"All the rain in the sky can't put out your fire. Of all the stars out tonight, you shine brighter."

Problem 1. Circle True or False. (1pt each)

a. ⟨True or False⟩ If \( \lim_{x \to 0} f(x) \) and \( \lim_{x \to 0} g(x) \) does not exist, then \( \lim_{x \to 0} (f(x) + g(x)) \) also does not exist.

**Solution:** False, consider \( f(x) = \frac{1}{x} \) and \( g(x) = -\frac{1}{x} \).

b. ⟨True or False⟩ If \( \lim_{x \to 0} f(x) \) and \( \lim_{x \to 0} g(x) \) does exist and are finite, then \( \lim_{x \to 0} (f(x) + g(x)) \) also does exist.

**Solution:** True, by the limit laws.

c. ⟨True or False⟩ If \( f \) is continuous at 0, then \( |f| \) is also continuous at 0.

**Solution:** True, since the composition of continuous functions is continuous.

d. ⟨True or False⟩ If \( |f| \) is continuous at 0, then \( f \) is also continuous at 0.

**Solution:** False, consider \( f(x) = \frac{|x|}{x} \).

Problem 2. Find the following limits [it may not exist] (3pts each):

(a) \( \lim_{x \to 1} \left[ \frac{1}{x - 1} + \frac{1}{x^2 - 3x + 2} \right] \)

**Solution:** \( \lim_{x \to 1} \left[ \frac{1}{x - 1} + \frac{1}{x^2 - 3x + 2} \right] = \lim_{x \to 1} \left[ \frac{x - 1}{(x - 1)(x - 2)} \right] = \frac{1}{1 - 2} = -1 \)

(b) \( \lim_{x \to 1} \frac{(x - 2)^2 + 1}{x - 1} \)

**Solution:** This does not exist since \( \lim_{x \to 1} \frac{(x - 2)^2 + 1}{x - 1} = \lim_{x \to 1} \frac{2}{x - 1} = \text{DNE} \).
(c) \( \lim_{x \to 2} \frac{\sqrt{x + 2} - x}{x^2 - 4} \)

Solution: \( \lim_{x \to 2} \frac{\sqrt{x + 2} - x}{x^2 - 4} = \lim_{x \to 2} \frac{(\sqrt{x + 2} - x)(\sqrt{x + 2} + x)}{(x^2 - 4)(\sqrt{x + 2} + x)} = \lim_{x \to 2} \frac{x + 2 - x^2}{(x^2 - 4)(\sqrt{x + 2} + x)} \)

\[ = \lim_{x \to 2} \frac{-(x - 2)(x + 1)}{(x - 2)(x + 2)(\sqrt{x + 2} + x)} = \lim_{x \to 2} \frac{-(x + 1)}{(x + 2)(\sqrt{x + 2} + x)} = -\frac{3}{16} \]

Problem 3. Prove \( \lim_{x \to 1} 3x^3 = 3 \). (5pts)

Proof. Without using limit laws, note we want to solve:

By the definition, we want to show that for any \( \epsilon > 0 \) (condition 1), we can find \( \delta > 0 \) (condition 2) such that

\[ 0 < |x - 1| < \delta \Rightarrow |3x^3 - 3| < \epsilon \] (condition 3)

We first solve for \( \delta \):

\[ |3x^3 - 3| < \epsilon \Rightarrow 3|x - 1||x^2 + x + 1| < \epsilon \Rightarrow |x - 1| < \epsilon/(3|x^2 + x + 1|) \]

Note we let \( \delta \leq 1 \) (i just picked 1, but you can pick any other constant), then we know that

\[ |x - 1| < \delta \leq 1 \Rightarrow |x - 1| \leq 1 \Rightarrow x \in [0, 2] \]

Therefore, this means that \( |x^2 + x + 1| \in [1, 7] \). So, this means that \( \epsilon/21 < \epsilon/(3|x^2 + x + 1|) \).

Therefore, \( \delta = \min(1, \epsilon/21) \).

Now, we prove our statement.

First, let \( \epsilon > 0 \) be any positive number (condition 1) and let \( \delta = \min(1, \epsilon/21) > 0 \) (condition 2). We claim that condition 3 is true:

\[ |x - 1| < \delta \Rightarrow |3x^3 - 3| < \epsilon \]

Since \( |x - 1| < \delta = \min(1, \epsilon/21) \leq 1 \), we conclude that \( x \in [0, 2] \Rightarrow |x^2 + x + 1| \in [1, 7] \).

Therefore, \( |3x^2 - 3| = 3|x - 1||x^2 + x + 1| < 3(\epsilon/21)(7) = \epsilon \).

So, \( |3x^3 - 3| < \epsilon \) and this concludes our proof.