( |x| < \sqrt{N} \), so we pick \( \delta = \sqrt{N} \).

We check that indeed condition 3 holds: \( |x| < \delta = \sqrt{N} \Rightarrow \frac{1}{x^2} > N \)

Therefore, we get our limit.

(e) \( \lim_{x \to \infty} e^{-x} = 0 \)

**Solution:** By definition, this means that for any \( \epsilon > 0 \), we can find \( N > 0 \) such that
\[ x > N \Rightarrow |e^{-x} - 0| < \epsilon \]

First, let \( \epsilon > 0 \) be any positive number. Note that we can solve for \( N \) by:
\[ |e^{-x}| < \epsilon \Rightarrow e^{-x} < \epsilon \Rightarrow e^x > \frac{1}{\epsilon} \Rightarrow x > -\ln(\epsilon) \]

So, we choose \( N = -\ln(\epsilon) \). However, note that if \( \epsilon \geq 1 \), then \( N < 0 \), so we run into problems. But, note that \( N = 1 \) would work when \( \epsilon \geq 1 \).

So, let \( N = \max(-\ln(\epsilon), 1) \).

We check that indeed condition 3 holds: \( x > N \Rightarrow |e^{-x} - 0| < \epsilon. \)