You do not need to simplify expressions that only involve arithmetic evaluations of numbers. Note: Formulas will be given to you on the quiz.

“But seek first the kingdom and his righteousness and all things will be given to you as well.”

Problem 1. Circle True or False.

a. (True or False) Applying Simpson’s rule to numerically integrate \( f(x) = x^2 \) returns a perfect approximation (no approximation error).

Solution: True. Simpson’s Rule perfectly integrates quadratics since its fourth derivative is 0, so the error term is 0.

b. (True or False) If a function \( f(x) \) is convex everywhere, then the Trapezoidal rule will under-estimate the true integral.

Solution: False, the Trapezoidal rule will overestimate as the error term will be negative since \( f''(x) > 0 \).

Problem 2. a. Consider \( f(x) = x^3 \). Use the forwards-difference formula to estimate \( f'(0) \) with \( h = 0.1 \) and bound the approximation error.

Solution: The approximation is \( f'(0) \approx \frac{f(0.1) - f(0)}{0.1} = (0.1)^2 = 0.01 \). The approximation error is at most

\[
\frac{|0.1|}{2} \max_{\xi \in [0,0.1]} |f''(\xi)| = \frac{|0.1|}{2} \max_{\xi \in [0,0.1]} |6\xi| = 0.03
\]
b. Assume that the forwards difference formula, \( N_1(h) \), has error of the form \( M = N_1(h) + K_2 h^2 + O(h^3) \). Perform Richardson’s extrapolation to approximate \( f'(0) \) with \( O(h^3) \) error using \( N_1(0.1) \) and \( N_1(0.05) \)

**Solution:** We can calculate easily that \( N_1(0.1) = 0.01 \) and \( N_1(0.05) = 0.0025 \). To get a \( O(h^3) \) error we see that

\[
f'(0) = N_1(0.1) + K_2(0.1)^2 + O((0.1)^3)
\]
\[
f'(0) = N_1(0.05) + K_2(0.05)^2 + O((0.1)^3)
\]
\[
\Rightarrow 4f'(0) = 4N_1(0.05) + K_2(0.1)^2 + O((0.1)^3)
\]

Subtracting, we get

\[
\Rightarrow 3f'(0) = 4N_1(0.05) - N_1(0.1) + O((0.1)^3).
\]

Therefore, the extrapolated answer is \( \frac{1}{3}(4N_1(0.05) - N_1(0.1)) = 0 \).

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**Problem 3.** c. Find \( c_1, x_0, x_1 \) such that the quadrature formula \( \int_0^2 f(x) \, dx = f(x_0) + c_1 f(x_1) \) has the highest precision possible.

**Solution:** We know that the highest possible precision is 2 since we have 3 unknowns.

When \( f(x) = 1 \), we get \( 2 = 1 + c_1 \Rightarrow c_1 = 1 \)

When \( f(x) = x \), we get \( 2 = x_0 + c_1 x_1 = x_0 + x_1 \)

When \( f(x) = x^2 \), we get \( 8/3 = x_0^2 + c_1 x_1^2 = x_0^2 + x_1^2 \)

Therefore, \( x_0^2 + (2 - x_0)^2 = 8/3 \Rightarrow 2x_0^2 - 4x_0 + 4 - 8/3 = 0 \)

So, \( x_0 = \frac{4 \pm \sqrt{16 - 8(4 - 8/3)}}{4} = 1 \pm \sqrt{1 - (2 - 4/3)} = 1 \pm \sqrt{1/3} \). And \( x_1 = 2 - x_0 = 1 \pm \sqrt{1/3} \).